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Real-Time Computing for a Parameterized Feedback Control Problem of Boussinesq Equations by POD and Deep Learning

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Abstract. An efficient real-time computational method for a feedback control problem of the Boussinesq equations is studied. We consider a simple and effective feedback control law based on the relationship between the control and adjoint variables in the optimality system. We investigate a closure type modeling in reduced order model (ROM) of this problem for real-time computing. In order to improve the existing well-known POD-ROM method, the deep learning technique, which is currently being actively researched, is studied and applied. Computational results presented show that the suggested methods work well.

AMS subject classifications: 65M10, 78A480

Key words: Optimal control, feedback control, Boussinesq, finite element, POD, LSTM.

1. Introduction

Many mathematicians and scientists have long worked on the mathematical analysis and computations of optimal control problems for fluid flows. Also, feedback control problem has been studied for efficient real-time computations. In this article, we study efficient computations for a linear feedback control problem of the Boussinesq equations describing viscous incompressible fluid flow coupled with thermodynamics. Dynamics and approximations for linear feedback controls for tracking velocities in Navier-Stokes in [14] and Bénard flows in [20, 21] were considered. Their goals were to steer over time a candidate velocity field $\boldsymbol{u}$ and fluid temperature θ to a target velocity field $\boldsymbol{U} \in \mathbf{L}^2(\Omega)$ and fluid temperature $\Theta \in L^2(\Omega)$ by appropriately controlling the body forces of the velocity and temperature field. For real-time and efficient numerical computations, we study the reduced order modeling technique that has been researched for the past 30 years and the deep learning method that has been very actively researched recently.

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First, we consider an optimal control problem for the non-dimensionalized Boussinesq equations, given $T > 0$, the target velocity $\boldsymbol{U}$, and the target temperature Θ, seek $(\boldsymbol{u}, \boldsymbol{f}, \theta, \tau)$ such that the cost functional

$$\begin{aligned}\mathscr{J}(\boldsymbol{u},\boldsymbol{f}) = &\frac{1}{2}\int_0^T\int_\Omega \left(|\boldsymbol{u}-\boldsymbol{U}|^2 + |\theta-\Theta|^2\right) d\Omega dt + \int_0^T\int_\Omega \left(\frac{\alpha_1}{2}|\boldsymbol{f}|^2 + \frac{\alpha_2}{2}|\tau|^2\right) d\Omega dt \\ &+ \frac{\delta_1}{2}\int_\Omega |\boldsymbol{u}(T)-\boldsymbol{U}(T)|^2 d\Omega + \frac{\delta_2}{2}\int_\Omega |\theta(T)-\Theta(T)|^2 d\Omega \end{aligned} \tag{1.1}$$

is minimized subject to constraints

$$\begin{aligned} &\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \beta\theta\mathbf{g} = \boldsymbol{f} && \text{in } (0,T)\times\Omega, \\ &\nabla\cdot\boldsymbol{u} = 0 && \text{in } (0,T)\times\Omega, \\ &\theta_t - \kappa\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta = \tau && \text{in } (0,T)\times\Omega, \\ &\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \quad \boldsymbol{u}(0,\boldsymbol{x}) = \mathbf{u}_0(\boldsymbol{x}), \quad \theta(t,\boldsymbol{x})|_{\partial\Omega} = 0, && \theta(0,\boldsymbol{x}) = \theta_0(\boldsymbol{x}), \end{aligned} \tag{1.2}$$

where Ω is a bounded open set in $\mathbb{R}^2$ denoted by $\partial\Omega$. Here $\boldsymbol{u}$ is the velocity vector, p is the pressure, θ is the temperature of the fluid, $\boldsymbol{f}$ is a source field, τ is a heat source. The functions $\mathbf{u}_0$ and θ_0 are given, $\mathbf{g}$ is a unit vector in the direction of gravitational acceleration, $\beta > 0$ is any positive number, $\nu > 0$ is the kinematic viscosity and $\kappa > 0$ is the thermal conductivity parameter. Using the Lagrange multipliers method, one can obtain the following optimality system: seek $(\boldsymbol{u}, p, \theta, \boldsymbol{w}, r, \psi)$ such that

$$\begin{aligned} &\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \beta\theta\mathbf{g} = \boldsymbol{f} && \text{in } (0,T)\times\Omega, \\ &\nabla\cdot\boldsymbol{u} = 0 && \text{in } (0,T)\times\Omega, \\ &\theta_t - \kappa\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta = \tau && \text{in } (0,T)\times\Omega, \end{aligned} \tag{1.3a}$$

$$\begin{aligned} &-\boldsymbol{w}_{t},\boldsymbol{v} + \nu\Delta\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + \nabla r - \beta\psi\mathbf{g} = \boldsymbol{u} - \boldsymbol{U} && \text{in } (0,T)\times\Omega, \\ &\nabla\cdot\boldsymbol{w} = 0 && \text{in } (0,T)\times\Omega, \\ &-\psi_t + \kappa\Delta\psi + (\boldsymbol{w}\nabla)\psi = \theta - \Theta && \text{in } (0,T)\times\Omega, \end{aligned} \tag{1.3b}$$

$$\boldsymbol{w} = -\alpha_1\boldsymbol{f}, \quad \psi = -\alpha_2\tau \tag{1.3c}$$

with the homogeneous boundary conditions, initial velocity and temperature $(\boldsymbol{u}_0, \theta_0)$ for the state equations, and the final conditions

$$\boldsymbol{w}(T,\boldsymbol{x}) = \delta_1\big(\boldsymbol{u}(T) - \boldsymbol{U}(T)\big), \quad \psi(T,\boldsymbol{x}) = \delta_2\big(\theta(T) - \Theta(T)\big)$$

for the adjoint equations. The optimal system (1.3) is a system of nonlinear partial differential equations consisting of nonlinear state equations (1.3a), linear adjoint equations (1.3b), and an optimality condition (1.3c). The state equations are forward in time and the adjoint equations are backward in time. For these reasons, it is known that the numerical computation of the optimality system is almost impossible or the amount of computation is