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An Adaptive Moving Mesh Method for Simulating Finite-Time Blowup Solutions of the Landau-Lifshitz -Gilbert Equation

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Abstract. We present a moving mesh finite element method to study the finite-time blowup solution of the Landau-Lifshitz-Gilbert (LLG) equation, considering both the heat flow of harmonic map and the full LLG equation. Our approach combines projection methods for solving the LLG equation with an iterative grid redistribution method to generate adaptive meshes. Through iterative remeshing, we successfully simulate blowup solutions with maximum gradient magnitudes up to 10^4 and minimum mesh sizes of 10^{-5}. We investigate the self-similar patterns and blowup rates of these solutions, and validate our numerical findings by comparing them to established analytical results from a recent study.

AMS subject classifications: 35Q60, 35B44, 65M50
Key words: Landau-Lifshitz-Gilbert equation, adaptive mesh, blowup solution.

1. Introduction

The Landau-Lifshitz-Gilbert equation is a mathematical model that describes magnetization dynamics in ferromagnetic materials. It was first introduced by Lifshitz and Landau in 1935 [3, 20]

$$\mathbf{m}_t = -\alpha\mathbf{m}\times(\mathbf{m}\times\Delta\mathbf{m}) - \beta\mathbf{m}\times\Delta\mathbf{m} \quad \text{in } \Omega. \tag{1.1}$$

Here, $\mathbf{m}=(m_1,m_2,m_3)^T\in\mathbb{R}^3$ is the magnetization vector, and Ω is in $\mathbb{R}^d$. The coefficient $\alpha>0$ represents the phenomenological damping parameter, and $\beta\in\mathbb{R}$ is the ferromagnetic ratio. By using the vector identity (and $|\mathbf{m}|=1$), we can rewrite Eq. (1.1) as

$$\mathbf{m}_t = \alpha(\Delta\mathbf{m}+|\nabla\mathbf{m}|^2\mathbf{m}) - \beta\mathbf{m}\times\Delta\mathbf{m} \quad \text{in } \Omega. \tag{1.2}$$

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A special case of Eq. (1.1) is when $\alpha = 1$ and $\beta = 0$, which corresponds to the heat flow of the harmonic map

$$\mathbf{m}_t = \Delta\mathbf{m} + |\nabla\mathbf{m}|^2\mathbf{m}. \tag{1.3}$$

Another special case is when $\alpha = 0$ and $\beta = -1$, resulting in the Schrodinger map flow

$$\mathbf{m}_t = \mathbf{m} \times \Delta\mathbf{m}.$$

For the heat flow of the harmonic map (1.3), Struwe [26] proved the existence and uniqueness of weak solutions with finitely many singularities on a two-dimensional Riemann surface. Further extensions and generalizations regarding the analysis of existence and uniqueness have been developed [6,13,14,23]. Analytical properties of extensions to higher dimensions ($d \geq 3$) have been studied [7–9,27].

Regarding the full Landau-Lifshitz-Gilbert equation (1.1), Alouges and Soyeur [1] established the existence and non-uniqueness of weak solutions when $d = 3$. Carbou and Fabrie [4,5] proved local and global existence with small data, as well as uniqueness of regular solutions when $d = 3$. They also demonstrated the global existence of regular solutions for small data when $d = 2$. Guo and Hong [15] proved the existence of global smooth solutions when $d = 2$ and demonstrated the global existence of weak solutions in a closed Riemannian manifold with dimension $d \geq 3$.

Adaptive mesh methods are widely used for solving partial differential equations with solutions that are nearly singular, such as shock waves and boundary layers. When numerical methods are applied, the rapid variation in solutions leads to inefficiency when discretizing functions using a uniform mesh, as an overly fine mesh is required to resolve the solution behavior in nearly singular regions. In such situations, a nonuniform adaptive mesh is preferred, where a higher proportion of points are clustered in areas with significant solution variation. This approach reduces computational cost while maintaining the accuracy of numerical solutions. There are three types of mesh refinement methods: h-refinement, p-refinement, and r-refinement.

The h-refinement locally increases the number of elements in the current mesh wherever a higher density is needed. The p-refinement increases the order of approximation polynomials to achieve better numerical convergence. The r-refinement, also known as the moving mesh method, involves relocating the vertices of the mesh based on specific criteria to improve the accuracy of numerical solutions while keeping the number of vertices and their connections unchanged. The adaptive mesh is, therefore, the image of the coordinate transformation (a one-to-one mapping) between the computational and physical domains. This paper focuses on the theories and applications of r-refinement methods (moving mesh methods) to LLG equations.

The variational approaches are the most commonly used methods when dealing with moving mesh problems in two or higher dimensions. As initially proposed by Winslow [31], the adaptive mesh is derived from the solution of a system of Poisson-like equations, which is equivalent to finding the minimizer of the corresponding functional. Specifically, one can obtain the mesh mapping by minimizing the following functional: