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A Compact Difference Scheme for Fourth-Order Temporal Multi-Term Fractional Wave Equations and Maximum Error Estimates

Guang-Hua Gao* and Rui Liu

College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, P.R. China.

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Abstract. A spatial compact difference scheme for a class of fourth-order temporal multi-term fractional wave equations is developed. The original problem is reduced to a lower order system and the corresponding time fractional derivatives are approximated by the $L1$-formula. The unconditional stability and convergence of the difference scheme are proved by the energy method. Numerical experiments support theoretical results.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Compact difference scheme, multi-term fractional derivatives, spatial fourth-order derivative, stability, convergence.

1. Introduction

Fractional calculus finds applications in various fields of science and technology, including physics, signal and image processing, control, mechanics, dynamic systems, biology, environmental science, materials and economics [26]. However, the non-locality and history dependence cause serious difficulties in finding solutions of fractional differential equations (FDEs). Therefore, considerable efforts have been spent on developing approximation methods for their solution. Nevertheless, the existing works mainly focus on equations with a single time fractional derivative term and some of the works on finite difference methods are reviewed here.

Murillo and Yuste [20] considered an explicit difference method for fractional diffusion-wave equations in the Caputo form, discretising time-fractional derivatives by the $L2$-formula. The method has first-order accuracy in time and fractional Von-Neumann approach was used in the stability analysis. Sun and Wu [29] studied a finite difference

*Corresponding author. *Email addresses:* gaoguanghua1107@163.com (G.H. Gao), 976754568@qq.com (R. Liu)

method, where time-fractional derivatives in the diffusion-wave equations are approximated by the $L1$-formula. Zhang *et al.* [36] proposed a compact alternating direction implicit (ADI) scheme for two-dimensional fractional wave equations and proved its unconditional stability in the H^1-norm. Using weighted and shifted Grünwald difference operator, Wang and Vong [31] developed numerical schemes with temporal second-order and spatial fourth-order accuracy for modified anomalous fractional diffusion-wave equations. Similar approach is used in the study of an ADI scheme for two-dimensional problems [32]. New difference schemes for fractional diffusion-wave equations with reaction term presented in [5] are based on the second-order Grünwald-Letnikov discretisation of the time-fractional derivatives. Huang *et al.* [13] considered the equivalence of time-fractional diffusion-wave equations with partial integro-differential equations in the construction of finite difference schemes for time-fractional diffusion-wave equation and proved their stability and convergence by the energy method. Arshad *et al.* [2] also relied on integral equations when considering a fourth-order difference method for time-space fractional differential equations. Zeng [34] utilised second-order fractional trapezoidal and generalised Newton-Gregory formulas in solving time-fractional diffusion-wave equations. Liu *et al.* [18] proposed a different approximation for the time-fractional derivative on the time level $t = t_n$ instead of the usual approximation on a half-one level in the $L1$-formula. Determining solutions of time-fractional diffusion-wave equations on a two-dimensional unbounded spatial domain, Brunner *et al.* [3] introduced artificial boundary conditions.

The modelling of various phenomena leads to fractional diffusion-wave equations with the fourth-order spatial derivative. Agrawal [1] found a general solution of such equations on bounded domains. On the other hand, Guo *et al.* [8] studied a local discontinuous Galerkin method. Li and Wong [16] used quintic splines in spatial derivative approximation to develop a numerical method of order higher than four. They also discretised the time-fractional derivative by weighted and shifted Grünwald-Letnikov formulas [15]. Hu and Zhang [10–12] considered various finite difference schemes for fourth-order fractional diffusion-wave equations. Zhang and Pu [35] proposed a compact difference scheme for fourth-order fractional sub-diffusion systems and Vong and Wang [30] considered a compact finite difference scheme for same type of systems with the first kind of Dirichlet boundary conditions. Ji *et al.* [14] studied another compact difference scheme for a similar problem and Yao and Wang [33] investigated a compact difference scheme to problems with Neumann boundary conditions.

All the above mentioned works deal with fourth-order fractional diffusion-wave equations containing a single time-fractional derivative term. It is worth noting that multi-term fractional derivatives are used in visco-elastic damping [9], frequency-dependent loss and dispersion [19]. As far as numerical methods are concerned, Liu *et al.* [19] reduced a multi-term fractional differential equation to a system with several single-term equations, employing then a fractional predictor and corrector method. Dehghan *et al.* [4] applied a compact finite difference approximation in time and Galerkin spectral approximation in space to a multi-term fractional wave equation and Zhou *et al.* [37] studied a weak Galerkin finite element method for multi-term time-fractional diffusion equation. Using variable separation and the $L1$-formula, Shen *et al.* [23] determined analytical and numerical solutions of