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A Second-Order Accurate Implicit Difference Scheme for Time Fractional Reaction-Diffusion Equation with Variable Coefficients and Time Drift Term

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Abstract. Two implicit finite difference schemes combined with the Alikhanov's $L2$-1_σ-formula are applied to one- and two-dimensional time fractional reaction-diffusion equations with variable coefficients and time drift term. The unconditional stability and L_2-convergence of the methods are established. It is shown that the convergence order of the methods is equal to 2 both in time and space. Numerical experiments confirm the theoretical results. Moreover, since the arising linear systems can be ill-conditioned, three preconditioned iterative methods are employed.

AMS subject classifications: 65M06, 65M12, 65N06

Key words: Caputo fractional derivative, $L2$-1_σ-formula, finite difference scheme, time fractional reaction-diffusion equation, iterative method.

1. Introduction

In the past decades, fractional calculus received considerable attention because of numerous applications such as modeling of HIV infection [7], human heart [43], entropy [39], hydrology [2], anomalous diffusion in complex systems [20] and in some other fields [3, 19, 40]. Fractional diffusion equations (FDEs) represent an important tool and are actively used in such studies — cf. [4, 8, 11, 14, 16, 17, 21, 22, 31, 53] and references therein.

Let us note that at any given point, the solutions of equations with fractional operators depend on their behavior in the entire domain — i.e. the fractional operators are

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nonlocal. Therefore, FDEs are more suited for description of materials and processes with memory than the usual integer-order equations. On the other hand, the nonlocality causes various problems — e.g. analytical solutions of FDEs are known only in certain special cases [38]. Therefore, the development of efficient numerical methods for the equations mentioned becomes a fundamental task. Up to now, a variety of numerical methods for FDEs have been proposed — e.g. finite difference method [4, 11, 14, 15, 17, 51], finite element method [24–26], collocation method [31], meshless method [6] and spectral method [32]. It is worth noting that the finite difference schemes are one of the most popular approaches, especially for space and time FDEs. Thus Meerschaert and Tadjeran [33] applied an implicit Euler method based on the standard Grünwald-Letnikov formula to discretise space-fractional advection-dispersion equations with the first-order accuracy. However, the corresponding implicit difference scheme (IDS) turns out to be unstable and in order to overcome instability, they introduced an unconditionally stable shifted Grünwald-Letnikov formula. Later on, second-order approximations of space FDEs have been considered. In particular, Sousa and Li [45] derived an unconditionally stable weighted average finite difference formula for a one-dimensional fractional differential equation (FDE). It converges with the rate $\mathscr{O}(\tau + h^2)$, where τ and h are time step and mesh sizes, respectively. Tian *et al.* [47] proposed a class of second-order approximations, termed as weighted and shifted Grünwald difference (WSGD) operators and used to solve two-sided one-dimensional space FDEs. The convergence rate of this implicit difference scheme is $\mathscr{O}(\tau^2 + h^2)$. Adopting the same ideas and utilising the quasi-compact numerical technique, Zhou *et al.* [52] developed a numerical approximate scheme with the convergence rate $\mathscr{O}(\tau^2+h^3)$. Subsequently, Hao *et al.* [18] applied a new fourth-order difference approximation, using weighted average of the shifted Grünwald formulae and compact numerical technique, to a two-sided one-dimensional space FDE. They showed that this quasi-compact difference scheme is unconditionally stable. Moreover, it converges in the L_2-norm with the optimal order $\mathscr{O}(\tau^2+h^4)$. On the other hand, for the time FDEs, various difference schemes have been initially obtained using the $L1$-formula [9, 12, 34]. Later on, Gao *et al.* [13] applied the fractional numerical differentiation $L1$-2-formula to time-fractional sub-diffusion equations and obtained a solution with the accuracy $\mathscr{O}(\tau^{3-\alpha} + h^2)$, $0 < \alpha < 1$. Alikhanov [1] proposed a modified scheme with the second-order accuracy. The stability of the scheme was also proven and numerical examples suggest that it approximates the α-order Caputo fractional derivative with the second-order accuracy. Yan *et al.* [48] used this modified scheme and developed a fast high-order accurate numerical method (named $FL2$-1_σ) for speedy evaluation of the Caputo fractional derivative. The scheme substantially reduces the storage and computational cost. Nevertheless, the results concerning numerical methods for space-time FDEs are still scarce — cf. [14, 28, 29, 44] and references therein. Here we apply a second-order implicit difference scheme to the initial-boundary value problem for the following one-dimensional time fractional reaction-diffusion equation (TFRDE) with variable coefficients and a time drift term:

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} + D_{0,t}^{\alpha} u(x,t) = \mathscr{L}u(x,t) + f(x,t), \quad 0 \le x \le L, \quad 0 \le t \le T, \\
&u(x,0) = u_0(x), \quad 0 \le x \le L, \quad u(0,t) = \phi_1(t), \quad u(L,t) = \phi_2(t), \quad 0 \le t \le T,
\end{aligned}
\tag{1.1}
$$