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A New C-Eigenvalue Localisation Set for Piezoelectric-Type Tensors

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Abstract. A new inclusion set for localisation of the C-eigenvalues of piezoelectric tensors is established. Numerical experiments show that it is better or comparable to the methods known in literature.

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Key words: C-eigenvalue, C-eigenvector, piezoelectric tensor, C-eigenvalue localisation theorem.

1. Introduction

Third order tensors play an important role in physics and engineering, including nonlinear optics [10,12], properties of crystals [6,11,19,20,22,26] and liquid crystals [5,9,24]. In particular, piezoelectric tensors find wide applications in converse piezoelectric and piezoelectric effects [4]. Chen *et al.* [4] specify the piezoelectric-type tensors as follows.

Definition 1.1 (cf. Chen *et al.* [4]). A third order n-dimensional tensor $\mathscr{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ is called the piezoelectric-type tensor if the last two indices of $\mathscr{A}$ are symmetric — i.e. if $a_{ijk} = a_{ikj}$ for all $j,k \in [n]$, where $[n] := \{1,2,\ldots,n\}$.

Qi [21] and Lim [18] introduced the notion of eigenvalues for higher order tensors. It is worth noting that the eigenvalues of the third order symmetric traceless-tensors are widely used in the theory of liquid crystals [5,9,24]. Following these ideas, Chen *et al.* [4] defined C-eigenvalues and C-eigenvectors for piezoelectric-type tensors, which turn out to be useful in the study of piezoelectric and converse piezoelectric effects in solid crystals.

Definition 1.2 (cf. Chen *et al.* [4]). Let $\mathscr{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a third-order n-dimensional tensor. A number $\lambda \in \mathbb{R}$ is called the C-eigenvalue of $\mathscr{A}$ if there are $x, y \in \mathbb{R}^n$ such that

$$\mathscr{A}yy = \lambda x, \quad x\mathscr{A}y = \lambda y, \quad x^\top x = 1, \quad y^\top y = 1, \tag{1.1}$$

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where

$$(\mathscr{A}yy)_i = \sum_{k,j\in[n]} c_{ikj}y_k y_j, \quad (x\mathscr{A}y)_i = \sum_{k,j\in[n]} c_{kji}x_k y_j.$$

The vectors x and y are referred to as associated left and right C-eigenvectors, respectively.

By $\sigma(\mathscr{A})$ we denote the C-spectrum of the piezoelectric-type tensor $\mathscr{A}$ — i.e. the set of all C-eigenvalues of the piezoelectric-type tensor $\mathscr{A}$. The C-spectral radius of $\mathscr{A}$ is defined by

$$\rho(\mathscr{A}) := \max\{|\lambda| : \lambda \in \sigma(\mathscr{A})\}.$$

For a piezoelectric tensor $\mathscr{A}$, Chen *et al.* [4] proved the existence of C-eigenvalues associated with left and right C-eigenvectors. They also showed that the largest C-eigenvalue of the piezoelectric tensor represents the highest piezoelectric coupling constant and it can be determined as

$$\lambda^* = \max\left\{x\mathscr{A}yy : x^\top x = 1,\, y^\top y = 1\right\},$$

where

$$x\mathscr{A}yy := \sum_{i,k,j\in[n]} c_{ijk}x_i y_j y_k.$$

However, the practical calculation of λ^* is a challenging problem because of the uncertainty with the C-eigenvectors x and y in actual operations. On the other hand, we can capture all eigenvalues of a high order tensor by the eigenvalue localisation. In particular, for real symmetric tensors, Qi [21] considers an eigenvalue localisation set, which is an extension of the Geršgorin matrix eigenvalue inclusion theorem for matrices [23]. For general tensors, Li *et al.* [16] proposed Brauer-type eigenvalue inclusion sets. Later on, various eigenvalue localisation sets and their applications have been studied in Refs. [1,2,8,13,14,17,25,27].

Recently, C. Li and Y. Li [15] introduced two intervals to estimate all C-eigenvalues of a piezoelectric-type tensor.

Theorem 1.1 (cf. C. Li & Y. Li [15]). *If λ is a C-eigenvalue of the piezoelectric-type tensor $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$, then*

$$\lambda \in [-\rho, \rho],$$

where

$$\rho = \max_{i,j\in[n]} \left\{R_i^{(1)}(\mathscr{C})R_j(\mathscr{C})\right\}^{1/2},$$
$$R_i^{(1)}(\mathscr{C}) = \sum_{l,k\in[n]} |c_{ilk}|, R_j(\mathscr{C}) = \sum_{l,k\in[n]} |c_{lkj}|, \quad [n] = \{1,2,\ldots,n\}.$$

Theorem 1.2 (cf. C. Li & Y. Li [15]). *If λ is a C-eigenvalue of the piezoelectric-type tensor $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$ and S is a subset of $[n]$, then*

$$\lambda \in \left[-\rho_s, \rho_s\right],$$

where

$$\rho_s = \max_{i,j\in[n]} \frac{1}{2}\left\{R_j^{\Delta_S}(\mathscr{C}) + \left(R_j^{\Delta_S}(\mathscr{C})\right)^2 + 4R_i^{(1)}(\mathscr{C})\left(R_j^{\overline{\Delta}_S}(\mathscr{C})\right)^{1/2}\right\},$$
$$\Delta_S = \{(i,j) : i \in S \text{ or } j \in S\}, \quad \overline{\Delta}_S = \{(i,j) : i \notin S \text{ and } j \notin S\},$$

and

$$R_j^{\Delta_S}(\mathscr{C}) = \sum_{l,k\in\Delta_S} |c_{lkj}|, R_j^{\overline{\Delta}_S}(\mathscr{C}) = \sum_{l,k\in\overline{\Delta}_S} |c_{lkj}|.$$

Moreover,

$$\lambda \in [-\rho_{\min}, \rho_{\min}],$$

where $\rho_{\min} = \min_{S\subseteq[n]} \rho_s$.

Theorem 1.3 (cf. C. Li & Y. Li [15]). *If λ is a C-eigenvalue of the piezoelectric-type tensor $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$, then*

$$\lambda \in [-\rho_{\min}, \rho_{\min}] \subseteq [-\rho, \rho],$$

where ρ and $\rho_{\min}$ are defined in Theorems 1.1 *and* 1.2*, respectively.*

On the other hand, Che *et al.* [3] proposed another localisation set for C-eigenvalues.

Theorem 1.4 (cf. Che *et al.* [3]). *Let $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor. Then*

$$\sigma(\mathscr{C}) \subseteq \Gamma(\mathscr{C}) = \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}),$$

where $\Gamma_j(\mathscr{C}) = \{z \in \mathbb{C}; |z| \leq R_j(\mathscr{C})\}$ and $R_j(\mathscr{C}) = \sum_{l,k\in[n]} |c_{lkj}|$.

Theorem 1.5 (cf. Che *et al.* [3]). *If $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$ is a piezoelectric-type tensor, then*

$$\sigma(\mathscr{C}) \subseteq \mathscr{L}(\mathscr{C}) = \bigcup_{j\in[n]} \left(\bigcap_{k\in[n],k\neq j} \mathscr{L}_{j,k}(\mathscr{C})\right),$$

where

$$\mathscr{L}_{j,k}(\mathscr{C}) = \left\{z \in \mathbb{C} : \left(|z| - R_j(\mathscr{C}) + R_j^k(\mathscr{C})\right)|z| \leq R_j^k(\mathscr{C})R_k(\mathscr{C})\right\}$$

and $R_j^k(\mathscr{C}) = \sum_{l\in[n]} |c_{lkj}|$.

Theorem 1.6 (cf. Che *et al.* [3]). *Let $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor. Then*

$$\sigma(\mathscr{C}) \subseteq \mathscr{M}(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \left(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C})\right),$$

where

$$\mathscr{M}_{i,k}(\mathscr{C}) = \left\{z \in \mathbb{C} : \left(|z| - \left(R_i(\mathscr{C}) - R_i^k(\mathscr{C})\right)\right)\left(|z| - R_k^k(\mathscr{C})\right) \leq R_i^k(\mathscr{C})\left(R_k(\mathscr{C}) - R_k^k(\mathscr{C})\right)\right\}$$

and

$$\mathscr{H}_{i,k}(\mathscr{C}) = \left\{z \in \mathbb{C} : |z| - \left(R_i(\mathscr{C}) - R_i^k(\mathscr{C})\right) \leq 0, |z| - R_k^k(\mathscr{C}) \leq 0\right\}.$$

Comparing the sets above, one can show that $\mathscr{L}(\mathscr{C}) \subseteq \Gamma(\mathscr{C})$ and $\mathscr{M}(\mathscr{C}) \subseteq \Gamma(\mathscr{C})$, i.e. the sets $\mathscr{L}(\mathscr{C})$ and $\mathscr{M}(\mathscr{C})$ are more tight than $\Gamma(\mathscr{C})$.

In this work, we present a new C-eigenvalue inclusion set, which is more accurate than the set $\Gamma(\mathscr{C})$ in Theorem 1.4. Moreover, numerical examples show that in some cases, it is superior to all the sets $\Gamma(\mathscr{C})$, $\mathscr{M}(\mathscr{C})$ and $\mathscr{L}(\mathscr{C})$.

2. New C-Eigenvalue Localisation Sets

In this section, we propose a new localisation set for C-eigenvalues and establish the relationship between this set and the set $\Gamma(\mathscr{C})$ from Theorem 1.4.

Theorem 2.1. *If $\mathscr{C} = (c_{ijk}) \in \mathbb{R}^{n\times n\times n}$ is a piezoelectric-type tensor, then*

$$\sigma(\mathscr{C}) \subseteq \Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \Big(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C})\Big),$$

where

$$\hat{\Upsilon}_{i,k}(\mathscr{C}) = \left\{z \in \mathbb{R} : |z| - R_i(\mathscr{C}) + R_i^k(\mathscr{C}) \leq 0, |z| - R_k(\mathscr{C}) + R_k^i(\mathscr{C}) \leq 0\right\},$$

and

$$\tilde{\Upsilon}_{i,k}(\mathscr{C}) = \left\{z \in \mathbb{R} : \left[|z| - R_i(\mathscr{C}) + R_i^k(\mathscr{C})\right]\left[|z| - R_k(\mathscr{C}) + R_k^i(\mathscr{C})\right] \leq R_i^k(\mathscr{C}) R_k^i(\mathscr{C})\right\}.$$

Proof. Consider a C-eigenvalue λ of $\mathscr{C}$ with corresponding left C-eigenvector $x = (x_1, x_2, \ldots, x_n)^\top$ and right C-eigenvector $y = (y_1, y_2, \ldots, y_n)^\top$, so that

$$\mathscr{C}yy = \lambda x, \quad x\mathscr{C}y = \lambda y, \quad x^\top x = 1, \quad y^\top y = 1. \tag{2.1}$$

The assumption

$$|y_p| \geq |y_q| \geq \max_{i\in N, i\neq p,q} |y_i|$$

yields $0 < |y_p| \leq 1$. If follows from (2.1) that

$$\lambda y_p = \sum_{l,k\in[n]} c_{lkp} x_l y_k = \sum_{\substack{l,k\in[n],\\ k\neq q}} c_{lkp} x_l y_k + \sum_{l\in[n]} c_{lqp} x_l y_q.$$

Since $0 \leq |x_i| \leq 1$ for any $i \in [n]$, we obtain

$$|\lambda| \leq \sum_{\substack{l,k\in[n],\\ k\neq q}} |c_{lkp}| \frac{|y_k|}{|y_p|} + \sum_{l\in[n]} |c_{lqp}| \frac{|y_q|}{|y_p|} \leq \sum_{\substack{l,k\in[n],\\ k\neq q}} |c_{lkp}| + \sum_{l\in[n]} |c_{lqp}| \frac{|y_q|}{|y_p|}.$$

Writing the last inequality as

$$|\lambda| - \sum_{\substack{l,k\in[n],\\ k\neq q}} |c_{lkp}| \leq \sum_{l\in[n]} |c_{lqp}| \frac{|y_q|}{|y_p|}, \tag{2.2}$$

we obtain the following estimates:

1. If $|y_q|=0$, then $|\lambda|-\sum_{\substack{l,k\in[n],\\ k\neq q}}|c_{lkp}|\leq 0$.
2. If $|\lambda|-R_q(\mathscr{C})+R_q^p(\mathscr{C})\geq 0$, then $\lambda\in\tilde{\Upsilon}_{p,q}(\mathscr{C})$.
3. If $|\lambda|-R_q(\mathscr{C})+R_q^p(\mathscr{C})\leq 0$, then $\lambda\in\hat{\Upsilon}_{p,q}(\mathscr{C})$.

On the other hand, if $|y_q|>0$, then

$$\lambda y_q=\sum_{l,k\in[n]}c_{lkq}x_l y_k=\sum_{\substack{l,k\in[n],\\ k\neq p}}c_{lkq}x_l y_k+\sum_{l\in[n]}c_{lpq}x_l y_p,$$

and

$$|\lambda|\leq\sum_{\substack{l,k\in[n],\\ k\neq p}}|c_{lkq}|\frac{|y_k|}{|y_q|}+\sum_{l\in[n]}|c_{lpq}|\frac{|y_p|}{|y_q|}\leq\sum_{\substack{l,k\in[n],\\ k\neq p}}|c_{lkq}|+\sum_{l\in[n]}|c_{lpq}|\frac{|y_p|}{|y_q|}.$$

Writing it in the form

$$|\lambda|-\sum_{\substack{l,k\in[n],\\ k\neq p}}|c_{lkq}|\leq\sum_{l\in[n]}|c_{lpq}|\frac{|y_p|}{|y_q|},\tag{2.3}$$

we note that if $|\lambda|-\sum_{\substack{l,k\in[n],\\ k\neq q}}|c_{lkp}|\leq 0$ or $|\lambda|-R_q(\mathscr{C})+R_q^p(\mathscr{C})\leq 0$, then multiplying (2.2) and (2.3), we arrive at the inequality

$$\Bigg(|\lambda|-\sum_{\substack{l,k\in[n],\\ k\neq q}}|c_{lkp}|\Bigg)\Bigg(|\lambda|-\sum_{\substack{l,k\in[n],\\ k\neq p}}|c_{lkq}|\Bigg)\leq\sum_{l\in[n]}|c_{lqp}|\sum_{l\in[n]}|c_{lpq}|.$$

It can be written as

$$\Big[|\lambda|-R_p(\mathscr{C})+R_p^q(\mathscr{C})\Big]\Big[|\lambda|-R_q(\mathscr{C})+R_q^p(\mathscr{C})\Big]\leq R_p^q(\mathscr{C})R_q^p(\mathscr{C}),$$

so that $\lambda\in\tilde{\Upsilon}_{p,q}(\mathscr{C})\subseteq\Upsilon(\mathscr{C})$. If

$$|\lambda|-\sum_{\substack{l,k\in[n],\\ k\neq q}}|c_{lkp}|\leq 0\quad\text{and}\quad|\lambda|-R_q(\mathscr{C})+R_q^p(\mathscr{C})\leq 0,$$

then $\lambda\in\hat{\Upsilon}_{p,q}(\mathscr{C})\subseteq\Upsilon(\mathscr{C})$, as required. □

Remark 2.1. For a real tensor $\mathscr{C}=(c_{ijk})\in\mathbb{R}^{n\times n\times n}$, $n\geq 2$, the set $\Gamma(\mathscr{C})$ consists of n sets $\Gamma_j(\mathscr{C})$, the set $\mathscr{L}(\mathscr{C})$ of $n(n-1)$ sets $\mathscr{L}_{j,k}(\mathscr{C})$, the set $\mathscr{M}(\mathscr{C})$ of $n(n-1)$ sets $\mathscr{M}_{i,k}(\mathscr{C})$ and $n(n-1)$ sets $\mathscr{H}_{i,k}(\mathscr{C})$, and the set $\Upsilon(\mathscr{C})$ of $n(n-1)$ sets $\hat{\Upsilon}_{i,k}(\mathscr{C})$ and $n(n-1)$ sets $\tilde{\Upsilon}_{i,k}(\mathscr{C})$. It is worth noting that for large n, the set $\Upsilon(\mathscr{C})$ locates all eigenvalues of $\mathscr{C}$ more accurately than $\Gamma(\mathscr{C})$, but $\Gamma(\mathscr{C})$ can be determined with less computational effort.

Comparing the sets $\Gamma(\mathscr{C})$ and $\Upsilon(\mathscr{C})$, we obtain the following result.

Theorem 2.2. *If $\mathscr{C}=(c_{ijk})\in\mathbb{R}^{n\times n\times n}$ is a piezoelectric-type tensor, then*

$$\sigma(\mathscr{C})\subseteq\Upsilon(\mathscr{C})\subseteq\Gamma(\mathscr{C}).$$

Proof. For any $z\in\Upsilon(\mathscr{C})$, there exist $i,k\in[n], i\neq k$ such that $z\in\hat{\Upsilon}_{i,k}(\mathscr{C})$ or $z\in\tilde{\Upsilon}_{i,k}(\mathscr{C})$. Next, we consider two cases.

Case I. If $z\in\hat{\Upsilon}_{i,k}(\mathscr{C})$, i.e. if

$$|z|-R_i(\mathscr{C})+R_i^k(\mathscr{C})\leq 0 \quad \text{and} \quad |z|-R_k(\mathscr{C})+R_k^i(\mathscr{C})\leq 0,$$

then

$$|z|\leq R_i(\mathscr{C}) \quad \text{and} \quad |z|\leq R_k(\mathscr{C}),$$

so that $z\in\Gamma(\mathscr{C})$.

Case II. If $z\in\tilde{\Upsilon}_{i,k}(\mathscr{C})$, then

$$\left[|z|-R_i(\mathscr{C})+R_i^k(\mathscr{C})\right]\left[|z|-R_k(\mathscr{C})+R_k^i(\mathscr{C})\right]\leq R_i^k(\mathscr{C})R_k^i(\mathscr{C}). \tag{2.4}$$

Assuming first that $R_i^k(\mathscr{C})R_k^i(\mathscr{C})=0$, we obtain

$$|z|-R_i(\mathscr{C})+R_i^k(\mathscr{C})\leq 0 \quad \text{or} \quad |z|-R_k(\mathscr{C})+R_k^i(\mathscr{C})\leq 0.$$

It follows that

$$|z|\leq R_i(\mathscr{C}) \quad \text{or} \quad |z|\leq R_k(\mathscr{C}).$$

Hence, $z\in\Gamma(\mathscr{C})$.

If we now assume that $R_i^k(\mathscr{C})R_k^i(\mathscr{C})>0$, then (2.4) yields

$$\frac{|z|-R_i(\mathscr{C})+R_i^k(\mathscr{C})}{R_i^k(\mathscr{C})}\cdot\frac{|z|-R_k(\mathscr{C})+R_k^i(\mathscr{C})}{R_k^i(\mathscr{C})}\leq 1,$$

and at least one of the inequalities

$$\frac{|z|-R_i(\mathscr{C})+R_i^k(\mathscr{C})}{R_i^k(\mathscr{C})}\leq 1, \quad \frac{|z|-R_k(\mathscr{C})+R_k^i(\mathscr{C})}{R_k^i(\mathscr{C})}\leq 1$$

holds. It follows that $z\in\Gamma_i(\mathscr{C})\bigcup\Gamma_k(\mathscr{C})$ and combining both cases, we finish the proof. □

3. Numerical Examples

In this section, we provide the results of numerical experiments to show that our approach locates C-eigenvalues much better than other methods. The piezoelectric tensors used in the examples, arise in piezoelectric materials with symmetries and have been previously studied in Refs. [3,4,7,22].

Example 3.1. Consider the piezoelectric tensor $\mathscr{A}_{VFeSb}$ [7] with the entries

$$a_{ijk} = \begin{cases} a_{123} = a_{213} = a_{312} = -3.68180667, \\ a_{ijk}, \quad \text{otherwise}. \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{VFeSb}$ is about 4.25138 and Theorems 1.4-1.6 show that

$$\Gamma(\mathscr{C}) = \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \leq 7.3636\},$$

$$\mathscr{L}(\mathscr{C}) = \bigcup_{j\in[n]} \Bigg(\bigcap_{k\in[n],k\neq j} \mathscr{L}_{j,k}(\mathscr{C}) \Bigg) = \{z \in \mathbb{C} : |z| \leq 7.3636\},$$

$$\mathscr{M}(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \Big(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \leq 7.3636\}.$$

From Theorem 2.1, we get

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \Big(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \leq 7.3636\}.$$

Example 3.2. Consider the piezoelectric tensor $\mathscr{A}_{SiO2}$ [6, 22] with the entries

$$a_{ijk} = \begin{cases} a_{111} = -a_{122} = a_{212} = -0.13685, \\ a_{123} = -a_{213} = -0.009715, \\ a_{ijk}, \quad \text{otherwise}. \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{SiO2}$ is about 0.1375 and Theorems 1.4-1.6 show that

$$\Gamma(\mathscr{C}) = \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \leq 0.2834\},$$

$$\mathscr{L}(\mathscr{C}) = \bigcup_{j\in[n]} \Bigg(\bigcap_{k\in[n],k\neq j} \mathscr{L}_{j,k}(\mathscr{C}) \Bigg) = \{z \in \mathbb{C} : |z| \leq 0.2744\},$$

$$\mathscr{M}(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \Big(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \leq 0.2834\}.$$

From Theorem 2.1, we have

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \Big(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \leq 0.2834\}.$$

Example 3.3. Consider the piezoelectric tensor $\mathscr{A}_{Cr2AgBiO8}$ [7] with the entries

$$a_{ijk} = \begin{cases} a_{123} = a_{213} = -0.22163, \\ a_{113} = -a_{223} = 2.608665, \\ a_{311} = -a_{322} = 0.152485, \\ a_{312} = -0.37153, \\ a_{ijk}, \quad \text{otherwise}. \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{Cr2AgBiO8}$ is about 2.6258 and Theorems 1.4-1.6 show that

$$\Gamma(\mathscr{C}) = \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \le 5.6606\},$$
$$\mathscr{L}(\mathscr{C}) = \bigcup_{j\in[n]} \left(\bigcap_{k\in[n],k\neq j} \mathscr{L}_{j,k}(\mathscr{C}) \right) = \{z \in \mathbb{C} : |z| \le 4.8058\},$$
$$\mathscr{M}(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \left(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \right) = \{z \in \mathbb{C} : |z| \le 4.7861\}.$$

From Theorem 2.1, we have

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \left(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \right) = \{z \in \mathbb{C} : |z| \le 4.7335\}.$$

Example 3.4. Consider the piezoelectric tensor $\mathscr{A}_{RbTaO3}$ [7] with the entries

$$a_{ijk} = \begin{cases} a_{113} = a_{223} = -8.40955, \\ a_{222} = -a_{212} = -a_{211} = -5.412525, \\ a_{311} = -a_{322} = -4.3031, \\ a_{333} = -5.14766, \\ a_{ijk}, \quad \text{otherwise.} \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{RbTaO3}$ is about 12.4234 and Theorems 1.4-1.6 show that

$$\Gamma(\mathscr{C}) = \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \le 23.5377\},$$
$$\mathscr{L}(\mathscr{C}) = \bigcup_{j\in[n]} \left(\bigcap_{k\in[n],k\neq j} \mathscr{L}_{j,k}(\mathscr{C}) \right) = \{z \in \mathbb{C} : |z| \le 23.5377\},$$
$$\mathscr{M}(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \left(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \right) = \{z \in \mathbb{C} : |z| \le 23.5377\}.$$

From Theorem 2.1, we have

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \left(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \right) = \{z \in \mathbb{C} : |z| \le 23.5377\}.$$

Example 3.5. Consider the piezoelectric tensor $\mathscr{A}_{NaBiS2}$ [7] with the entries

$$a_{ijk} = \begin{cases} a_{113} = -8.90808, a_{223} = -0.00842, a_{311} = -7.11526, \\ a_{322} = -0.6222, a_{333} = -7.93831, \\ a_{ijk}, \quad \text{otherwise.} \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{NaBiS2}$ is about 11.6674 and Theorems 1.4-1.6 show that

$$\begin{aligned}\Gamma(\mathscr{C}) &= \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \leq 16.8548\},\\ \mathscr{L}(\mathscr{C}) &= \bigcup_{j\in[n]} \Bigg(\bigcap_{k\in[n],k\neq j} \mathscr{L}_{j,k}(\mathscr{C}) \Bigg) = \{z \in \mathbb{C} : |z| \leq 16.5640\},\\ \mathscr{M}(\mathscr{C}) &= \bigcup_{i,k\in[n],k\neq i} \Big(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \leq 16.8464\}.\end{aligned}$$

From Theorem 2.1, we have

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \big(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \big) = \{z \in \mathbb{C} : |z| \leq 16.8464\}.$$

Example 3.6. Consider the piezoelectric tensor $\mathscr{A}_{LiBiB2O5}$ [7] with the entries

$$a_{ijk} = \begin{cases} a_{123} = 2.35682, a_{112} = 0.34929, a_{211} = 0.16101, a_{222} = 0.12562,\\ a_{233} = 0.1361, a_{213} = -0.05587, a_{323} = 6.91074, a_{312} = 2.57812,\\ a_{ijk}, \quad \text{otherwise.} \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{LiBiB2O5}$ is about 7.7376 and Theorems 1.4-1.6 show that

$$\begin{aligned}\Gamma(\mathscr{C}) &= \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \leq 12.3206\},\\ \mathscr{L}(\mathscr{C}) &= \bigcup_{j\in[n]} \Bigg(\bigcap_{k\in[n],k\neq j} \mathscr{L}_{j,k}(\mathscr{C}) \Bigg) = \{z \in \mathbb{C} : |z| \leq 11.0127\},\\ \mathscr{M}(\mathscr{C}) &= \bigcup_{i,k\in[n],k\neq i} \Big(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \leq 11.0038\}.\end{aligned}$$

From Theorem 2.1, we have

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\neq i} \big(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \big) = \{z \in \mathbb{C} : |z| \leq 10.9998\}.$$

Example 3.7. Consider the piezoelectric tensor $\mathscr{A}_{KBi2F7}$ [7] with the entries

$$a_{ijk} = \begin{cases} a_{111} = 12.64393, a_{122} = 1.08802, a_{133} = 4.14350, a_{123} = 1.59052,\\ a_{113} = 1.96801, a_{112} = 0.22465, a_{211} = 2.59187, a_{222} = 0.08263,\\ a_{233} = 0.81041, a_{223} = 0.51165, a_{213} = 0.71432, a_{212} = 0.10570,\\ a_{311} = 1.51254, a_{322} = 0.68235, a_{333} = -0.23019, a_{323} = 0.19013,\\ a_{313} = 0.39030, a_{312} = 0.08381. \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{KBi2F7}$ is about 20.2351 and Theorems 1.4-1.6 show that

$$\Gamma(\mathscr{C}) = \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \le 20.2351\},$$
$$\mathscr{L}(\mathscr{C}) = \bigcup_{j\in[n]} \Big(\bigcap_{k\in[n],k\ne j} \mathscr{L}_{j,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \le 18.8793\},$$
$$\mathscr{M}(\mathscr{C}) = \bigcup_{i,k\in[n],k\ne i} \Big(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \le 19.8830\}.$$

From Theorem 2.1, we have

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\ne i} \big(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \big) = \{z \in \mathbb{C} : |z| \le 19.8319\}.$$

Example 3.8. Consider the piezoelectric tensor $\mathscr{A}_{BaNiO3}$ [7] with the entries

$$a_{ijk} = \begin{cases} a_{113} = a_{223} = 0.038385, a_{311} = a_{322} = 6.89822, a_{333} = 27.4628, \\ a_{ijk}, \quad \text{otherwise}. \end{cases}$$

According to [4], the largest C-eigenvalue of $\mathscr{A}_{BaNiO3}$ is about 27.4628 and Theorems 1.4-1.6 show that

$$\Gamma(\mathscr{C}) = \bigcup_{j\in[n]} \Gamma_j(\mathscr{C}) = \{z \in \mathbb{C} : |z| \le 27.5396\},$$
$$\mathscr{L}(\mathscr{C}) = \bigcup_{j\in[n]} \Big(\bigcap_{k\in[n],k\ne j} \mathscr{L}_{j,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \le 27.5109\},$$
$$\mathscr{M}(\mathscr{C}) = \bigcup_{i,k\in[n],k\ne i} \Big(\mathscr{M}_{i,k}(\mathscr{C}) \bigcup \mathscr{H}_{i,k}(\mathscr{C}) \Big) = \{z \in \mathbb{C} : |z| \le 27.5013\}.$$

From Theorem 2.1, we have

$$\Upsilon(\mathscr{C}) = \bigcup_{i,k\in[n],k\ne i} \big(\hat{\Upsilon}_{i,k}(\mathscr{C}) \bigcup \tilde{\Upsilon}_{i,k}(\mathscr{C}) \big) = \{z \in \mathbb{C} : |z| \le 27.5013\}.$$

Let λ^* be the largest C-eigenvalue of the piezoelectric tensor and $[-\rho_{\Gamma}, \rho_{\Gamma}]$, $[-\rho_{\mathscr{L}}, \rho_{\mathscr{L}}]$, $[-\rho_{\mathscr{M}}, \rho_{\mathscr{M}}]$ and $[-\rho_{\Upsilon}, \rho_{\Upsilon}]$ are the intervals generated by Theorems 1.4, 1.5, 1.6 and 2.1, respectively. We note that in all examples, Theorem 2.2 always provides the best result among the methods tested. Table 1 lists the results obtained by methods [3,4,15] and by Theorem 2.1. It indicates that ρ_{Υ} is more precise than ρ, $\rho_{\min}$, ρ_{Γ} and $\rho_{\mathscr{M}}$. Moreover, in some cases ρ_{Υ} is tighter than $\rho_{\mathscr{L}}$. Thus, the C-eigenvalue localisation theorem obtained in this work improves the known results in [3,4,15].

4. Conclusion

We derived a new inclusion set for localisation of the C-eigenvalues of piezoelectric tensors. Numerical experiments show that it is better than the known set $\Gamma(\mathscr{C})$ and is comparable or better than the sets $\mathscr{M}(\mathscr{C})$ and $\mathscr{L}(\mathscr{C})$.

Table 1: Numerical comparison of Theorem 2.1 and the related results [3,4,15].

	$\mathscr{A}_{VFeSb}$	$\mathscr{A}_{SiO2}$	$\mathscr{A}_{Cr2AgBiO8}$	$\mathscr{A}_{RbTaO3}$	$\mathscr{A}_{NaBiS2}$	$\mathscr{A}_{LiBiB2O5}$	$\mathscr{A}_{KBi2F7}$	$\mathscr{A}_{BaNiO3}$
λ^*	4.2514	0.1375	2.6258	12.4234	11.6674	7.7376	13.5021	27.4628
ρ	7.3636	0.2882	5.6606	30.0911	17.3288	15.2911	22.6896	38.8162
$\rho_{\min}$	7.3636	0.2834	5.6606	23.5377	16.8548	12.3206	20.2351	35.3787
ρ_{Γ}	7.3636	0.2834	5.6606	23.5377	16.8548	12.3206	20.2351	27.5396
$\rho_{\mathscr{L}}$	7.3636	0.2744	4.8058	23.5377	16.5460	11.0127	18.8973	27.5109
$\rho_{\mathscr{M}}$	7.3636	0.2834	4.7861	23.5377	16.8464	11.0038	19.8830	27.5013
ρ_{Υ}	7.3636	0.2834	4.7335	23.5377	16.8464	10.9998	19.8319	27.5013

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