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An Adaptive Multigrid Method for Semilinear Elliptic Equations

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Abstract. An adaptive multigrid method for semilinear elliptic equations based on adaptive multigrid methods and on multilevel correction methods is developed. The solution of a semilinear problem is reduced to a series of linearised elliptic equations on the sequence of adaptive finite element spaces and semilinear elliptic problems on a very low dimensional space. The corresponding linear elliptic equations are solved by an adaptive multigrid method. The convergence and optimal complexity of the algorithm is proved and illustrating numerical examples are provided. The method requires only the Lipschitz continuity of the nonlinear term. This approach can be extended to other nonlinear problems, including Navier-Stokes problems and phase field models.

AMS subject classifications: 65F15, 65N15, 65N25, 65N30, 65N50

Key words: Semilinear elliptic problem, adaptive multigrid method, convergence, optimal complexity.

1. Introduction

This paper focuses on the adaptive finite element method (AFEM) for semilinear elliptic equations widely used in physics and scientific computing. AFEM was proposed by Babuška and his collaborators in [2, 3]. The corresponding theoretical analysis is well-developed — e.g. Dörfler [14] introduced a marking strategy and proved strict energy error reduction for the Laplace problem in the case of fine initial meshes. Morin *et al.* [26] considered interior node property and data oscillation and proved that there is no strict energy error reduction in general. Mekchay and Nochetto [25] obtained a similar result for second order elliptic operators by developing a total error concept. The standard AFEM has been later examined by Cascon *et al.* [8]. We note that the AFEM is also successfully applied to nonlinear elliptic equations — cf. [16, 17]. For more information about the AFEM the reader can consult Refs. [12, 27, 29]. Further development of adaptive finite element methods led to adaptive

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multigrid methods, which turns out to be fully compatible with multigrid structures. Thus, Brandt [4,5] introduced a multilevel adaptive technique, McCormick [23] developed a fast adaptive composite grid method (FAC). Besides, various topics related to adaptive multigrid methods have been discussed in Refs. [7,10,15,24,30,31,33].

Xie *et al.* [9,19–21,32] developed a multilevel correction technique and constructed an optimal algorithm for eigenvalue problems. The main idea of the multilevel correction consists in transforming eigenvalue problems into boundary value problems via multilevel finite element spaces with subsequent correction of approximate solutions by solving a low dimensional eigenvalue problem. This allows to avoid solving eigenvalue problems in fine spaces and improve the efficiency of the numerical approach.

The aim of this work is to construct a numerical method for semilinear elliptic equations, which would combine the advantages of adaptive finite element and adaptive multigrid methods with the multilevel correction method. In particular, the solution of the said equations is constructed via solutions of associated linearised elliptic equations on a series of adaptively refined partitions and a small scale semilinear elliptic equation. The dimension of the corresponding semilinear equation is fixed during the whole adaptive process and the solving time can be ignored if the size of the mesh becomes sufficiently small. In addition, for the corresponding linearised elliptic problems, multigrid iteration steps are performed only on newly refined elements and their neighbors — cf. [4, 15, 30, 31] and references therein. To investigate the convergence and complexity of the method we adopt the approach in [8,12,16]. However, in contrast to the existing adaptive methods for semilinear elliptic equations [13], our approach requires only the Lipschitz continuity of the nonlinear term instead of the boundedness of its second derivatives. Another advantage of the method is that it can be applied to linear and nonlinear eigenvalue problems.

The rest of the paper is arranged as follows. In Section 2, we recall basic notation and standard AFEM for the second order linear elliptic equations. In Section 3, we introduce an adaptive multigrid method for semilinear elliptic equations. The convergence and complexity of the method are studied in Section 4. Section 5 includes the results of numerical experiments, which demonstrate the efficiency of the method and confirm theoretical analysis. Finally, concluding remarks are given in Section 6.

2. Adaptive Finite Element Method for Linear Elliptic Equations

Let us start with notation and useful results in adaptive finite element methods for the second order linear elliptic equations. We denote by C generic positive constants which may be different at different occurrences. The symbols $\lesssim$, $\gtrsim$ and $\approx$ are, respectively, referred to the inequalities $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ with constants C_1, c_2, c_3, C_3 independent of the mesh size. If $\Omega \subset \mathscr{R}^d$, $d = 2,3$ is a bounded domain with the Lipschitz boundary $\partial\Omega$, we use the standard notation $W^{s,p}(\Omega)$ for Sobolev spaces and the corresponding norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ — cf. [1], and if $p = 2$, we write

$$H^s(\Omega) = W^{s,2}(\Omega), \quad H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \right\},$$