East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-271.290324

Vol. **14**, No. 3, pp. 460-489
August 2024

Recovering the Source Term in Elliptic Equation via Deep Learning: Method and Convergence Analysis

Chenguang Duan[1,2], Yuling Jiao[1,2], Jerry Zhijian Yang[1,2,*] and Pingwen Zhang[1,2,3]

[1] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.*
[2] *Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China.*
[3] *School of Mathematical Sciences, Peking University, Beijing 100871, China.*

Received 26 October 2023; Accepted (in revised version) 29 March 2024.

Abstract. In this paper, we present a deep learning approach to tackle elliptic inverse source problems. Our method combines Tikhonov regularization with physics-informed neural networks, utilizing separate neural networks to approximate the source term and solution. Firstly, we construct a population loss and derive stability estimates. Furthermore, we conduct a convergence analysis of the empirical risk minimization estimator. This analysis yields a prior rule for selecting regularization parameters, determining the number of observations, and choosing the size of neural networks. Finally, we validate our proposed method through numerical experiments. These experiments also demonstrate the remarkable robustness of our approach against data noise, even at high levels of up to 50%.

AMS subject classifications: 65N15, 65N20, 65N21

Key words: Inverse source problem, deep neural network, stability estimate, convergence rate.

1. Introduction

Inverse source problems have attracted considerable interest in various scientific and engineering domains, which arise in practical applications over natural phenomena such as pollution source identification [4,5,10,45], dislocation problems [6] and inverse problems of gravimetry [29]. Additionally, they have found extensive use in a range of biomedical imaging techniques, including photo-acoustic and thermo-acoustic tomography, optical tomography [2], electroencephalography (EEG) [38], magnetoencephalography (MEG) [26], and bioluminescence tomography (BLT) [60]. Of particular relevance to this paper is the modeling of the seawater intrusion phenomenon [7,17], where the source term represents the pumping wells of freshwater within the context of seawater intrusion.

[*]Corresponding author. *Email address:* `zjyang.math@whu.edu.cn` (J.-Z. Yang)

In this work, we aim at identifying the unknown source density in elliptic problems from interior measurements. Let $\Omega \subseteq \mathbb{R}^d$ $(d \geq 1)$ be a simply connected bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the following second order elliptic equation with Neumann boundary condition:

$$\begin{aligned} -\Delta u + V(x)u &= f(x) \quad \text{in } \Omega, \\ \partial_n u &= g(x) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the potential function V and the boundary flux g are given. Further, the potential function V has a positive lower bound, that is, $V(x) \geq V_0 > 0$ for each $x \in \Omega$. Let $f^\dagger$ be the ground truth space-dependent source density and $u^\dagger$ be the solution of (1.1) corresponding to the source density $f^\dagger$. The elliptic inverse source problems aim to recover the unknown source density from finite number of random samples generated from the following noisy model:

$$y^\delta(x) = u^\dagger(x) + \xi(x), \quad z^\delta(x) = \nabla u^\dagger(x) + \zeta(x), \quad x \sim U(\Omega), \tag{1.2}$$

where $\xi(x)$ and $\zeta_j(x)$, $j \in [d]$ are noise terms, and $U(\Omega)$ represents the uniform distribution on Ω. Further, we assume that

$$\|\xi\|_{L^\infty(\Omega)} \leq \delta, \quad \|\zeta_j\|_{L^\infty(\Omega)} \leq \delta, \quad j \in [d], \tag{1.3}$$

where δ is known as the noise level in the context of inverse problems. Notice that $H^1(\Omega)$-norm measurement in (1.2) is stronger than the usual $L^2(\Omega)$-norm measurement, the technical motivation for which is the necessity to establish stability estimates (see Theorem 2.1) for reconstructions. However, $H^1(\Omega)$-norm measurement also makes sense, which has been used in [36, 44]. For example, in the context of inverse problems of gravimetry [29], the gravitational force $\nabla u^\dagger$ can be measured directly, and the measurement of gravitational field $u^\dagger$ can be perceived by the noisy measurement of gravitational force. Besides, if only the $L^2(\Omega)$-norm measurement is available, the measurement of gradient can be obtained by some numerical differentiation methods after pre-smoothing the raw noisy data of $u^\dagger$. In addition, the provable $H^1(\Omega)$-norm estimation can also be obtained from noisy $L^2(\Omega)$-measurements via the finite element method [28] or deep Sobolev regression [18].

There have been extensive study devoted to the uniqueness and stability of inverse source problems [7, 8]. The uniqueness can be obtained by means of Holmgren's theorem and the regularity of the forward problem, as it was done in [8]. Further, the Lipschitz stability estimates for inverse source problems are proposed in [7].

Due to the ill-posed nature of inverse source problems [7, 21, 27, 56], constructing accurate and stable numerical approximations can be challenging. Several reconstruction methods have been developed to address this issue [1,28,39,40,45,58,61]. One popular approach involves reformulating the inverse source problem as an output least-squares PDE-constrained optimization problem, complemented with Tikhonov regularization [14, 30]. By formulating it as an optimization problem, classical optimization algorithms can then be employed for solution. In practical computation, one still needs to discretize Tikhonov functional and the PDE constraint, which is often achieved by the Galerkin finite element