East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-280.090123

Vol. **14**, No. 4, pp. 675-730
November 2024

Double and Triple-Poles Soliton Solutions of Kundu-Type Equation with Zero/Nonzero Boundary Conditions

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Received 12 October 2022; Accepted (in revised version) 9 January 2023.

Abstract. The present work studies the double-poles and triple-poles soliton solutions of the Kundu-type equation with zero boundary conditions (ZBCs) and non-zero boundary conditions (NZBCs) via the Riemann-Hilbert (RH) method. We construct the RH problem with ZBCs and NZBCs both analyzing the discrete spectral and combining with the analyticity, symmetries, as well as asymptotic behavior of the modified Jost function and the scattering matrix. In the case that the reflection coefficient is double-poles and triple-poles, the inverse scattering transformation (IST) are established and solved by the RH problem with ZBCs and NZBCs, and the reconstruction formula, trace formula and theta conditions. The general formulas of double-poles and triple-poles soliton solutions with ZBCs and NZBCs are explicitly realized through expresses of determinants. The dynamic analysis for the double-poles and triple-poles soliton solutions of ZBCs/NZBCs are vividly described in the form of images.

AMS subject classifications: 65M10, 78A48

Key words: Kundu-type equation, Riemann-Hilbert method, double-poles solutions, triple-poles solutions, zero/nonzero boundary condition.

1. Introduction

In this paper, we focus on the Kundu-type equation [7, 20, 21]

$$\mathrm{i}u_t + u_{xx} - 2\mathrm{i}(2\beta - 1)|u|^2 u_x - \mathrm{i}(4\beta - 1)u^2 u_x^* + \beta(4\beta - 1)|u|^4 u = 0, \tag{1.1}$$

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where u is the complex envelope of wave and $u^* = u^*(x,t)$ the complex conjugate of $u = u(x,t)$. Among them, as a special case of complex Ginzburg-Landau equation, the Kundu-type equation (1.1) is used to describe some phenomena in physics and mechanics [7, 20, 21, 39]. The Kundu equation was first proposed by Kundu while studying the nonlinear Schrödinger (NLS)-type equations [20, 21], but there are differences between the expression forms of the Eq. (1.1) and the Kundu-Eckhaus equation. The Eq. (1.1) can be used to describe the propagation of ultrashort femtosecond pulses in optical fibers due to it is added with three nonlinear terms [19]. In addition, the Eq. (1.1) can be transformed into three kinds of derivative NLS equations. For $\beta = 0$, the Kundu-type equation (1.1) reduces to the first type of derivative NLS equation — i.e. the Kaup-Newell equation [13, 18, 23, 35–37, 50, 59]. For $\beta = 1/4$, the Kundu-type equation (1.1) reduces to the second type of derivative NLS equation — i.e. the Chen-Lee-Liu equation [51, 57, 58]. For $\beta = 1/2$, the Kundu-type equation (1.1) reduces to the third type of derivative NLS equation — i.e. the Gerjikov-Ivanov equation [14, 34, 45, 49]. In [38], Qiao studied the process of transforming the spectral problem into a finite dimensional completely integrable Hamiltonian system, and obtained the involutive solutions of the generalized derivative NLS equation. In [7], Fan successfully constructed the N-fold Darboux transformation (DT) of the Kundu-type equation (1.1). The solitary wave solutions of the Kundu equation are obtained by using algebraic curve method [8]. A weak conservative difference scheme for the nonlinear Kundu equation are proposed [22]. The implicit higher-order rogue wave solutions for the Eq. (1.1) are constructed by applying its Darboux transformation [46]. Recently, the Riemann-Hilbert (RH) method were used to constructed N-soliton solution of the Kundu-type equation (1.1) with zero boundary conditions (ZBCs) at infinity [44]. Hu *et al.* [16] proved that the potential function of Kundu equation can be represented by the uniqueness of the solution of RH problem.

As one of the most effective methods to study the initial value problem (IVP) of integrable nonlinear evolution equations, the inverse scattering transformation (IST) method was first proposed by Gardner *et al.* [9], and applied to the IVP of the classical KdV equation. After that, the method is popularized and applied in the corresponding fields. Among them, the original IST methods can be summarized as solving on the Gel'fand-Levitan-Marchenko (GLM) integral equation. In 1984, Zakharov *et al.* [54] developed a RH formula to replace the GLM integral equation, thereby simplifying the IST method. After that RH formula was adopted more and more extensively in the integrable equations by many researchers [11, 26, 28–30, 42, 47, 55, 60]. It has further promoted the development of RH method in the field of the integrable systems, and has still attracted the attention of many researchers until now [4, 5, 10, 12, 15, 17, 24, 25, 27, 31, 32, 40, 41, 48]. For direct scattering, the corresponding soliton solutions are obtained by constructing the corresponding RH problem. Similarly, by solving the RH problem under reflection-less case, the corresponding multi-soliton solutions are obtained. If the inverse reflection coefficient has multiple poles, which makes the RH problem becomes non-regular, therefore it can not be solved by using Plemelj formula. We solved the problem with the adoption of methods proposed by Ablowitz *et al.* [1, 2], Biondini and Kraus [3], Demontis *et al.* [6], and regularized the RH problem by subtracting pole contribution and asymptotic behavior. In recent years, re-