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# Global Behavior of a Higher-Order Nonlinear Difference Equation with Many Arbitrary Multivariate Functions

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**Abstract.** Let $k \geq 0$ and $l \geq 2$ be integers, $c$ a nonnegative number and $f$ an arbitrary multivariate function such that $f(x_1, x_2, x_3, \cdots, x_l) \geq x_1 + x_2$ for $x_1, x_2 \geq 0$. This work deals with the higher-order nonlinear difference equation

$$z_{n+1} = \frac{(c+1)z_n z_{n-k} + c[f(z_n, z_{n-k}, w_3, \cdots, w_l)) - z_n - z_{n-k}] + 2c^2}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l)) + c}, \quad n \geq 0,$$

where $z_{-k}, z_{-k+1}, \cdots, z_0$ are positive initial values and $w_i$, $3 \leq i \leq l$, arbitrary functions of variables $z_{n-k}, z_{n-k+1}, \cdots, z_n$. All solutions of this equation are classified into three groups, according to their asymptotic behavior, and a decreasing and increasing characteristic of oscillatory solutions is also explored. Finally, the global asymptotic stability of the positive equilibrium solution $\bar{z} = c$ is exhibited by establishing a strong negative feedback property.

**AMS subject classifications**: 39A11, 39A30, 39A10

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## 1. Introduction

Many problems in probability, biology, computer science, digital signal processing and economics involve difference equations. Difference equations are connected with differential equations as discrete mathematics is connected with continuous mathematics. Differential equations, even supposedly elementary ones, can often be hard and overwhelming. By contrast, elementary difference equations are relatively easy to deal with.

Nevertheless, the solution structure is well studied only for linear difference equations [3], whereas for nonlinear ones, various properties of solutions are usually observed and guessed only via numerical simulations but not by rigorous mathematical analysis [4,13,15,28]. Therefore, it is fundamentally important to provide a qualitative analysis on nonlinear difference equations, especially their global behavior. This is the main topic of the current study. For related analytical studies on rational difference equations, the reader may consult Refs. [1,5,6,8,9,11,16,18,26]. Asymptotic behavior of eigenfunctions also plays a crucial role in determining scattering data in Riemann-Hilbert problems and matrix spectral problems [22,23] and representing algebro-geometric solutions of integrable equations [19,20].

Let $k \geq 0$ and $l \geq 2$ be integers, $c$ a nonnegative number and $f$ an arbitrary multivariate function such that

$$f(x_1, x_2, x_3, \cdots, x_l) \geq x_1 + x_2, \quad \text{when} \;\; x_1, x_2 \geq 0. \tag{1.1}$$

We would like to study a higher-order nonlinear difference equation involving many arbitrary multivariate functions, — viz.

$$z_{n+1} = \frac{(c+1)z_n z_{n-k} + c[f(z_n, z_{n-k}, w_3, \cdots, w_l) - z_n - z_{n-k}] + 2c^2}{z_n z_{n-k} + f(z_n, z_{n-k}, w_3, \cdots, w_l) + c}, \quad n \geq 0, \tag{1.2}$$

with positive initial values $z_{-k}, z_{-k+1}, \cdots, z_0$ and arbitrary multivariate functions $w_i$, $3 \leq i \leq l$, of variables $z_{n-k}, z_{n-k+1}, \cdots, z_n$. The positivity of the initial values and the property (1.1) guarantee the existence of positive solutions for the Eq. (1.2). Moreover, a direct computation can show that the Eq. (1.2) possesses only one positive equilibrium (steady state) solution $\bar{z} = c$.

Observe that the transformation

$$z_n = \frac{c}{y_n}, \quad n \geq -k, \tag{1.3}$$

puts (1.2) into the equivalent difference equation

$$y_{n+1} = \frac{c(y_n y_{n-k} + c) + f(c/y_n, c/y_{n-k}) y_n y_{n-k}}{c(2y_n y_{n-k} - y_n - y_{n-k} + 1) + f(c/y_n, c/y_{n-k}) y_n y_{n-k} + c^2}, \quad n \geq 0, \tag{1.4}$$

where $f = f(x_1, x_2)$ is assumed. Obviously, under (1.3), the positive equilibrium solution $\bar{z} = c$ of (1.2) becomes the positive equilibrium solution $\bar{y} = 1$ of the Eq. (1.4). Upon taking a reduction with $c = 1$ and $f(x_1, x_2) = x_1 + x_2$, we obtain the rational difference equation

$$y_{n+1} = \frac{(y_n + 1)(y_{n-k} + 1)}{2(y_n y_{n-k} + 1)}, \quad n \geq 0. \tag{1.5}$$