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An Adaptive Projection Algorithm for Solving Nonlinear Monotone Equations with Convex Constraints

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Abstract. In this paper, we are concerned with the problem of solving nonlinear monotone equations with convex constraints in Euclidean spaces. By combining diagonal Barzilai-Borwein method, hyperplane projection method, and adaptive extrapolation technique, an adaptive projection method is constructed. This new method is globally convergent under the assumption of continuity of the underlying map and nonemptiness of the solution set. If this map is Lipschitz continuous and satisfies the local error bound condition, this algorithm has local linear convergence rate. Numerical results show the efficiency of the proposed algorithm.

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Key words: Monotone equation, convex constraint, hyperplane projection method, diagonal Barzilai-Borwein method, local error bound condition.

1. Introduction

In the Euclidean space $\mathbb{R}^n$, a map $F:\mathbb{R}^n\to\mathbb{R}^n$ is called monotone if it satisfies

$$\langle \mathbf{x}-\mathbf{z}, F(\mathbf{x})-F(\mathbf{z})\rangle \geq 0, \quad \forall \mathbf{x},\mathbf{z}\in\mathbb{R}^n, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^n$, i.e. $\langle \mathbf{x},\mathbf{z}\rangle = \mathbf{x}^T\mathbf{z}$ for $\mathbf{x},\mathbf{z}\in\mathbb{R}^n$. For simplicity, let $\|\cdot\|$ denote the 2-norm of $\mathbb{R}^n$ and $\mathbf{0}$ denote the zero vector of $\mathbb{R}^n$. Given a continuous monotone map $F:\mathbb{R}^n\to\mathbb{R}^n$ and a closed convex set $C\subset\mathbb{R}^n$, we consider solving the following convex-constrained monotone equation:

$$F(\mathbf{x})=\mathbf{0}, \quad \text{s.t.} \quad \mathbf{x}\in C. \tag{1.2}$$

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Nonlinear monotone equations with convex constraints have application in various fields such as the subproblem in the generalized proximal algorithms with Bregman distance [6], some l_1-norm regularized optimization problems in compressive sensing [4], the neural network problem [14], the chemical equilibrium problem [11], and so on.

In the past few decades, many iterative methods have been proposed for solving nonlinear monotone equations. In [15], Solodov and Svaiter proposed a Newton-type method for solving nonlinear monotone equations without constraints, which is globally convergent and has local superlinear convergence rate under some mild assumptions. In [16], Wang extended the work of Solodov and Svaiter to solve nonlinear monotone equations with convex constraints, the global and local superlinear convergence properties were proved. These two methods require the solution of a linear system at each iteration to generate a search direction, which may be computationally costly for large-scale problems.

To avoid solving linear systems, Zhou and Li proposed some limited memory BFGS-type methods [23, 24]. Specially, various conjugate gradient type methods have been proposed for solving nonlinear monotone equations with convex constraints [1, 2, 5, 7–10, 12, 16–19, 22]. In addition, Yu *et al.* proposed a spectral gradient projection method [21]. All these methods do not require the computation of (sub)gradients of the component functions that define the problem, they rely only on function evaluations, thus these methods belong to derivative-free methods. These derivative-free methods are globally convergent if the solution set is nonempty and the underlying monotone map is continuous or Lipschitz continuous. Local linear convergence rate can be obtained under different regularity assumptions.

To solve problem (1.2), we devise an adaptive derivative-free projection method. In [13, 20], some variable metric proximal gradient type methods with diagonal Barzilai-Borwein stepsizes have been proposed for solving convex optimization problem in composite form. Inspired by the ideas of these methods, we apply the diagonal Barzilai-Borwein method to generate the search direction. In addition, we combine the hyperplane projection method with adaptive extrapolation technique to generate a candidate iterative point. Then a new iterative point can be obtained by projecting this candidate point to the constraint set. This method is globally convergent if the solution set is nonempty. Specially, local linear convergence rate can be achieved if Lipschitz continuity and local error bound condition are satisfied by the underlying map. Numerical experiments show the efficiency of this new method.

The rest of this paper is organized as follows. In Section 2, we propose an adaptive projection method. The global convergence and local linear convergence rate of this algorithm are established under some mild assumptions in Section 3. In Section 4, some numerical results are presented. Finally, some concluding remarks are given in Section 5.

2. An Adaptive Projection Algorithm for Monotone Equations

In this section, we construct an adaptive projection method to solve the convex-constrained monotone equation (1.2). Given a point $\mathbf{z} \in \mathbb{R}^n$, if $F(\mathbf{z}) \neq \mathbf{0}$, then the following set: