A New C-Eigenvalue Localisation Set for Piezoelectric-Type Tensors

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Abstract. A new inclusion set for localisation of the C-eigenvalues of piezoelectric tensors is established. Numerical experiments show that it is better or comparable to the methods known in literature.

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Key words: C-eigenvalue, C-eigenvector, piezoelectric tensor, C-eigenvalue localisation theorem.

1. Introduction

Third order tensors play an important role in physics and engineering, including nonlinear optics [10,12], properties of crystals [6,11,19,20,22,26] and liquid crystals [5,9,24]. In particular, piezoelectric tensors find wide applications in converse piezoelectric and piezoelectric effects [4]. Chen et al. [4] specify the piezoelectric-type tensors as follows.

Definition 1.1 (cf. Chen et al. [4]). A third order $n$-dimensional tensor $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ is called the piezoelectric-type tensor if the last two indices of $A$ are symmetric — i.e. if $a_{ijk} = a_{ikj}$ for all $j,k \in [n]$, where $[n] := \{1,2,\ldots,n\}$.

Qi [21] and Lim [18] introduced the notion of eigenvalues for higher order tensors. It is worth noting that the eigenvalues of the third order symmetric traceless-tensors are widely used in the theory of liquid crystals [5,9,24]. Following these ideas, Chen et al. [4] defined C-eigenvalues and C-eigenvectors for piezoelectric-type tensors, which turn out to be useful in the study of piezoelectric and converse piezoelectric effects in solid crystals.

Definition 1.2 (cf. Chen et al. [4]). Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a third-order $n$-dimensional tensor. A number $\lambda \in \mathbb{R}$ is called the C-eigenvalue of $A$ if there are $x,y \in \mathbb{R}^n$ such that

$$A yy = \lambda x, \quad x A y = \lambda y, \quad x^\top x = 1, \quad y^\top y = 1,$$

where

$$\lambda = \frac{\text{trace}(A yy)}{\text{trace}(x A y)}.$$
where

\[(\mathcal{A}y y)_i = \sum_{k, j \in [n]} c_{ikj} y_k y_j, \quad (x \mathcal{A} y)_i = \sum_{k, j \in [n]} c_{kj} x_k y_j.\]

The vectors \(x\) and \(y\) are referred to as associated left and right \(C\)-eigenvectors, respectively.

By \(\sigma(\mathcal{A})\) we denote the \(C\)-spectrum of the piezoelectric-type tensor \(\mathcal{A}\) — i.e., the set of all \(C\)-eigenvalues of the piezoelectric-type tensor \(\mathcal{A}\). The \(C\)-spectral radius of \(\mathcal{A}\) is defined by

\[\rho(\mathcal{A}) := \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.\]

For a piezoelectric tensor \(\mathcal{A}\), Chen et al. [4] proved the existence of \(C\)-eigenvalues associated with left and right \(C\)-eigenvectors. They also showed that the largest \(C\)-eigenvalue of the piezoelectric tensor represents the highest piezoelectric coupling constant and it can be determined as

\[\lambda^* = \max \{x \mathcal{A} y y : x^\top x = 1, \quad y^\top y = 1\},\]

where

\[x \mathcal{A} y y := \sum_{i, k, j \in [n]} c_{ijk} x_i y_j y_k.\]

However, the practical calculation of \(\lambda^*\) is a challenging problem because of the uncertainty with the \(C\)-eigenvectors \(x\) and \(y\) in actual operations. On the other hand, we can capture all eigenvalues of a high order tensor by the eigenvalue localisation. In particular, for real symmetric tensors, Qi [21] considers an eigenvalue localisation set, which is an extension of the Geršgorin matrix eigenvalue inclusion theorem for matrices [23]. For general tensors, Li et al. [16] proposed Brauer-type eigenvalue inclusion sets. Later on, various eigenvalue localisation sets and their applications have been studied in Refs. [1, 2, 8, 13, 14, 17, 25, 27]. Recently, C. Li and Y. Li [15] introduced two intervals to estimate all \(C\)-eigenvalues of a piezoelectric-type tensor.

**Theorem 1.1** (cf. C. Li & Y. Li [15]). If \(\lambda\) is a \(C\)-eigenvalue of the piezoelectric-type tensor \(\mathcal{A} = (c_{ijk}) \in \mathbb{R}^{n \times n \times n}\), then

\[\lambda \in [-\rho, \rho],\]

where

\[\rho = \max_{i, j \in [n]} \left\{R_i^{(1)}(\mathcal{A}) R_j(\mathcal{A})\right\}^{1/2},\]

\[R_i^{(1)}(\mathcal{A}) = \sum_{l, k \in [n]} |c_{l|k}| R_j(\mathcal{A}) = \sum_{l, k \in [n]} |c_{l|k}|, \quad [n] = \{1, 2, \ldots, n\}.\]

**Theorem 1.2** (cf. C. Li & Y. Li [15]). If \(\lambda\) is a \(C\)-eigenvalue of the piezoelectric-type tensor \(\mathcal{A} = (c_{ijk}) \in \mathbb{R}^{n \times n \times n}\) and \(S\) is a subset of \([n]\), then

\[\lambda \in [-\rho_s, \rho_s],\]

where

\[\rho_s = \max_{i, j \in [n]} \left\{R_i^{(1)}(\mathcal{A}) R_j(\mathcal{A})\right\}^{1/2},\]

\[R_i^{(1)}(\mathcal{A}) = \sum_{l, k \in [n]} |c_{l|k}| R_j(\mathcal{A}) = \sum_{l, k \in [n]} |c_{l|k}|, \quad [n] = \{1, 2, \ldots, n\}.\]
where

\[ \rho_s = \max_{i,j \in [n]} \frac{1}{2} \left\{ R^\Delta_i(\mathcal{C}) + \left( R^\Delta_j(\mathcal{C}) \right)^2 + 4R^{(1)}_i(\mathcal{C}) \left( R^\Delta_j(\mathcal{C}) \right)^{1/2} \right\}, \]

\[ \Delta_S = \{(i,j) : i \in S \text{ or } j \in S\}, \quad \Delta_S = \{(i,j) : i \notin S \text{ and } j \notin S\}, \]

and

\[ R^\Delta_j(\mathcal{C}) = \sum_{l,k \in \Delta_S} |c_{lk}|, R^\Delta_j(\mathcal{C}) = \sum_{l,k \in \Delta_S} |c_{lk}|. \]

Moreover,

\[ \lambda \in [-\rho_{\min}, \rho_{\min}], \]

where \( \rho_{\min} = \min_{S \subseteq [n]} \rho_s \).

**Theorem 1.3** (cf. C. Li & Y. Li [15]). If \( \lambda \) is a C-eigenvalue of the piezoelectric-type tensor \( \mathcal{C} = (c_{ijk}) \in \mathbb{R}^{n \times n \times n} \), then

\[ \lambda \in [-\rho_{\min}, \rho_{\min}] \subseteq [-\rho, \rho], \]

where \( \rho \) and \( \rho_{\min} \) are defined in Theorems 1.1 and 1.2, respectively.

On the other hand, Che et al. [3] proposed another localisation set for C-eigenvalues.

**Theorem 1.4** (cf. Che et al. [3]). Let \( \mathcal{C} = (c_{ijk}) \in \mathbb{R}^{n \times n \times n} \) be a piezoelectric-type tensor. Then

\[ \sigma(\mathcal{C}) \subseteq \Gamma(\mathcal{C}) = \bigcup_{j \in [n]} \Gamma_j(\mathcal{C}), \]

where \( \Gamma_j(\mathcal{C}) = \{z \in \mathbb{C} : |z| \leq R_j(\mathcal{C})\} \) and \( R_j(\mathcal{C}) = \sum_{l,k \in [n]} |c_{lk}|. \)

**Theorem 1.5** (cf. Che et al. [3]). If \( \mathcal{C} = (c_{ijk}) \in \mathbb{R}^{n \times n \times n} \) is a piezoelectric-type tensor, then

\[ \sigma(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{C}) \subseteq \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(\mathcal{C}) \right), \]

where

\[ \mathcal{L}_{j,k}(\mathcal{C}) = \left\{ z \in \mathbb{C} : \left( |z| - R_j(\mathcal{C}) + R^k_j(\mathcal{C}) \right) |z| \leq R^k_j(\mathcal{C})R_k(\mathcal{C}) \right\}, \]

and \( R^k_j(\mathcal{C}) = \sum_{l \in [n]} |c_{lk}|. \)

**Theorem 1.6** (cf. Che et al. [3]). Let \( \mathcal{C} = (c_{ijk}) \in \mathbb{R}^{n \times n \times n} \) be a piezoelectric-type tensor. Then

\[ \sigma(\mathcal{C}) \subseteq \mathcal{M}(\mathcal{C}) \subseteq \bigcup_{i,k \in [n], k \neq i} \left( \mathcal{M}_{i,k}(\mathcal{C}) \cup \mathcal{K}_{i,k}(\mathcal{C}) \right), \]

where

\[ \mathcal{M}_{i,k}(\mathcal{C}) = \left\{ z \in \mathbb{C} : \left( |z| - (R_i(\mathcal{C}) - R^k_i(\mathcal{C})) \right) \left( |z| - R^k_i(\mathcal{C}) \right) \leq R^k_i(\mathcal{C})R_k(\mathcal{C}) - R^k_i(\mathcal{C}) \right\}, \]

and

\[ \mathcal{K}_{i,k}(\mathcal{C}) = \left\{ z \in \mathbb{C} : |z| - (R_i(\mathcal{C}) - R^k_i(\mathcal{C})) \leq 0, |z| - R^k_i(\mathcal{C}) \leq 0 \right\}. \]
Comparing the sets above, one can show that \( \mathcal{L}(\mathcal{E}) \subseteq \Gamma(\mathcal{E}) \) and \( \mathcal{M}(\mathcal{E}) \subseteq \Gamma(\mathcal{E}) \), i.e. the sets \( \mathcal{L}(\mathcal{E}) \) and \( \mathcal{M}(\mathcal{E}) \) are more tight than \( \Gamma(\mathcal{E}) \).

In this work, we present a new C-eigenvalue inclusion set, which is more accurate than the set \( \Gamma(\mathcal{E}) \) in Theorem 1.4. Moreover, numerical examples show that in some cases, it is superior to all the sets \( \Gamma(\mathcal{E}) \), \( \mathcal{M}(\mathcal{E}) \) and \( \mathcal{L}(\mathcal{E}) \).

2. New C-Eigenvalue Localisation Sets

In this section, we propose a new localisation set for C-eigenvalues and establish the relationship between this set and the set \( \Gamma(\mathcal{E}) \) from Theorem 1.4.

**Theorem 2.1.** If \( \mathcal{E} = (c_{ijk}) \in \mathbb{R}^{n \times n \times n} \) is a piezoelectric-type tensor, then

\[
\sigma(\mathcal{E}) \subseteq \Upsilon(\mathcal{E}) = \bigcup_{i,k \in [n], k \neq i} \left( \hat{\Upsilon}_{i,k}(\mathcal{E}) \cup \hat{\Upsilon}_{i,k}^*(\mathcal{E}) \right),
\]

where

\[
\hat{\Upsilon}_{i,k}(\mathcal{E}) = \left\{ z \in \mathbb{R} : |z| - R_i(\mathcal{E}) + \bar{R}_i^k(\mathcal{E}) \leq 0, |z| - R_k(\mathcal{E}) + \bar{R}_i^k(\mathcal{E}) \leq 0 \right\},
\]

and

\[
\hat{\Upsilon}_{i,k}^*(\mathcal{E}) = \left\{ z \in \mathbb{R} : \left[ |z| - R_i(\mathcal{E}) + \bar{R}_i^k(\mathcal{E}) \right] \left[ |z| - R_k(\mathcal{E}) + \bar{R}_i^k(\mathcal{E}) \right] \leq \bar{R}_i^k(\mathcal{E}) R_i^k(\mathcal{E}) \right\}.
\]

**Proof.** Consider a C-eigenvalue \( \lambda \) of \( \mathcal{E} \) with corresponding left C-eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \) and right C-eigenvector \( y = (y_1, y_2, \ldots, y_n)^T \), so that

\[
\mathcal{E}y = \lambda x, \quad x^T \mathcal{E}x = \lambda y, \quad x^T x = 1, \quad y^T y = 1. \tag{2.1}
\]

The assumption

\[
|y_p| \geq |y_q| \geq \max_{i \in [n], i \neq p,q} |y_i|
\]

yields \( 0 < |y_p| \leq 1 \). If follows from (2.1) that

\[
\lambda y_p = \sum_{l,k \in [n]} c_{lk p} x_l y_k = \sum_{l,k \in [n], k \neq q} c_{lk p} x_l y_k + \sum_{l \in [n]} c_{lq p} x_l y_q.
\]

Since \( 0 \leq |x_i| \leq 1 \) for any \( i \in [n] \), we obtain

\[
|\lambda| \leq \sum_{l,k \in [n], k \neq q} |c_{lk p}| \frac{|y_k|}{|y_p|} + \sum_{l \in [n]} |c_{lq p}| \frac{|y_q|}{|y_p|} \leq \sum_{l,k \in [n], k \neq q} |c_{lk p}| + \sum_{l \in [n]} |c_{lq p}| \frac{|y_q|}{|y_p|}.
\]

Writing the last inequality as

\[
|\lambda| - \sum_{l,k \in [n], k \neq q} |c_{lk p}| \leq \sum_{l \in [n]} |c_{lq p}| \frac{|y_q|}{|y_p|}, \tag{2.2}
\]

we obtain the following estimates:
1. If \(|y_q| = 0\), then \(|\lambda| - \sum_{l,k \in [n], k \neq q} |c_{lkp}| \leq 0\).

2. If \(|\lambda| - R_q(\mathcal{C}) + R_q^p(\mathcal{C}) \geq 0\), then \(\lambda \in \tilde{\mathcal{T}}_{p,q}(\mathcal{C})\).

3. If \(|\lambda| - R_q(\mathcal{C}) + R_q^p(\mathcal{C}) \leq 0\), then \(\lambda \in \tilde{\mathcal{T}}_{p,q}(\mathcal{C})\).

On the other hand, if \(|y_q| > 0\), then

\[
\lambda y_q = \sum_{l,k \in [n]} c_{l,k q} x_l y_k = \sum_{l,k \in [n], k \neq p} c_{l,k q} x_l y_k + \sum_{l \in [n]} c_{l,p q} x_l y_p,
\]

and

\[
|\lambda| \leq \sum_{l,k \in [n], k \neq p} |c_{l,k q}| \frac{|y_k|}{|y_q|} + \sum_{l \in [n]} |c_{l,p q}| \frac{|y_p|}{|y_q|} \leq \sum_{l,k \in [n], k \neq p} |c_{l,k q}| + \sum_{l \in [n]} |c_{l,p q}| \frac{|y_p|}{|y_q|}.
\]

Writing it in the form

\[
|\lambda| - \sum_{l,k \in [n], k \neq p} |c_{l,k q}| \leq \sum_{l \in [n]} |c_{l,p q}| \frac{|y_p|}{|y_q|},
\]

we note that if \(|\lambda| - \sum_{l,k \in [n], k \neq p} |c_{l,k q}| \leq 0\) or \(|\lambda| - R_q(\mathcal{C}) + R_q^p(\mathcal{C}) \leq 0\), then multiplying (2.2) and (2.3), we arrive at the inequality

\[
\left( |\lambda| - \sum_{l,k \in [n], k \neq p} |c_{l,k q}| \right) \left( |\lambda| - \sum_{l,k \in [n], k \neq p} |c_{l,k q}| \right) \leq \sum_{l \in [n]} |c_{l,q p}| \sum_{l \in [n]} |c_{l,p q}|.
\]

It can be written as

\[
\left[ |\lambda| - R_q(\mathcal{C}) + R_q^p(\mathcal{C}) \right] \left[ |\lambda| - R_q(\mathcal{C}) + R_q^p(\mathcal{C}) \right] \leq R_q(\mathcal{C}) R_q^p(\mathcal{C}),
\]

so that \(\lambda \in \tilde{\mathcal{T}}_{p,q}(\mathcal{C}) \subseteq \mathcal{T}(\mathcal{C})\). If

\[
|\lambda| - \sum_{l,k \in [n], k \neq q} |c_{l,k p}| \leq 0 \quad \text{and} \quad |\lambda| - R_q(\mathcal{C}) + R_q^p(\mathcal{C}) \leq 0,
\]

then \(\lambda \in \tilde{\mathcal{T}}_{p,q}(\mathcal{C}) \subseteq \mathcal{T}(\mathcal{C})\), as required. \(\square\)

**Remark 2.1.** For a real tensor \(\mathcal{C} = (c_{i,j,k}) \in \mathbb{R}^{n \times n \times n}\), \(n \geq 2\), the set \(\Gamma(\mathcal{C})\) consists of \(n\) sets \(\Gamma_j(\mathcal{C})\), the set \(\mathcal{L}(\mathcal{C})\) of \(n(n-1)\) sets \(\mathcal{L}_{j,k}(\mathcal{C})\), the set \(\mathcal{M}(\mathcal{C})\) of \(n(n-1)\) sets \(\mathcal{M}_{i,k}(\mathcal{C})\) and \(n(n-1)\) sets \(\mathcal{N}_{i,k}(\mathcal{C})\), and the set \(\mathcal{T}(\mathcal{C})\) of \(n(n-1)\) sets \(\tilde{\mathcal{T}}_{i,k}(\mathcal{C})\) and \(n(n-1)\) sets \(\tilde{\mathcal{T}}_{i,k}(\mathcal{C})\). It is worth noting that for large \(n\), the set \(\mathcal{T}(\mathcal{C})\) locates all eigenvalues of \(\mathcal{C}\) more accurately than \(\Gamma(\mathcal{C})\), but \(\Gamma(\mathcal{C})\) can be determined with less computational effort.
Comparing the sets $\Gamma(\mathcal{E})$ and $\Upsilon(\mathcal{E})$, we obtain the following result.

**Theorem 2.2.** If $\mathcal{E} = (\epsilon_{ijk}) \in \mathbb{R}^{n \times n \times n}$ is a piezoelectric-type tensor, then

$$\sigma(\mathcal{E}) \subseteq \Upsilon(\mathcal{E}) \subseteq \Gamma(\mathcal{E}).$$

**Proof.** For any $z \in \Upsilon(\mathcal{E})$, there exist $i, k \in [n], i \neq k$ such that $z \in \hat{T}_{i,k}(\mathcal{E})$ or $z \in \tilde{T}_{i,k}(\mathcal{E})$.

Next, we consider two cases.

**Case I.** If $z \in \hat{T}_{i,k}(\mathcal{E})$, i.e. if

$$|z| - R_i(\mathcal{E}) + R_k(\mathcal{E}) \leq 0 \quad \text{and} \quad |z| - R_k(\mathcal{E}) + R_i(\mathcal{E}) \leq 0,$$

then

$$|z| \leq R_i(\mathcal{E}) \quad \text{and} \quad |z| \leq R_k(\mathcal{E}),$$

so that $z \in \Gamma(\mathcal{E})$.

**Case II.** If $z \in \tilde{T}_{i,k}(\mathcal{E})$, then

$$\left[ |z| - R_i(\mathcal{E}) + R_k(\mathcal{E}) \right] \left[ |z| - R_k(\mathcal{E}) + R_i(\mathcal{E}) \right] \leq R_i(\mathcal{E}) R_k(\mathcal{E}). \quad (2.4)$$

Assuming first that $R_i(\mathcal{E}) R_k(\mathcal{E}) = 0$, we obtain

$$|z| - R_i(\mathcal{E}) + R_k(\mathcal{E}) \leq 0 \quad \text{or} \quad |z| - R_k(\mathcal{E}) + R_i(\mathcal{E}) \leq 0.$$

It follows that

$$|z| \leq R_i(\mathcal{E}) \quad \text{or} \quad |z| \leq R_k(\mathcal{E}).$$

Hence, $z \in \Gamma(\mathcal{E})$.

If we now assume that $R_i(\mathcal{E}) R_k(\mathcal{E}) > 0$, then (2.4) yields

$$\frac{|z| - R_i(\mathcal{E}) + R_k(\mathcal{E})}{R_i(\mathcal{E})} \cdot \frac{|z| - R_k(\mathcal{E}) + R_i(\mathcal{E})}{R_k(\mathcal{E})} \leq 1,$$

and at least one of the inequalities

$$\frac{|z| - R_i(\mathcal{E}) + R_k(\mathcal{E})}{R_i(\mathcal{E})} \leq 1, \quad \frac{|z| - R_k(\mathcal{E}) + R_i(\mathcal{E})}{R_k(\mathcal{E})} \leq 1$$

holds. It follows that $z \in \Gamma_i(\mathcal{E}) \cup \Gamma_k(\mathcal{E})$ and combining both cases, we finish the proof. □

3. Numerical Examples

In this section, we provide the results of numerical experiments to show that our approach locates $C$-eigenvalues much better than other methods. The piezoelectric tensors used in the examples, arise in piezoelectric materials with symmetries and have been previously studied in Refs. [3, 4, 7, 22].
Example 3.1. Consider the piezoelectric tensor \( \mathcal{A}_{VFeSb} \) [7] with the entries
\[
a_{ijk} = \begin{cases} 
a_{123} = a_{213} = a_{312} = -3.68180667, \\
a_{ijk}, \text{ otherwise.}
\end{cases}
\]

According to [4], the largest \( C \)-eigenvalue of \( \mathcal{A}_{VFeSb} \) is about 4.25138 and Theorems 1.4-1.6 show that
\[
\Gamma(\mathcal{C}) = \bigcup_{j \in [n]} \Gamma_j(\mathcal{C}) = \{ z \in \mathbb{C} : |z| \leq 7.3636 \},
\]
\[
\mathcal{L}(\mathcal{C}) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(\mathcal{C}) \right) = \{ z \in \mathbb{C} : |z| \leq 7.3636 \},
\]
\[
\mathcal{M}(\mathcal{C}) = \bigcup_{i,j,k \in [n], k \neq i} \left( \mathcal{M}_{i,j,k}(\mathcal{C}) \bigcup \mathcal{H}_{i,j,k}(\mathcal{C}) \right) = \{ z \in \mathbb{C} : |z| \leq 7.3636 \}.
\]

From Theorem 2.1, we get
\[
\Upsilon(\mathcal{C}) = \bigcup_{i,j,k \in [n], k \neq i} \left( \hat{T}_{i,j,k}(\mathcal{C}) \bigcup \hat{T}_{i,j,k}(\mathcal{C}) \right) = \{ z \in \mathbb{C} : |z| \leq 7.3636 \}.
\]

Example 3.2. Consider the piezoelectric tensor \( \mathcal{A}_{SiO2} \) [6, 22] with the entries
\[
a_{ijk} = \begin{cases} 
a_{111} = -a_{122} = a_{212} = -0.13685, \\
a_{123} = -a_{213} = -0.009715, \\
a_{ijk}, \text{ otherwise.}
\end{cases}
\]

According to [4], the largest \( C \)-eigenvalue of \( \mathcal{A}_{SiO2} \) is about 0.1375 and Theorems 1.4-1.6 show that
\[
\Gamma(\mathcal{C}) = \bigcup_{j \in [n]} \Gamma_j(\mathcal{C}) = \{ z \in \mathbb{C} : |z| \leq 0.2834 \},
\]
\[
\mathcal{L}(\mathcal{C}) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(\mathcal{C}) \right) = \{ z \in \mathbb{C} : |z| \leq 0.2744 \},
\]
\[
\mathcal{M}(\mathcal{C}) = \bigcup_{i,j,k \in [n], k \neq i} \left( \mathcal{M}_{i,j,k}(\mathcal{C}) \bigcup \mathcal{H}_{i,j,k}(\mathcal{C}) \right) = \{ z \in \mathbb{C} : |z| \leq 0.2834 \}.
\]

From Theorem 2.1, we have
\[
\Upsilon(\mathcal{C}) = \bigcup_{i,j,k \in [n], k \neq i} \left( \hat{T}_{i,j,k}(\mathcal{C}) \bigcup \hat{T}_{i,j,k}(\mathcal{C}) \right) = \{ z \in \mathbb{C} : |z| \leq 0.2834 \}.
\]

Example 3.3. Consider the piezoelectric tensor \( \mathcal{A}_{Cr2AgBiO8} \) [7] with the entries
\[
a_{ijk} = \begin{cases} 
a_{123} = a_{213} = -0.22163, \\
a_{113} = -a_{223} = 2.608665, \\
a_{311} = -a_{322} = 0.152485, \\
a_{321} = -0.37153, \\
a_{ijk}, \text{ otherwise.}
\end{cases}
\]
According to [4], the largest $C$-eigenvalue of $\mathcal{A}_{\text{Gr2AgBiO8}}$ is about 2.6258 and Theorems 1.4-1.6 show that

$\Gamma(\mathcal{C}) = \bigcup_{j \in [n]} \Gamma_j(\mathcal{C}) = \{z \in \mathbb{C} : |z| \leq 5.6606\},$

$\mathcal{L}(\mathcal{C}) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 4.8058\},$

$\mathcal{M}(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \mathcal{M}_{i,k}(\mathcal{C}) \cup \mathcal{H}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 4.7861\}.$

From Theorem 2.1, we have

$\mathcal{N}(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \hat{\mathcal{N}}_{i,k}(\mathcal{C}) \cup \hat{\mathcal{N}}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 4.7335\}.$

Example 3.4. Consider the piezoelectric tensor $\mathcal{A}_{\text{RbTaO3}}$ [7] with the entries

\[
a_{ij} = \begin{cases} 
a_{113} = a_{223} = -8.40955, \\
a_{222} = -a_{212} = -a_{211} = -5.412525, \\
a_{311} = -a_{322} = -4.3031, \\
a_{333} = -5.14766, \\
a_{ijk}, \text{ otherwise.}
\end{cases}
\]

According to [4], the largest $C$-eigenvalue of $\mathcal{A}_{\text{RbTaO3}}$ is about 12.4234 and Theorems 1.4-1.6 show that

$\Gamma(\mathcal{C}) = \bigcup_{j \in [n]} \Gamma_j(\mathcal{C}) = \{z \in \mathbb{C} : |z| \leq 23.5377\},$

$\mathcal{L}(\mathcal{C}) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 23.5377\},$

$\mathcal{M}(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \mathcal{M}_{i,k}(\mathcal{C}) \cup \mathcal{H}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 23.5377\}.$

From Theorem 2.1, we have

$\mathcal{N}(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \hat{\mathcal{N}}_{i,k}(\mathcal{C}) \cup \hat{\mathcal{N}}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 23.5377\}.$

Example 3.5. Consider the piezoelectric tensor $\mathcal{A}_{\text{NaBiS2}}$ [7] with the entries

\[
a_{ij} = \begin{cases} 
a_{113} = -8.90808, a_{223} = -0.00842, a_{311} = -7.11526, \\
a_{322} = -0.6222, a_{333} = -7.93831, \\
a_{ijk}, \text{ otherwise.}
\end{cases}
\]
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According to [4], the largest C-eigenvalue of $\mathcal{A}_{\text{NaBiS2}}$ is about 11.6674 and Theorems 1.4-1.6 show that

$$\Gamma(C) = \bigcup_{j \in [n]} \Gamma_j(C) = \{z \in \mathbb{C} : |z| \leq 16.8548\},$$

$$\mathcal{L}(C) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(C) \right) = \{z \in \mathbb{C} : |z| \leq 16.5640\},$$

$$\mathcal{M}(C) = \bigcup_{i,k \in [n], k \neq i} \left( \mathcal{M}_{i,k}(C) \cup \mathcal{N}_{i,k}(C) \right) = \{z \in \mathbb{C} : |z| \leq 16.8464\}.$$

From Theorem 2.1, we have

$$\Upsilon(C) = \bigcup_{i,k \in [n], k \neq i} \left( \hat{T}_{i,k}(C) \cup \hat{T}_{i,k}(C) \right) = \{z \in \mathbb{C} : |z| \leq 16.8464\}.$$

**Example 3.6.** Consider the piezoelectric tensor $\mathcal{A}_{\text{LiBiB205}}$ [7] with the entries

$$a_{ij} = \begin{cases} 
a_{123} = 2.35682, a_{112} = 0.34929, a_{211} = 0.16101, a_{222} = 0.12562, 
a_{233} = 0.1361, a_{213} = -0.05587, a_{323} = 6.91074, a_{312} = 2.57812, 
a_{ijk}, \text{ otherwise.} 
\end{cases}$$

According to [4], the largest C-eigenvalue of $\mathcal{A}_{\text{LiBiB205}}$ is about 7.7376 and Theorems 1.4-1.6 show that

$$\Gamma(C) = \bigcup_{j \in [n]} \Gamma_j(C) = \{z \in \mathbb{C} : |z| \leq 12.3206\},$$

$$\mathcal{L}(C) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(C) \right) = \{z \in \mathbb{C} : |z| \leq 11.0127\},$$

$$\mathcal{M}(C) = \bigcup_{i,k \in [n], k \neq i} \left( \mathcal{M}_{i,k}(C) \cup \mathcal{N}_{i,k}(C) \right) = \{z \in \mathbb{C} : |z| \leq 11.0038\}.$$n

From Theorem 2.1, we have

$$\Upsilon(C) = \bigcup_{i,k \in [n], k \neq i} \left( \hat{T}_{i,k}(C) \cup \hat{T}_{i,k}(C) \right) = \{z \in \mathbb{C} : |z| \leq 10.9998\}.$$

**Example 3.7.** Consider the piezoelectric tensor $\mathcal{A}_{\text{KBi2F7}}$ [7] with the entries

$$a_{ij} = \begin{cases} 
a_{111} = 12.64393, a_{122} = 1.08802, a_{133} = 4.14350, a_{123} = 1.59052, 
a_{113} = 1.96801, a_{112} = 0.22465, a_{211} = 2.59187, a_{222} = 0.08263, 
a_{233} = 0.81041, a_{223} = 0.51165, a_{213} = 0.71432, a_{212} = 0.10570, 
a_{311} = 1.51254, a_{322} = 0.68235, a_{333} = -0.23019, a_{323} = 0.19013, 
a_{313} = 0.39030, a_{312} = 0.08381. 
\end{cases}$$
According to [4], the largest $C$-eigenvalue of $\mathcal{A}_{KBI2F7}$ is about 20.2351 and Theorems 1.4-1.6 show that

$$
\Gamma(\mathcal{C}) = \bigcup_{j \in [n]} \Gamma_j(\mathcal{C}) = \{z \in \mathbb{C} : |z| \leq 20.2351\},
$$

$$
\mathcal{L}(\mathcal{C}) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 18.8793\},
$$

$$
\mathcal{M}(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \mathcal{M}_{i,k}(\mathcal{C}) \cup \mathcal{H}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 19.8830\}.
$$

From Theorem 2.1, we have

$$
\Upsilon(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \hat{\mathcal{H}}_{i,k}(\mathcal{C}) \cup \hat{\mathcal{H}}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 19.8319\}.
$$

Example 3.8. Consider the piezoelectric tensor $\mathcal{A}_{BaNiO3}$ [7] with the entries

$$
a_{ijk} = \begin{cases} 
    a_{113} = a_{223} = 0.038385, & a_{311} = a_{322} = 6.89822, a_{333} = 27.4628, \\
    a_{ijk}, & \text{otherwise}.
\end{cases}
$$

According to [4], the largest $C$-eigenvalue of $\mathcal{A}_{BaNiO3}$ is about 27.4628 and Theorems 1.4-1.6 show that

$$
\Gamma(\mathcal{C}) = \bigcup_{j \in [n]} \Gamma_j(\mathcal{C}) = \{z \in \mathbb{C} : |z| \leq 27.5396\},
$$

$$
\mathcal{L}(\mathcal{C}) = \bigcup_{j \in [n]} \left( \bigcap_{k \in [n], k \neq j} \mathcal{L}_{j,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 27.5109\},
$$

$$
\mathcal{M}(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \mathcal{M}_{i,k}(\mathcal{C}) \cup \mathcal{H}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 27.5013\}.
$$

From Theorem 2.1, we have

$$
\Upsilon(\mathcal{C}) = \bigcup_{i,k \in [n], k \neq i} \left( \hat{\mathcal{H}}_{i,k}(\mathcal{C}) \cup \hat{\mathcal{H}}_{i,k}(\mathcal{C}) \right) = \{z \in \mathbb{C} : |z| \leq 27.5013\}.
$$

Let $\lambda^+$ be the largest $C$-eigenvalue of the piezoelectric tensor and $[-\rho_\gamma, \rho_\gamma]$, $[-\rho_\gamma, \rho_\gamma]$, $[-\rho_\gamma, \rho_\gamma]$ and $[-\rho_\gamma, \rho_\gamma]$ are the intervals generated by Theorems 1.4, 1.5, 1.6 and 2.1, respectively. We note that in all examples, Theorem 2.2 always provides the best result among the methods tested. Table 1 lists the results obtained by methods [3,4,15] and by Theorem 2.1. It indicates that $\rho_\gamma$ is more precise than $\rho$, $\rho_{\min}$, $\rho_r$ and $\rho_\gamma$. Moreover, in some cases $\rho_r$ is tighter than $\rho_\gamma$. Thus, the $C$-eigenvalue localisation theorem obtained in this work improves the known results in [3,4,15].

4. Conclusion

We derived a new inclusion set for localisation of the $C$-eigenvalues of piezoelectric tensors. Numerical experiments show that it is better than the known set $\Gamma(\mathcal{C})$ and is comparable or better than the sets $\mathcal{M}(\mathcal{C})$ and $\mathcal{L}(\mathcal{C})$. 
A New C-Eigenvalue Localisation Set for Piezoelectric-Type Tensors

### Table 1: Numerical comparison of Theorem 2.1 and the related results [3, 4, 15].

<table>
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<tr>
<th>$\lambda^*$</th>
<th>$\rho^\gamma$</th>
<th>$\rho^\gamma_{FBLO}$</th>
<th>$\rho^\gamma_{CT^{2}ABIO}$</th>
<th>$\rho^\min_{3BIO}$</th>
<th>$\rho^\gamma_{RIB2O5}$</th>
<th>$\rho^\gamma_{KB2B7}$</th>
<th>$\rho^\gamma_{RNIO3}$</th>
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### References


