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An Adaptive Nonlinear Least-Squares Finite Element Method for a Pucci Equation in Two Dimensions

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Abstract. We present an adaptive nonlinear least-squares finite element method for a two dimensional Pucci equation. The efficiency of the method is demonstrated by a numerical experiment.

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Key words: Pucci equation, nonlinear least-squares, finite element, adaptive.

1. Introduction

The Pucci equation is a fully nonlinear second order elliptic partial differential equation that first appeared in the study of linear uniformly elliptic equations in nondivergence form — cf. [13, 35, 36], and has found applications in optimal designs (cf. [14]) and population models — cf. [12, 37].

Let Ω be a bounded convex polygon in $\mathbb{R}^2$. We consider in this paper the following Dirichlet boundary value problem for a Pucci equation:

$$
\begin{aligned}
&\alpha\lambda_{\max}(D^2u)+\lambda_{\min}(D^2u)=\psi && \text{in } \Omega,\\
&u=\phi && \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $\alpha>1$, $\lambda_{\max}(D^2u)$ (resp., $\lambda_{\min}(D^2u)$) is the maximum (resp., minimum) eigenvalue of D^2u (the Hessian of u), $\psi\in L^2(\Omega)$ and $\phi\in H^2(\Omega)$.

Remark 1.1. Throughout this paper we will follow the standard notation for differential operators, functions spaces and norms that can be found for example in [1, 7, 22].

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The numerical treatment of Pucci's equation began in [14, 19], followed by the work in [30]. The finite element methods in these papers were tested extensively but without convergence analysis. Finite difference methods for the viscosity solutions of the Pucci equation were investigated in [23, 34], where the convergence was established in the framework of [2] without convergence rate, and a second order consistent finite difference method was considered in [4].

Motivated by our work on the Monge-Ampère equation in [10], a nonlinear least-squares method was presented in [9] for the strong solutions of (1.1), where convergence with convergence rates was established. Our goal in this paper is to present an adaptive version of this nonlinear least-squares method and demonstrate its effectiveness through a numerical experiment.

The rest of the paper is organized as follows. We introduce the nonlinear least-squares method in Section 2 and briefly recall the theoretical results from [9]. The numerical result for the adaptive version is presented in Section 3. We end with some concluding remarks in Section 4.

2. A Nonlinear Least-Squares Finite Element Method

Let $S_{2\times 2}$ be the space of real 2×2 symmetric matrices and $P(M)$ be the Pucci operator defined on $S_{2\times 2}$ given by

$$P(M) = \alpha\lambda_{\max}(M) + \lambda_{\min}(M)$$

for a constant $\alpha > 1$. We can then write the boundary value problem (1.1) as

$$\begin{aligned} P(D^2 u) &= \psi \quad \text{in } \Omega, \\ u &= \phi \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

A unique strong solution $u \in H^2(\Omega)$ of (2.1) was established in [9] by using the uniform ellipticity of $P(D)$, the Miranda-Talenti inequality

$$\|D^2 v\|_{L^2(\Omega)} \le \|\Delta v\|_{L^2(\Omega)}, \quad \forall v \in H^2(\Omega)\cap H_0^1(\Omega),$$

that holds on convex domains [24,32,38] and the theory of Companato on near operators [15,31].

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size $h, V_h \subset H^1(\Omega)$ be the cubic Lagrange finite element space (cf. [7,18]) associated with $\mathcal{T}_h$, and Π_h be the nodal interpolation operator from $C(\bar{\Omega})$ to V_h.

The nonlinear least-squares method in [9] is given by

$$u_h = \underset{v_h \in L_h}{\operatorname{argmin}} J_h(v_h), \tag{2.2}$$

where the constraint set L_h is defined by

$$L_h = \{v_h \in V_h : \ v_h = \Pi_h \phi \text{ on } \partial\Omega\}, \tag{2.3}$$