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Multivariate Feedback Particle Filter Rederived from the Splitting-Up Scheme

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Abstract. The multivariate feedback particle filter (FPF) is formulated from the viewpoint of splitting-up methods. The essential difference between this formulation and the formal derivation is that instead of one-time control at a discrete time instant, we consider the updating stage as a stochastic flow of particles in each time interval. This allows to easily obtain a consistent stochastic flow by comparing the Kolmogorov forward equation of particles and the updating part of the Kushner's equation in the splitting-up method. Moreover, if an optimal stochastic flow exists, the convergence of the splitting-up method can be studied by passing to an FPF with a continuous time. To guarantee the existence of a stochastic flow, we validate the Poincaré inequality for the alternating distributions, given the time discretization and the observation path, under mild conditions on the nonlinear filtering system and the initial state. Besides, re-examining the original derivation of the FPF, we show that the optimal transport map between the prior and the posterior is an f-divergence invariant in the abstract Bayesian inference framework.

AMS subject classifications: 65M10, 78A48

Key words: Feedback particle filter, splitting-up method, stochastic flow, invariance property of f-divergence.

1. Introduction

The nonlinear filtering (NLF), often called the nonlinear estimation, is to give the state/signal X_t a proper estimation based on the observation history $\{z_s, 0 \le s \le t\}$ in some way, say the conditional expectation $\mathbb{E}[X_t|\mathscr{Z}_t]$, $\mathscr{Z}_t := \sigma(\{z_s,\, 0 \le s \le t\})$. It was first investigated in pioneering works of Wiener [35] and Kolmogorov [15]. The most influential work in filtering theory is the classic Kalman filter (KF) [16] and Kalman-Bucy filter [17], which are optimal for linear filtering problems. To deal with nonlinear and non-Gaussian problems, there are lots of derivatives of KF, including the extended Kalman filter [10],

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unscented Kalman filter [14], ensemble Kalman filter [8], etc. All these approaches aim to obtain the approximation of interested statistical quantities, say expectation, variance, etc., and are referred to as local approaches [23]. In contrast, global approaches approximate the posterior distribution of the state x_t conditioning on the history of observation, which satisfies the Kushner's equation [19]. It is clear that global approaches provide more accurate estimation of the state than local ones, but with much heavier computational burden as the trade-off.

One of the aims of the global approach is the numerical solution of the Kushner's equation or its unnormalized version — viz. the Duncan-Mortensen-Zakai (DMZ) equation [7, 29, 40]. Since such stochastic partial differential equations (SPDEs) cannot be solved in a closed form, Bensoussan *et al.* [3] proposed to approximate the solution of the DMZ equation by splitting it into two alternating processes — viz. prediction and updating, motivated by the Trotter decomposition from semigroup theory. One of them is the solution of a deterministic second order parabolic partial differential equation (PDE), while the other one satisfies a stochastic differential equation (SDE). Both are easier to be handled than the original DMZ equation. Ito *et al.* [13] pointed out that the Zakai's equation has serious deficiencies as a computational tool, due to its fast dissipation with respect to time and the effect of intermittency — i.e. the large peaks. They investigated a splitting-up method for the Kushner's equation. In fact, they showed the approximated solution converges in the weak and strong L^2-sense. The splitting-up method has also been extended to the NLF problems with the correlated noises [9, 25]. In 2008, Yau *et al.* [39] who proposed a feasible algorithm to the robust piecewise DMZ equation. Later on, the real-time performance of this algorithm for 1-dimensional time-varying system has been numerically validated by the second author of this paper and her co-authors [26, 27].

Besides these SPDE based algorithm, the most popular global approach is the so-called particle filter (PF) [1]. The PF is a simulation-based algorithm, which approximates the posterior distribution by the empirical distribution of the particles $\{X_t^i\}_{i=1,\dots,N}$. It is well-known that the traditional PF bears particle impoverish and degeneracy. A common remedy to avoid these shortcomings is to vigor the particles by resampling according to the importance weight at each time step. After the proper resampling strategy, the PF can propagate the posterior distribution with accuracy improved by increasing the number of the particles [5]. Nevertheless, the choice of the importance weight is crucial, problem-dependent and with no universal guidelines. Recently, the idea of transporting measures via coupling techniques has been introduced in the NLF [32] to avoid the resampling in PF. The couplings can be viewed as the nonlinear transformations from the priori distribution to the posterior distribution. It is well-known that the couplings are not unique. The uniqueness may be guaranteed by imposing some optimality conditions. Nevertheless, the major difficulty of this method is the intractability of the transport map. The transport maps induced by the flows of the ordinary differential equations are investigated in the particle flow [6] and the Gibbs flow [11]. Related ideas also appear in feedback particle filter (FPF) [38]. It is mentioned in [32, Section 5] that the consistency of the FPF may shed some light on that of the stochastic map filter in a continuous-time setting. This motivates us to rederive the FPF in the continuous setting.