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A Space-Time Petrov-Galerkin Spectral Method for Time Fractional Fokker-Planck Equation with Nonsmooth Solution

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Abstract. An STPG spectral method for TFFP equations with nonsmooth solutions is developed. The numerical scheme is based on generalised Jacobi functions in time and Legendre polynomials in space. The generalised Jacobi functions match the leading singularity of the corresponding problem. Therefore, the method performs better than methods with polynomial bases. The stability and convergence of the method are proved. Numerical experiments confirm the theoretical error estimates.

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Key words: Time fractional Fokker-Planck equation, nonsmooth solution, generalised Jacobi functions, space-time Petrov-Galerkin spectral method, error estimate.

1. Introduction

We consider the time fractional Fokker-Planck (TFFP) equation [22, 23]

$$\begin{aligned}&\partial_t u = {}_0^R D_t^{1-\alpha}\big\{\partial_x[p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + f(x,t)\big\},\\ &(x,t)\in(a,b)\times(0,T]\end{aligned} \tag{1.1}$$

with the initial and boundary conditions

$$\begin{aligned}&u(x,0) = u_0(x), \quad x\in(a,b),\\ &u(a,t) = u_a(t), \quad u(b,t) = u_b(t), \quad t\in I := [0,T],\end{aligned}$$

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where $u(x,t)$ is a probability density function, $p(x)$ a nonpositive and monotonically decreasing function in the interval $[a,b]$, $K_\alpha > 0$ a diffusion constant and ${}_0^RD_t^{1-\alpha}$, $0 < \alpha \leq 1$ the left Riemann-Liouville fractional derivative defined by

$$ {}_0^RD_t^{1-\alpha}u(t) := \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}ds, \quad t \in I. $$

If $\alpha = 1$, the Eq. (1.1) becomes the classical Fokker-Planck (FP) equation. However, classical FP equations do not properly describe anomalous diffusion processes in highly nonhomogeneous medium spaces [3,24]. Instead, fractional Fokker-Planck (FFP) equations should be used. For example, space fractional Fokker-Planck equations are adopted to describe Lévy flights, TFFP equations to characterise the traps, and the space and time fractional Fokker-Planck (STFFP) equations handle competition between Lévy flights and traps — [1,2,9,11,15,16,19,24,25].

Lately, numerical methods for the Eq. (1.1) attracted considerable attention. Thus finite difference methods are discussed in [4,5,7,13,31,32] and finite element methods and finite volume methods in [8,14,17,18,36]. Recently Zheng *et al.* [35] proposed a space-time spectral method based on Jacobi polynomials for temporal discretisation and on Fourier-like basis functions for spatial discretisation, whereas Yang *et al.* [33] suggested a spectral collocation method based on both temporal and spatial discretisations with a spectral expansion of Jacobi interpolation polynomials. For the STFFP equation, Zhang *et al.* [34] employed a time-space spectral method with Jacobi polynomials for temporal discretisation and Legendre polynomials for spatial discretisation. A pseudospectral method was discussed in [12,30].

It is worth noting that polynomial approximations are not efficient in the case of Caputo or Riemann-Liouville fractional differential equations because of initial or endpoint singularities in the solutions. As is shown in [33–35], the methods using Jacobi polynomials in the discretisation of time fractional derivatives, may not be of the highest accuracy — viz. these methods fail to achieve spectral accuracy when solutions are not smooth. Nevertheless, Chen *et al.* [6] showed that the Petrov-Galerkin method using generalised Jacobi functions (GJFs) is efficient for a class of prototypical fractional initial value problems and fractional boundary value problems of general order.

The aim of this work is to study a space-time Petrov-Galerkin (STPG) spectral method for TFFP equations with nonsmooth solutions. In order to match the singularities in the corresponding solutions, time fractional derivatives can be approximated by suitable GJFs [6,28,29], while Legendre polynomials are employed for space approximations [26]. Moreover, we also analyse the errors of the method proposed.

The remainder of this paper is organised as follows. In Section 2, equivalent equations for the Eq. (1.1) are provided and some functional spaces as well as projection operators are defined. In Section 3, we consider a Petrov-Galerkin method for the TFFP equation. Section 4 deals with error estimates for the STPG spectral scheme. Numerical results, presented in Section 5, support theoretical findings and show the effectiveness of the scheme. Our conclusions are in the last section.

2. Preliminaries

In this section we introduce necessary operators and spaces and rewrite the Eq. (1.1) in different forms. Besides, we recall properties of shifted GJFs and shifted Legendre polynomials. Note that in what follows, the notation $A \lesssim B$ means that $A \leq cB$ with a positive constant c independent of functions and discretisation parameters.

2.1. Equivalent Equations

According to [35], the Eq. (1.1) can be represented in the form

$$ {}_0^C D_t^\alpha u(x,t) = \partial_x[p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + f(x,t). \tag{2.1} $$

Noting that

$$ {}_0^R D_t^\alpha u(t) = {}_0^C D_t^\alpha u(t) + \frac{u(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \quad t \in I $$

and using the representation $u(x,t) = u^h(x,t) + u_0(x)$ with $u^h(x,0) := 0$, we write the Eq. (2.1) as

$$ {}_0^R D_t^\alpha u^h(x,t) = \partial_x[p(x)u^h(x,t)] + K_\alpha \partial_{xx} u^h(x,t) + g(x,t), \quad (x,t) \in \Omega, \tag{2.2} $$

where

$$ g(x,t) = f(x,t) + K_\alpha \partial_{xx} u_0(x) + \partial_x[p(x)u_0(x)]. $$

In what follows, we consider the Eq. (2.2) and, for simplicity, we set $\Lambda := (a,b) = (-1,1)$, so that the homogeneous initial and boundary conditions are

$$ u^h(x,0) = 0, \quad x \in \Lambda, \tag{2.3} $$

$$ u^h(\pm 1,t) = 0, \quad t \in I. \tag{2.4} $$

2.2. Functional spaces and their properties

For the bounded domain I, we define the space

$$ H^s(I) = \left\{ v \in L^2(I) | \exists\ \tilde{v} \in H^s(\mathbb{R}) \text{ such that } \tilde{v}|_I = v \right\} $$

with the norm:

$$ \|v\|_{s,I} = \inf_{\tilde{v} \in H^s(\mathbb{R}),\ \tilde{v}|_I = v} \|\tilde{v}\|_{s,\mathbb{R}}. $$

Let $C_0^\infty(I)$ be the set of smooth functions with compact support in I. Following [20], we denote by $H_0^s(I)$ the closure of $C_0^\infty(I)$ in the norm $\|\cdot\|_{s,I}$. Moreover, we consider the following spaces

$$ {}_0C^\infty(I) = \left\{ v | v \in C^\infty(I) \text{ with compact support in } (0,1] \right\}, $$

$$ {}_0^rC^\infty(I) = \left\{ v | v \in C^\infty(I) \text{ with compact support in } [0,1) \right\}. $$

By ${}_0H^s(I)$ we denote the closure of ${}_0C^\infty(I)$ in the norm $\|\cdot\|_{s,I}$. If X is a Sobolev space with the norm $\|\cdot\|_X$, then we consider the set

$$H^s(I;X) := \big\{v \| \|v(\cdot,t)\|_X \in H^s(I)\big\}, \quad {}_0H^s(I;X) := \big\{v \| \|v(\cdot,t)\|_X \in {}_0H^s(I)\big\},$$

and equip it with the norm

$$\|v\|_{H^s(I;X)} := \| \|v(\cdot,t)\|_X \|_{s,I}, \quad s \ge 0.$$

We also consider the set

$$B^s(\Omega) := H^s(I, L^2(\Lambda)) \cap L^2(I, H_0^1(\Lambda))$$

and equip it with the norm

$$\|v\|_{B^s(\Omega)} := \Big(\|v\|^2_{H^s(I,L^2(\Lambda))} + \|v\|^2_{L^2(I,H_0^1(\Lambda))}\Big)^{1/2}.$$

It is easily seen that $B^s(\Omega)$ is a Banach space.

We recall some definitions from [20]. Let $s > 0$. By $H_l^s(I)$ we denote the closure of ${}_0C^\infty(I)$ with respect to the semi-norm $|v|_{H_l^s(I)} := \|{}_0^RD_t^s v\|_{L^2(I)}$ and the norm $\|v\|_{H_l^s(I)} := (\|v\|_{L^2(I)} + |v|_{H_l^s(I)})^{1/2}$. And by $H_r^s(I)$ the closure of ${}_0^rC^\infty(I)$ with respect to the semi-norm $|v|_{H_r^s(I)} := \|{}_t^RD_T^s v\|_{L^2(I)}$ and the norm $\|v\|_{H_r^s(I)} := (\|v\|_{L^2(I)} + |v|_{H_r^s(I)})^{1/2}$. Besides, if $s \ne n+1/2$, then $H_c^s(I)$ refers to the closure of $C_0^\infty(I)$ with respect to the semi-norm $|v|_{H_c^s(I)} := |({}_0^RD_t^s v, {}_t^RD_T^s v)_I|^{1/2}$ and the norm $\|v\|_{H_c^s(I)} := (\|v\|_{L^2(I)} + |v|_{H_c^s(I)})^{1/2}$.

Lemma 2.1 (cf. Refs. [20,21,29]). *For $s > 0$ and $s \ne n+1/2$, the spaces $H_l^s(I)$, $H_r^s(I)$, $H_c^s(I)$ and $H_0^s(I)$ are equal in the sense that their seminorms and also the norms are equivalent. In particular, if $s = \alpha/2$, $0 < \alpha < 1$, then*

$$H_0^{\alpha/2}(I) = H^{\alpha/2}(I) = {}_0H^{\alpha/2}(I).$$

In addition, if $v \in H_l^s(I)$, then $\|v\|_{L^2(I)} \lesssim |v|_{H_l^s(I)}$, if $v \in H_r^s(I)$, then $\|v\|_{L^2(I)} \lesssim |v|_{H_r^s(I)}$, and if $\omega \in {}_0H^1(I)$ and $v \in {}_0H^{\alpha/2}(I)$, then

$$\big({}_0^RD_t^\alpha v, \omega\big)_I = \big({}_0^RD_t^{\alpha/2} v, {}_t^RD_T^{\alpha/2}\omega\big)_I.$$

2.3. Trial and test functions in time

Let $\alpha, \beta > -1$ and $\chi^{(\alpha,\beta)}(x) := (1-x)^\alpha(1+x)^\beta$. We consider the set of standard Jacobi polynomials $P_n^{(\alpha,\beta)}(x), x \in \Lambda$ of degree n. This is a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal set — i.e.

$$\int_{-1}^1 P_l^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\chi^{(\alpha,\beta)}(x)dx = \gamma_l^{(\alpha,\beta)}\delta_{l,m}, \tag{2.5}$$

where $\delta_{l,m}$ is the Kronecker function and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2l+\alpha+\beta+1}\frac{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{l!\Gamma(l+\alpha+\beta+1)}.$$

The shifted Jacobi polynomials of degree n are defined by

$$\tilde{P}_n^{(\alpha,\beta)}(t) = P_n^{(\alpha,\beta)}\left(\frac{2t-T}{T}\right), \quad t \in I, \quad n \geq 0. \tag{2.6}$$

It is easily seen that the polynomials $\tilde{P}_n^{(\alpha,\beta)}(t)$ form a complete $L^2_{\omega^{(\alpha,\beta)}}(I)$-orthogonal system with respect to the weight function $\omega^{(\alpha,\beta)}(t) = (T-t)^\alpha t^\beta$. It follows from (2.5) and (2.6) that

$$\int_I \tilde{P}_l^{(\alpha,\beta)}(t)\tilde{P}_m^{(\alpha,\beta)}(t)\omega^{(\alpha,\beta)}(t)dt = \left(\frac{T}{2}\right)^{\alpha+\beta+1}\gamma_l^{(\alpha,\beta)}\delta_{l,m}.$$

For any $\alpha, \beta > -1$, the shifted generalised Jacobi functions $J_n^{(\alpha,\beta)}$ and the corresponding approximation space $\mathfrak{F}_N^{(\alpha)}(I)$ are defined by

$$J_n^{(\alpha,\beta)}(t) = t^\beta \tilde{P}_n^{(\alpha,\beta)}(t), \quad t \in I, \quad n \geq 0, \tag{2.7}$$

$$\mathfrak{F}_N^{(\alpha)}(I) := \left\{t^\alpha \psi(t) : \psi(t) \in \mathscr{P}_N(I)\right\} = \operatorname{span}\left\{J_n^{(-\alpha,\alpha)}(t) = t^\alpha \tilde{P}_n^{(-\alpha,\alpha)}(t)\right\}. \tag{2.8}$$

In particular, the shifted Legendre polynomials $L_n(t)$, $t \in I$ have the form

$$L_n(t) = P_n^{0,0}\left(\frac{2t}{T} - 1\right), \tag{2.9}$$

and the set of all polynomials $L_n(t)$, $n = 0, 1, \ldots$ is a complete $L^2(I)$-orthogonal system with

$$\int_I L_l(t)L_m(t)dt = \frac{T}{2l+1}\delta_{l,m}. \tag{2.10}$$

The relations (2.7)-(2.9) and straightforward calculations in [6] show that

$${}_0^R D_t^\alpha J_n^{(-\alpha,\alpha)}(t) = \frac{\Gamma(n+\alpha+1)}{n!}L_n(t). \tag{2.11}$$

2.4. Projection operators

Let us consider two projection operators needed in what follows.

The operator $\Pi_M^{1,0} : H_0^1(\Lambda) \to V_M$ is defined by

$$\left((\Pi_M^{1,0} v - v)', \phi'\right)_\Lambda = 0 \quad \text{for all} \quad \phi \in V_M. \tag{2.12}$$

In order to characterise the regularity of a function u in time variable t, we consider a non-uniformly weighted space — viz.

$$\mathfrak{B}^s_{\alpha,\beta}(I) := \left\{v \in L^2_{\omega^{(\alpha,-\beta)}(I)} : {}_0^R D_t^{\beta+r} v \in L^2_{\omega^{(\alpha+\beta+r,r)}}(I), \ 0 \leq r \leq s, \ s \in \mathbb{N}_0\right\}.$$

The other projection operator $\pi_N^{(-\alpha,\alpha)} : L^2_{\omega^{(-\alpha,\alpha)}}(I) \to \mathfrak{F}_N^{\alpha}(I)$ is defined by

$$\left(\pi_N^{(-\alpha,\alpha)}v - v, \psi\right)_{\omega^{(-\alpha,\alpha)}} = 0 \quad \text{for all} \quad \psi \in \mathfrak{F}_N^{(\alpha)}(I).$$

We recall [6] that

$$\left({}_0^R D_t^{\alpha}(\pi_N^{(-\alpha,\alpha)}v - v), p\right) = 0 \quad \text{for all} \quad p \in \mathscr{P}_N(I). \tag{2.13}$$

We also need the approximation results below.

Lemma 2.2 (cf. Refs. [10,27]). *If $v \in H_0^1(\Lambda)$ and $\partial_x^k v \in L^2_{\chi^{k-1}}(\Lambda)$ for any $1 \le k \le r$, then*

$$\left\|\partial_x^{\mu}(\Pi_M^{1,0}v - v)\right\|_{\Lambda} \lesssim M^{\mu-r}\left\|\partial_x^r v\right\|_{\Lambda,\chi^{r-1}}, \quad \mu \le r, \quad \mu = 0,1.$$

Lemma 2.3 (cf. Refs. [10,27]). *If $\alpha \in (0,1)$ and $v \in \mathfrak{B}^s_{-\alpha,\alpha}(I)$ for an integer $0 \le s \le N$, then*

$$\begin{aligned}
&\left\|\pi_N^{(-\alpha,\alpha)}v - v\right\|_{\omega^{(-\alpha,\alpha)}} \lesssim N^{-(\alpha+s)}\left\|{}_0^R D_t^{\alpha+s}v\right\|_{\omega^{(s,s)}}, \\
&\left\|{}_0^R D_t^{\alpha}(\pi_N^{(-\alpha,\alpha)}v - v)\right\|_I \lesssim N^{-s}\left\|{}_0^R D_t^{\alpha+s}v\right\|_{\omega^{(s,s)}}.
\end{aligned}$$

3. STPG Spectral Method for TFFP Equation

Let $S_M := \text{span}\{L_0(x), L_1(x), \ldots, L_M(x)\}$ be the set of Legendre polynomials. We consider the standard polynomial space

$$V_M := \{u \in S_M(\Lambda) : u(\pm 1) = 0\}$$

and the fractional polynomial space $\mathfrak{F}_N^{(\alpha)}(I)$ defined in (2.8). In the STPG spectral method for the Eq. (2.2), we are looking for an element $u_L^h(x,t) := u_{MN}^h \in V_M \otimes \mathfrak{F}_N^{(\alpha)}$, such that

$$\left({}_0^R D_t^{\alpha}u_L^h, v\right)_{\Omega} + K_{\alpha}\left(\partial_x u_L^h, \partial_x v\right)_{\Omega} - \left(\partial_x(pu_L^h), v\right)_{\Omega} = (g, v)_{\Omega} \quad \text{for all} \quad v \in V_M \otimes \mathscr{P}_N. \tag{3.1}$$

The next task is to construct an appropriate basis using the functions from V_M and $\mathfrak{F}_N^{(\alpha)}(I)$ in order to efficiently solve the Eq. (3.1).

Lemma 3.1 (cf. Shen [26]). *If*

$$\begin{aligned}
&c_k = \frac{1}{\sqrt{4k+6}}, \quad \phi_k(x) = c_k(L_k(x) - L_{k+2}(x)), \\
&a_{jk} = (\partial_x\phi_k(x), \partial_x\phi_j(x)), \quad b_{jk} = (\phi_k(x), \phi_j(x)), \quad c_{jk} = (\partial_x\phi_k(x), \phi_j(x)),
\end{aligned}$$

then

$$a_{jk} = \begin{cases} 1, & k = j, \\ 0, & k \ne j, \end{cases} \quad , \quad b_{jk} = b_{kj} = \begin{cases} c_k c_j\left(\dfrac{2}{2j+1} + \dfrac{2}{2j+5}\right), & k = j, \\ -c_k c_j\dfrac{2}{2k+1}, & k = j+2, \\ 0, & \textit{otherwise}, \end{cases}$$

$$c_{jk} = -c_{kj} = \begin{cases} 2c_k c_j, & k = j+1, \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$V_M = \operatorname{span}\{\phi_0(x), \phi_1(x), \ldots, \phi_{M-2}(x)\}.$$

Note that in what follows we use the notation

$$B := (b_{jk})_{0\le j,k\le M-2}, \quad C := (c_{jk})_{0\le j,k\le M-2}.$$

Moreover, let $J_n^{(-\alpha,\alpha)}(t)$ be the functions defined in (2.7). As a test space, we use the space of scaled Legendre polynomials

$$\mathscr{P}_N(I) = \operatorname{span}\left\{L_n^{(\alpha)}(t) := C_{n,\alpha} L_n(t) : 0 \le n \le N\right\}, \tag{3.2}$$

where $C_{n,\alpha} = (n!(2n+1))/(T \cdot \Gamma(n+\alpha+1))$, and we consider the Eq. (3.1) with the set of basis functions above. Thus the numerical solution is sought in the form

$$u_L^h(x,t) = \sum_{m=0}^{M-2} \sum_{n=0}^{N} \tilde{u}_{nm}^h \phi_m(x) J_n^{(-\alpha,\alpha)}(t). \tag{3.3}$$

Substituting (3.3) into (3.1) and using the functions $v_L = \phi_p(x) L_q^{(\alpha)}(t)$ yields

$$\begin{aligned} \sum_{m=0}^{M-2} \sum_{n=0}^{N} \tilde{u}_{nm}^h \Big\{ &\left(\phi_m, \phi_p\right) \left({}_0^R D_t^\alpha J_n^{(-\alpha,\alpha)}, L_q^{(\alpha)}\right) - K_\alpha \left(\phi_m'', \phi_p\right) \left(J_n^{(-\alpha,\alpha)}, L_q^{(\alpha)}\right) \\ &- \left(\partial_x (p\phi_m), \phi_p\right) \left(J_n^{(-\alpha,\alpha)}, L_q^{(\alpha)}\right) \Big\} = \left(g, \phi_p L_q^{(\alpha)}\right)_\Omega. \end{aligned}$$

Set

$$\begin{aligned} &g_{nm} = \left(g, \phi_m(x) L_n^{(\alpha)}(t)\right)_\Omega, && F = (g_{nm})_{0\le n\le N, 0\le m\le M-2}, \\ &c_{pm}^x = \left(p(x)\partial_x \phi_m, \phi_p\right)_\Lambda, && C^x = \left(c_{pm}^x\right)_{0\le p\le M-2, 0\le m\le M-2}, \\ &b_{pm}^x = \left(\partial_x p(x) \phi_m, \phi_p\right)_\Lambda, && B^x = \left(b_{pm}^x\right)_{0\le p\le M-2, 0\le m\le M-2}, \\ &s_{qn}^t = \int_I {}_0^R D_t^\alpha J_n^{(-\alpha,\alpha)}(t) L_q^{(\alpha)}(t) dt, && m_{qn}^t = \int_I J_n^{(-\alpha,\alpha)}(t) L_q^{(\alpha)}(t) dt, \\ &\mathbf{S}^t = \left(s_{qn}^t\right)_{0\le q,n\le N}, \quad \mathbf{M}^t = \left(m_{qn}^t\right)_{0\le q,n\le N}, && U = \left(\tilde{u}_{nm}^h\right)_{0\le n\le N, 0\le m\le M-2}. \end{aligned} \tag{3.4}$$

It follows from (2.10), (2.11) and (3.2) that $\mathbf{S}^t = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix. Although matrix $\mathbf{M}^t$ is not sparse, Sheng and Shen [29] show that it can be accurately computed by Jacobi-Gauss quadrature with $(0,\alpha)$. The scalar products $(p(x)\partial_x \phi_m, \phi_p)_\Lambda$, $(\partial_x p(x)\phi_m, \phi_p)_\Lambda$ and $(g, v)_\Omega$ are computed by the Legendre-Gauss quadrature.

Lemma 3.1 and the representations (3.4) yield that the Eq. (3.1) is equivalent to the linear system

$$\mathbf{S}^t U B + K_\alpha \mathbf{M}^t U \mathbf{I} - \mathbf{M}^t U C^x - \mathbf{M}^t U B^x = F. \tag{3.5}$$

In particular, for $p(x)=p=$ constant, the system (3.5) takes the form

$$\mathbf{S}^t UB + K_\alpha \mathbf{M}^t U\mathbf{I} - p\mathbf{M}^t UC = F.$$

Finally, the numerical solution of (2.2) is obtained by $u_L = u_L^h + u_0$.

4. Stability and Convergence of STPG Method

We start with the stability analysis.

Lemma 4.1. *Assume that $u_0 \in H_0^1(\Lambda)$. If u_L^h is the solution of* (3.1)*, then*

$$\|u_L^h\|_\Omega^2 + \|\partial_x u_L^h\|_\Omega^2 \lesssim \|{}_0^RD_t^{\alpha/2} u_L^h\|_\Omega^2 + \|\partial_x u_L^h\|_\Omega^2 \lesssim \|u_0\|_\Lambda^2 + \|\partial_x u_0\|_\Lambda^2 + \|f\|_\Omega^2. \tag{4.1}$$

Proof. Substituting $v = u_L^h + u_0$ into the Eq. (3.1), using Lemma 2.1 and the relation

$$\left(\partial_x\left(u_L^h+u_0\right), u_L^h+u_0\right)_\Omega = 0,$$

we obtain

$$\begin{aligned}
&\left({}_0^RD_t^\alpha u_L^h, u_L^h\right)_\Omega + K_\alpha\left(\partial_x u_L^h, \partial_x u_L^h\right)_\Omega \\
=&-\left({}_0^RD_t^\alpha u_L^h, u_0\right)_\Omega - K_\alpha\left(\partial_x u_L^h, \partial_x u_0\right)_\Omega + p\left(\partial_x u_L^h, u_L^h+u_0\right)_\Omega \\
&+\left(f + K_\alpha \Delta u_0 + p\partial_x u_0, u_L^h + u_0\right)_\Omega \\
=&-\left({}_0^RD_t^\alpha u_L^h, u_0\right)_\Omega - K_\alpha\left(\partial_x u_L^h, \partial_x u_0\right)_\Omega + \left(f + K_\alpha\Delta u_0, u_L^h + u_0\right)_\Omega \\
\le&-\left({}_0^RD_t^{\alpha/2} u_L^h, {}_t^RD_T^{\alpha/2} u_0\right)_\Omega + K_\alpha\left\|\partial_x u_L^h\right\|_\Omega\left\|\partial_x u_0\right\|_\Lambda \\
&+\|f\|_\Omega\left\|u_L^h+u_0\right\|_\Omega + K_\alpha\|\partial_x u_0\|_\Lambda\Big(\|\partial_x u_L^h\|_\Omega + T\|\partial_x u_0\|_\Lambda\Big) \\
\le&\|{}_0^RD_t^{\alpha/2}u_L^h\|_\Omega\|{}_t^RD_T^{\alpha/2}u_0\|_\Omega + 2K_\alpha\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda \\
&+K_\alpha T\|\partial_x u_0\|_\Lambda^2 + \|f\|_\Omega\Big(\|u_L^h\|_\Omega + T\|u_0\|_\Lambda\Big) \\
=&\|{}_0^RD_t^{\alpha/2}u_L^h\|_\Omega\int_I t^{-\alpha/2}dt\|u_0\|_\Lambda + 2K_\alpha\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda \\
&+K_\alpha T\|\partial_x u_0\|_\Lambda^2 + \|f\|_\Omega\Big(\|u_L^h\|_\Omega + T\|u_0\|_\Lambda\Big) \\
=&\frac{T^{\alpha/2}}{1-\alpha/2}\|{}_0^RD_t^{\alpha/2}u_L^h\|_\Omega\|u_0\|_\Lambda + 2K_\alpha\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda \\
&+K_\alpha T\|\partial_x u_0\|_\Lambda^2 + \|f\|_\Omega\Big(\|u_L^h\|_\Omega + T\|u_0\|_\Lambda\Big) \\
\le&\frac{1}{2}\|{}_0^RD_t^{\alpha/2}u_L^h\|_\Omega^2 + \frac{T^\alpha}{2(1-\alpha/2)^2}\|u_0\|_\Lambda^2 + 2K_\alpha\|\partial_x u_L^h\|_\Omega\|\partial_x u_0\|_\Lambda \\
&+K_\alpha T\|\partial_x u_0\|_\Lambda^2 + \|f\|_\Omega\Big(\|u_L^h\|_\Omega + T\|u_0\|_\Lambda\Big).
\end{aligned}$$

Moreover, Lemma 2.1 and the Young's inequality yield

$$\begin{aligned}
&\left\|{}_0^R D_t^{\alpha/2} u_L^h\right\|_\Omega^2 + K_\alpha \left\|\partial_x u_L^h\right\|_\Omega^2 \\
=&\left({}_0^R D_t^{\alpha/2} u_L^h, {}_t^R D_T^{\alpha/2} u_L^h\right)_\Omega + K_\alpha \left\|\partial_x u_L^h\right\|_\Omega^2 \\
\le&\frac{1}{2}\left\|{}_0^R D_t^{\alpha/2} u_L^h\right\|_\Omega^2 + \frac{T^\alpha}{2(1-\alpha/2)^2}\left\|u_0\right\|_\Lambda^2 + \frac{K_\alpha}{2}\left\|\partial_x u_L^h\right\|_\Omega^2 \\
&+ (2+T)K_\alpha \left\|\partial_x u_0\right\|_\Lambda^2 + \left\|f\right\|_\Omega\left(\left\|u_L^h\right\|_\Omega + T\left\|u_0\right\|_\Lambda\right) \\
\le&\frac{1}{2}\left\|{}_0^R D_t^{\alpha/2} u_L^h\right\|_\Omega^2 + \frac{T^\alpha}{2(1-\alpha/2)^2}\left\|u_0\right\|_\Lambda^2 + \frac{K_\alpha}{2}\left\|\partial_x u_L^h\right\|_\Omega^2 \\
&+ (2+T)K_\alpha \left\|\partial_x u_0\right\|_\Lambda^2 + \left\|f\right\|_\Omega^2 + \frac{1}{2}\left\|u_L^h\right\|_\Omega^2 + \frac{T^2}{2}\left\|u_0\right\|_\Lambda^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2}\left\|{}_0^R D_t^{\alpha/2} u_L^h\right\|_\Omega^2 + \frac{K_\alpha}{2}\left\|\partial_x u_L^h\right\|_\Omega^2 \le& \left(\frac{T^\alpha}{2(1-\alpha/2)^2} + \frac{T^2}{2}\right)\left\|u_0\right\|_\Lambda^2 \\
&+ (2+T)K_\alpha \left\|\partial_x u_0\right\|_\Lambda^2 + \left\|f\right\|_\Omega^2 + \frac{1}{2}\left\|u_L^h\right\|_\Omega^2,
\end{aligned}$$

and using Lemma 2.1 once more, we arrive at the estimate (4.1). ☐

In order to study the convergence of the method, we introduce suitable functional spaces for the space-time approximation.

Assumption 4.1. The solution u^h of the Eq. (2.2) belongs to the space $H^\alpha(I;H_0^1(\Lambda)) \cap E^r(\Omega) \cap K^s(\Omega)$, where $E^r(\Omega)$, $r \ge 2$ and $K^s(\Omega)$, $s \ge 0$ are the spaces of measurable functions such that

$$\begin{aligned}
\|u\|_{E^r(\Omega)} &= \left(\left\|\partial_x^r u\right\|_{L^2_{\chi^{r-1}}(\Lambda, L^2(I))}^2 + \left\|\partial_x^{r-1}\left({}_0^R D_t^\alpha u\right)\right\|_{L^2_{\chi^{r-2}}(\Lambda, L^2(I))}^2\right)^{1/2} < \infty, \\
\|u\|_{K^s(\Omega)} &= \left(\left\|\partial_x {}_0^R D_t^{\alpha+s} u\right\|_{L^2\left(\Lambda, L^2_{\omega^{(s,s)}}(I)\right)}^2 + \left\|\partial_x^2\left({}_0^R D_t^{\alpha+s} u\right)\right\|_{L^2\left(\Lambda, L^2_{\omega^{(s,s)}}(I)\right)}^2\right)^{1/2} < \infty.
\end{aligned}$$

Theorem 4.1. *Assume that the solution u^h of the Eq.* (2.2) *satisfies Assumption* 4.1. *If u_L^h is the solution of the Eq.* (3.1)*, then*

$$\begin{aligned}
\left\|u^h - u_L^h\right\|_{B^\alpha(\Omega)} :=& \left\|{}_0^R D_t^\alpha (u^h - u_L^h)\right\|_\Omega + \left\|\partial_x (u^h - u_L^h)\right\|_\Omega \\
\lesssim& M^{1-r}\left\|u^h\right\|_{E^r(\Omega)} + N^{-(\alpha+s)}\left\|u^h\right\|_{K^s(\Omega)}.
\end{aligned} \tag{4.2}$$

Proof. Setting

$$\tilde{u}_L^h := \pi_N^{(-\alpha,\alpha)} \Pi_M^{1,0} u^h = \Pi_M^{1,0} \pi_N^{(-\alpha,\alpha)} u^h, \quad e_L := \tilde{u}_L^h - u_L^h,$$

we obtain from (3.1) that

$$
\begin{aligned}
\mathscr{A}(e_L,v) &= \big({}_0^RD_t^\alpha e_L, v\big)_\Omega + K_\alpha\big(\partial_x e_L, \partial_x v\big)_\Omega \\
&= \big({}_0^RD_t^\alpha(\tilde{u}_L^h - u^h), v\big)_\Omega + K_\alpha\left(\partial_x(\tilde{u}_L^h - u^h), \partial_x v\right)_\Omega \\
&\quad - \big(\partial_x(pu_L^h) - \partial_x(p\tilde{u}_L^h), v\big)_\Omega - \big(\partial_x(p\tilde{u}_L^h) - \partial_x(pu^h), v\big)_\Omega \quad \text{for all} \quad v \in V_M \otimes \mathscr{P}_N.
\end{aligned}
$$

The Eqs. (2.12), (2.13) yield

$$
\begin{aligned}
\mathscr{A}(e_L,v) &= \left({}_0^RD_t^\alpha(\Pi_M^{(1,0)}u^h - u^h), v\right)_\Omega + K_\alpha\left(\partial_x(\pi_N^{(-\alpha,\alpha)}u^h - u^h), \partial_x v\right)_\Omega \\
&\quad - \big(\partial_x(pu_L^h) - \partial_x(p\tilde{u}_L^h), v\big)_\Omega - \big(\partial_x(p\tilde{u}_L^h) - \partial_x(pu^h), v\big)_\Omega \quad \text{for all} \quad v \in V_M \otimes \mathscr{P}_N,
\end{aligned}
$$

and for $v = {}_0^RD_t^\alpha e_L$ we obtain

$$
\begin{aligned}
&\big({}_0^RD_t^\alpha e_L, {}_0^RD_t^\alpha e_L\big)_\Omega + K_\alpha\left(\partial_x e_L, \partial_x({}_0^RD_t^\alpha e_L)\right)_\Omega \\
=& \left({}_0^RD_t^\alpha(\Pi_M^{(1,0)}u^h - u^h), {}_0^RD_t^\alpha e_L\right)_\Omega + K_\alpha\left(\partial_x(\pi_N^{(-\alpha,\alpha)}u^h - u^h), \partial_x({}_0^RD_t^\alpha e_L)\right)_\Omega \\
&- \big(\partial_x(pu_L^h) - \partial_x(p\tilde{u}_L^h), {}_0^RD_t^\alpha e_L\big)_\Omega - \big(\partial_x(p\tilde{u}_L^h) - \partial_x(pu^h), {}_0^RD_t^\alpha e_L\big)_\Omega.
\end{aligned}
$$

The inequality (4.11a) from [29] shows that

$$
\begin{aligned}
\big\|{}_0^RD_t^\alpha e_L\big\|_\Omega^2 + \big\|\partial_x e_L\big\|_\Omega^2 \lesssim& \big\|{}_0^RD_t^\alpha(\Pi_M^{(1,0)}u^h - u^h)\big\|_\Omega \big\|{}_0^RD_t^\alpha e_L\big\|_\Omega \\
&+ \big\|\partial_x^2(\pi_N^{(-\alpha,\alpha)}u^h - u^h)\big\|_\Omega \big\|{}_0^RD_t^\alpha e_L\big\|_\Omega \\
&+ \big\|\partial_x u_L^h - \partial_x\tilde{u}_L^h\big\|_\Omega \big\|{}_0^RD_t^\alpha e_L\big\|_\Omega + \big\|\partial_x\tilde{u}_L^h - \partial_x u^h\big\|_\Omega \big\|{}_0^RD_t^\alpha e_L\big\|_\Omega,
\end{aligned}
$$

and recalling the definition of the norm $\|\cdot\|_{B^s(\Omega)}$, we arrive at the estimate

$$
\begin{aligned}
\|e_L\|_{B^\alpha(\Omega)} \lesssim& \big\|{}_0^RD_t^\alpha(\Pi_M^{(1,0)}u^h - u^h)\big\|_\Omega + \big\|\partial_x^2(\pi_N^{(-\alpha,\alpha)}u^h - u^h)\big\|_\Omega \\
&+ \big\|\partial_x u_L^h - \partial_x\tilde{u}_L^h\big\|_\Omega^2 + \big\|\partial_x\tilde{u}_L^h - \partial_x u^h\big\|_\Omega.
\end{aligned} \tag{4.3}
$$

The terms in the right-hand side of (4.3) can be estimated by Lemmas 2.2-2.3, viz.

$$
\begin{aligned}
&\big\|{}_0^RD_t^\alpha(\Pi_M^{(1,0)}u^h - u^h)\big\|_\Omega \lesssim M^{1-r}\big\|\partial_x^{r-1}({}_0^RD_t^\alpha u^h)\big\|_{L^2_{\chi^{r-2}}(\Lambda,L^2(I))}, \qquad (4.4)\\
&\big\|\partial_x^2(\pi_N^{(-\alpha,\alpha)}u^h - u^h)\big\|_\Omega \lesssim N^{-(\alpha+s)}\big\|\partial_x^2({}_0^RD_t^{\alpha+s}u^h)\big\|_{L^2(\Lambda,L^2_{\omega^{(s,s)}}(I))}, \\
&\big\|\partial_x\tilde{u}_L^h - \partial_x u^h\big\|_\Omega \lesssim \big\|\partial_x\Pi_M^{1,0}(u^h - \pi_N^{(-\alpha,\alpha)}u^h)\big\|_\Omega + \big\|\partial_x(u^h - \Pi_M^{1,0}u^h)\big\|_\Omega \\
&\qquad\lesssim \big\|\partial_x(u^h - \pi_N^{(-\alpha,\alpha)}u^h)\big\|_\Omega + \big\|\partial_x(u^h - \Pi_M^{1,0}u^h)\big\|_\Omega \\
&\qquad\lesssim N^{-(\alpha+s)}\big\|\partial_x({}_0^RD_t^{\alpha+s}u^h)\big\|_{L^2(\Lambda,L^2_{\omega^{(s,s)}}(I))} \\
&\qquad\quad + M^{1-r}\big\|\partial_x^r u^h\big\|_{L^2_{\chi^{r-1}}(\Lambda,L^2(I))}.
\end{aligned}
$$

The estimates (4.3) and (4.4) yield

$$\begin{aligned}\|e_L\|_{B^\alpha(\Omega)} \lesssim & M^{1-r}\left\|\partial_x^{r-1}\left({}_0^R D_t^\alpha u^h\right)\right\|_{L^2_{\chi^{r-2}}(\Lambda,L^2(I))} + N^{-(\alpha+s)}\left\|\partial_x^2\left({}_0^R D_t^{\alpha+s} u^h\right)\right\|_{L^2(\Lambda,L^2_{\omega^{(s,s)}}(I))} \\ & + N^{-(\alpha+s)}\left\|\partial_x\left({}_0^R D_t^{\alpha+s} u^h\right)\right\|_{L^2(\Lambda,L^2_{\omega^{(s,s)}}(I))} + M^{1-r}\left\|\partial_x^r u^h\right\|_{L^2_{\chi^{r-1}}(\Lambda,L^2(I))}. \end{aligned} \tag{4.5}$$

Since $u^h - u_L^h = u - \tilde{u}_L^h + e_L$, the estimate (4.5) and the triangle inequality leads to (4.2). $\square$

5. Numerical Experiments

We want to demonstrate by numerical experiments that the STPG spectral method for TFFP equation is spectrally accurate.

Example 5.1. Consider the TFFP equation

$$\begin{aligned} &\partial_t u = {}_0^R D_t^{1-\alpha}\left\{\partial_x[p(x)u(x,t)] + \partial_{xx}u(x,t) + f(x,t)\right\}, \quad (x,t) \in (-1,1)\times(0,2], \\ &f(x,t) = \Gamma(\alpha+1)\sin(\pi x) + \pi^2\sin(\pi x)t^\alpha + \pi\cos(\pi x)(x+1)t^\alpha + \sin(\pi x)t^\alpha, \\ &p(x) = -x-1 \end{aligned} \tag{5.1}$$

with the initial and boundary conditions (2.3)-(2.4).

The Eq. (5.1) has the solution $u(x,t) = \sin(\pi x)t^\alpha$. Table 1 shows that numerical errors in time reach machine accuracy very fast — viz. for $N = 4,6,8$. This happens because the GJFs basis almost matches the singularities of the solution. Besides, the numerical error in space decays exponentially as M grows — cf. Fig. 1.

Table 1: L^∞- and L^2-errors versus $N, M = 30$.

	$\alpha=0.1$		$\alpha=0.5$		$\alpha=0.9$	
N	$\|u^h-u_L^h\|_{L^\infty}$	$\|u^h-u_L^h\|_0$	$\|u^h-u_L^h\|_{L^\infty}$	$\|u^h-u_L^h\|_0$	$\|u^h-u_L^h\|_{L^\infty}$	$\|u^h-u_L^h\|_0$
4	1.2879e-14	3.9747e-14	1.6875e-14	5.1282e-14	2.2204e-14	6.6004e-14
6	1.2879e-14	3.9562e-14	1.7319e-14	5.2605e-14	2.2649e-14	6.8144e-14
8	1.3101e-14	3.9188e-14	1.6875e-14	5.0652e-14	2.2649e-14	6.6528e-14

Example 5.2. Consider the TFFP equation (5.1) with $p = -1$ and

$$f(x,t) = \frac{\Gamma(7+9/17)}{\Gamma(7+9/17-\alpha)}\sin(\pi x)t^{6+9/17-\alpha} + \pi^2\sin(\pi x)t^{6+9/17} + \pi\cos(\pi x)t^{6+9/17}.$$

It has the solution is $u(x,t) = \sin(\pi x)t^{6+9/17}$ and Figs. 2-3 show that numerical errors decay exponentially as M or N increase — i.e. the method is spectrally accurate in both time and space.

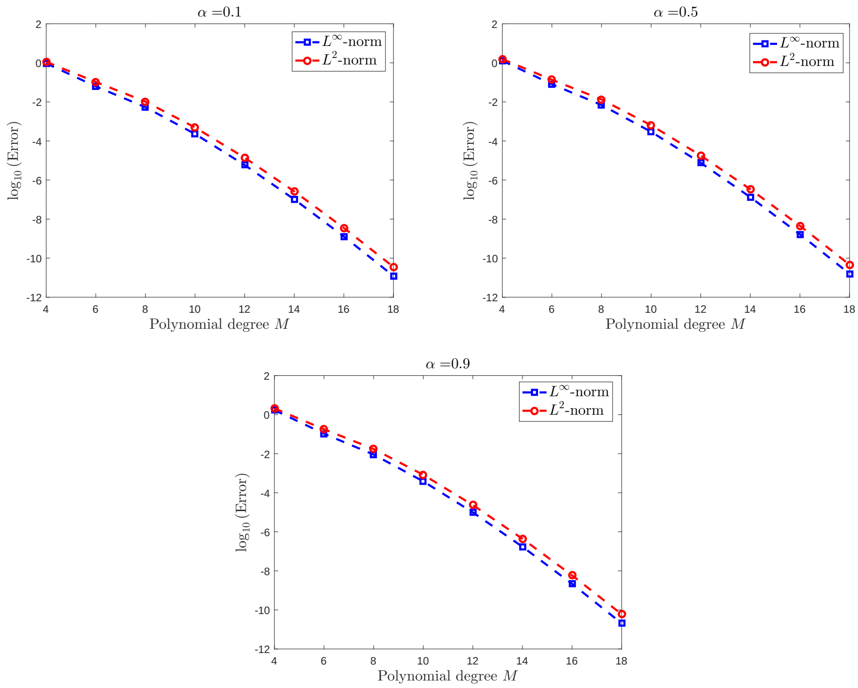

Figure 1: Example 5.1. L^∞- and L^2-errors in semi-log scale versus M and different α, $N = 30$.

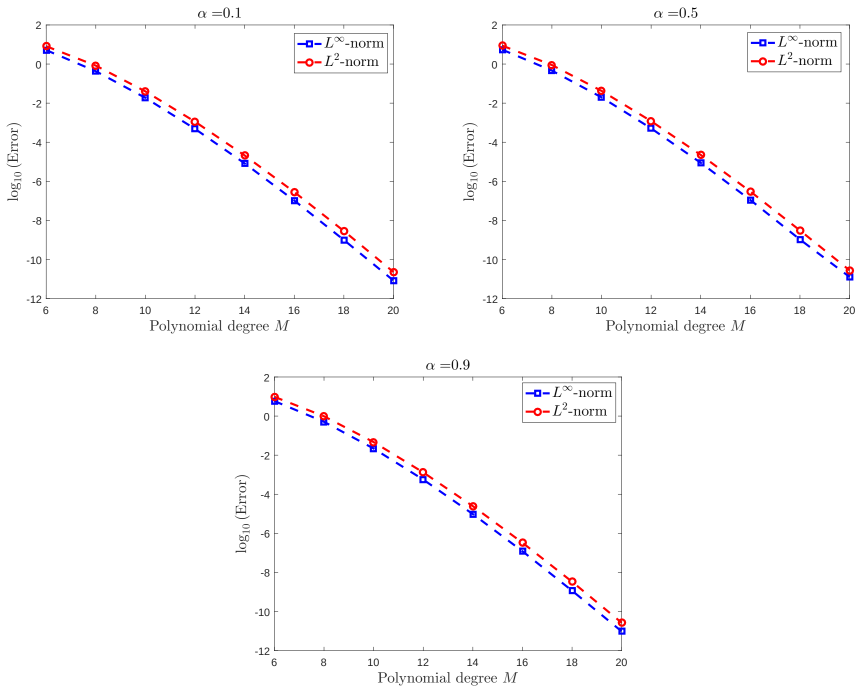

Figure 2: Example 5.2. L^∞- and L^2-errors in semi-log scale versus M and different α, $N = 30$.

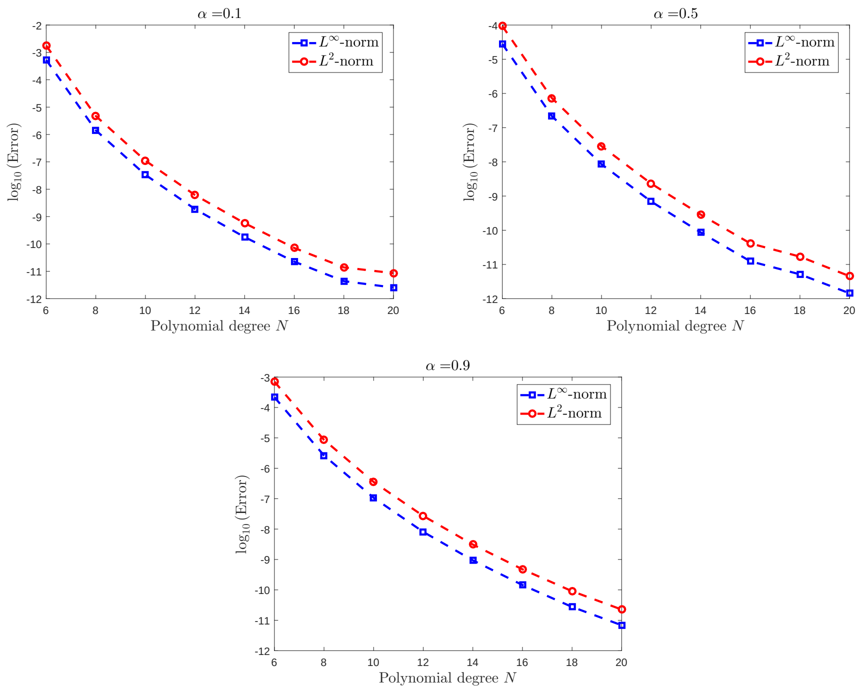

Figure 3: Example 5.2. L^∞- and L^2-errors in semi-log scale versus N and different α, $M=30$.

Example 5.3. Consider the TFFP equation (5.1) with the source function

$$f(x,t)=\sin(\pi x)\sin(\pi t)$$

and homogeneous initial conditions. The exact solution is not known but it is expected to have singularity at $t=0$.

First, we determine the numerical solution for $M=40$, $N=80$ and set it as the reference solution. Fig. 4 shows the exponential decay of L^2- and L^∞-errors in space in semi-log scale for different α. For time errors, the error accuracy is not as high as in Example 5.1-5.2, and higher degree polynomial functions are needed to achieve better accuracy in Fig. 5. But the accuracy improves for α close to 1.

Example 5.4. Consider the TFFP equation (5.1) with $p=-1$ and

$$f(x,t)=\Gamma(\alpha+2)\sin(\pi x)t+\pi^2\sin(\pi x)t^{1+\alpha}+\pi\cos(\pi x)t^{1+\alpha}.$$

It has solution $u(x,t)=\sin(\pi x)t^{1+\alpha}$.

Table 2 provides L^2- and L^∞-errors of the STPG spectral method and Zheng's method for $N=4,6,8$, $\alpha=0.5$ and $M=30$. The numerical errors of the method reach the machine accuracy for $N=4,6,8$. On the other hand, the accuracy of Zheng's method is low. Even for $N=90$, the L^2- and L^∞-errors still do not approach 4 as Fig. 6 shows. Thus for equations with nonsmooth solutions, the method above is superior to Zheng's method.

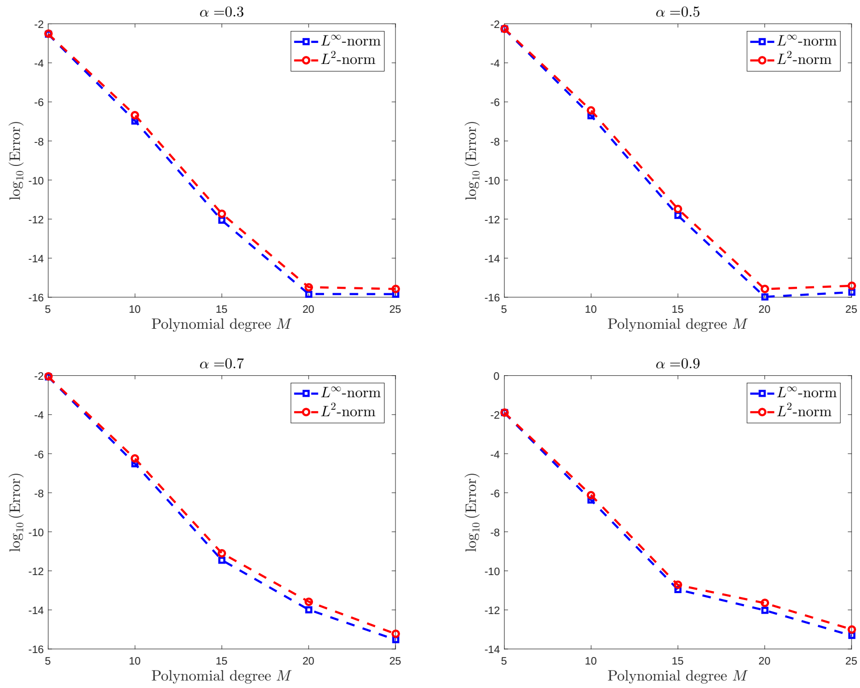

Figure 4: Example 5.3. L^∞- and L^2-errors in semi-log scale versus M and different α, $N = 80$.

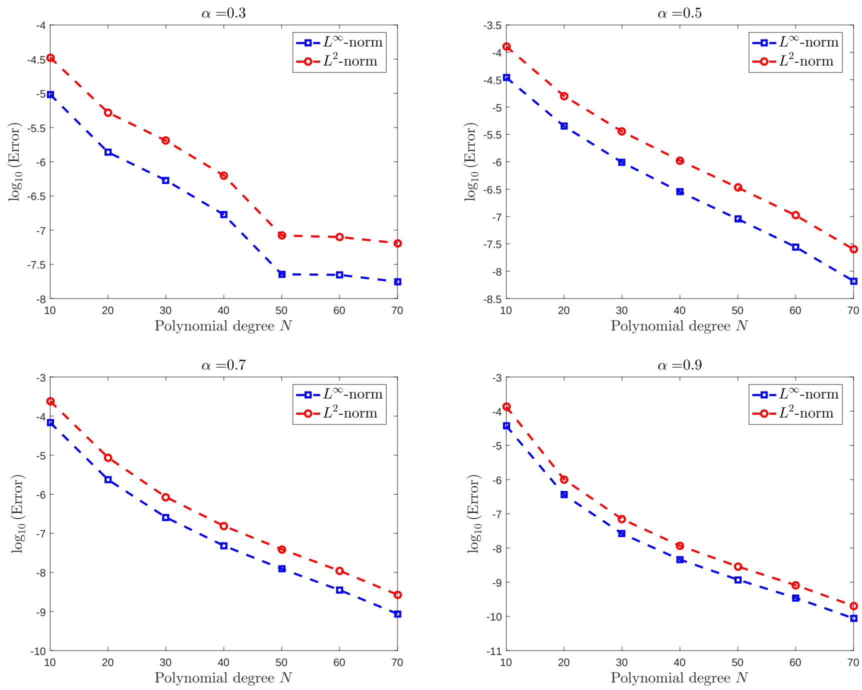

Figure 5: Example 5.3. L^∞- and L^2-errors in semi-log scale versus N and different α, $M = 40$.

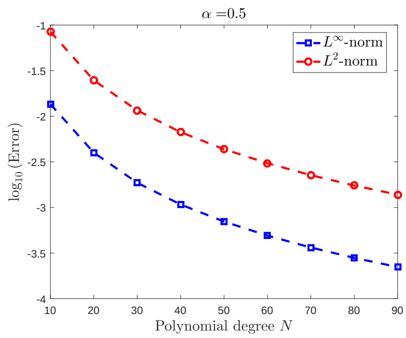

Figure 6: Example 5.4. L^∞- and L^2-errors in semi-log scale versus N, $\alpha=0.5, M=100$.

Table 2: L^∞- and L^2-errors of STPG spectral method and Zheng's method, $\alpha=0.5, M=30$.

	STPG spectral method		Zheng's method	
N	$\|\|u^h-u_L^h\|\|_{L^\infty}$	$\|\|u^h-u_L^h\|\|_0$	$\|\|u^h-u_L^h\|\|_{L^\infty}$	$\|\|u^h-u_L^h\|\|_0$
4	3.5527e-14	1.1400e-13	3.9668e-02	1.3304e-01
6	5.7732e-14	1.8958e-13	2.9136e-02	9.8337e-02
8	1.9096e-14	5.6652e-14	1.9432e-02	6.5405e-02

6. Conclusions

We developed an STPG spectral method for TFFP equations with nonsmooth solutions. The numerical scheme is based on GJFs in time and Legendre polynomials in space. The GJFs match the leading singularity of the corresponding problem. Therefore, the method performs better than ones with polynomial bases. Moreover, the employment of GJFs produces the identity stiffness matrix in calculation. We also discuss the errors of the method, and numerical results are consistent with theoretical error estimates.

However, if the singularity is complicated or unknown, the method does not achieve the spectral accuracy.

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