Cross-Kink Wave Solutions and Semi-Inverse Variational Method for (3 + 1)-Dimensional Potential-YTSF Equation

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Abstract. Periodic wave solutions of (3 + 1)-dimensional potential-Yu-Toda-Sasa-Fuku yama (YTSF) equation are constructed. Using the bilinear form of this equation, we chose ansatz as a combination of rational, trigonometric and hyperbolic functions. Density graphs of certain solutions in 3D and 2D situations show different cross-kink waveforms and new multi wave and cross-kink wave solutions. Moreover, we employ the semi-inverse variational principle (SIVP) in order to study the solitary, bright and dark soliton wave solutions of the YTSF equation.

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1. Introduction

Many nonlinear phenomena, which play an important role in applied sciences and engineering are modeled by nonlinear partial differential equations (NPDEs). Numerous examples of such equations can be found in plasma physics, elastic media, optical fibers, fluid dynamics, quantum mechanics, chemical physics, biotechnology, signal processing, solid state physics, and shallow water wave theory. However, their explicit analytic solutions are rarely available. Therefore, finding localised solutions and, more specifically, solitary wave solutions [1, 6, 7, 28–30, 33–36, 41, 55], lump-type solutions [5, 14–17, 19, 26–28, 32,
33,37,38,40,44,45,49], and also describing the interactions soliton-soliton, soliton-kink, kink-kink [9,18,46], as well as the interaction between solitary waves, lumps [47,53] and periodic wave solutions [2,8,31] is an interesting problem. The approaches used in these studies include exp-function method [4,28], homotopy perturbation technique [3], and inverse scattering method [39].

The nonlinear (3 + 1)-dimensional potential-Yu-Toda-Sasa-Fukuyama equation has the form

$$-4u_{xt} + u_{xxxx} + 4u_xu_{xx} + 2u_{xx}u_z + 3u_{yy} = 0. \quad (1.1)$$

It appears in fluid dynamics, plasma physics, weakly dispersive media and other physical applications. Various powerful methods for solving (3+1)-dimensional YTSF equation, such as $G'/G$-expansion method [43], generalized projective Riccati equation method [51], symmetry method [48], Korteweg-de Vries equation-based sub-equation method [42], extended homoclinic test technique [50], homoclinic test approach and three-wave method [13] have been considered. Applying the dependent variable transformation

$$\eta = x + \omega z, \quad u = 2(\ln f)_{\eta}, \quad f = f(\eta, y, t), \quad (1.2)$$

one can transform (1.1) into the nonlinear equation

$$-4u_{\eta t} + \omega u_{\eta\eta\eta\eta} + 6\omega u_{\eta}u_{\eta\eta} + 3u_{yy} = 0,$n

and consequent application of the mapping

$$u = 2(\ln f)_{\eta}, \quad f = f(\eta, y, t)$$

leads to the Hirota bilinear form

$$\left(-4D_\eta D_t + \omega D_\eta^4 + 3D_y^2\right)f \cdot f = 0 \quad (1.3)$$

with a bilinear operator $D$ and an unknown function $f = f(x, y, t)$, which has to be determined later on.

Suppose the Hirota derivatives for functions $f$ and $g$ can be written as

$$\prod_{i=1}^3 D_{j_i}^\beta_i f \cdot g = \prod_{i=1}^3 \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \eta'}\right)^\beta_i f(\eta)g(\eta') \bigg|_{\eta' = \eta},$$

where

$$j = (j_1, j_2, j_3) = (\eta, y, t), \quad j' = (j'_1, j'_2, j'_3) = (\eta', y', t')$$

and $\beta_1, \beta_2, \beta_3$ are arbitrary nonnegative integers. The corresponding bilinear formalism for the Eq. (1.3) is

$$-4f f_{\eta t} + 4f_\eta f_t + \omega \left( f f_{\eta\eta\eta\eta} - 4f_\eta f_{\eta\eta} + 3f_{\eta}^2 \right) + 3f f_{yy} - 3f_y^2 = 0. \quad (1.4)$$

For simplicity, we change $\eta$ to $x$, so that the Eq. (1.4) takes the form

$$-4f f_{xt} + 4f_x f_t + \omega \left( f f_{xxx} - 4f_x f_{xxx} + 3f_{xx}^2 \right) + 3f f_{yy} - 3f_y^2 = 0. \quad (1.5)$$
Lump solutions and their interactions are derived from exact rational soliton solutions for nonlinear evolution equations \([44]\), Kadomtsev-Petviashvili (KP) equation \([33]\), reduced \(p\)-gKP and \(p\)-gbKP equations \([26]\). The interaction between lumps and other solitary, periodic and kink solitons for \((2+1)\)-dimensional breaking soliton equation is studied in \([38]\). The lump and interaction between different types of lumps have been worked, for example, the variable-coefficient KP equation \([16]\), the periodic type and periodic cross-kink wave solutions \([15]\), the \((2+1)\)-dimensional Sawada-Kotera equation \([14]\), the \((2+1)\)-dimensional bSK equation \([17,37]\), the \((2+1)\)-dimensional generalized fifth-order KdV equation \([18]\), the \((2+1)\)-dimensional Burger equation \([46]\), the generalized \((3+1)\)-dimensional Shallow water-like equation \([53]\), and the B-KP equation \([28]\). Numerous studies based on the Hirota bilinear operator, investigate periodic solitary wave solutions of \((2+1)\)-dimensional extended Jimbo-Miwa equations \([31]\), interaction between lump and other solitary, periodic and kink solitons of \((2+1)\)-dimensional breaking soliton equation \([38]\), lump and interaction between various solutions of variable-coefficient Kadomtsev-Petviashvili equation \([16]\), and periodic type and periodic cross-kink wave solutions \([15]\). Other types of exact solutions such as lump solutions for a combined fourth-order nonlinear PDE \([21]\), lump and interaction solutions of linear PDEs \([22]\), interaction solutions for lumps arising in the Hirota-Satsuma-Ito equation \([23]\), solitonless solutions for a three-component coupled mKdV system \([24]\), nonlocal integrable using inverse scattering transforms \([25]\) have been also considered. On the other hand, the same quadratic function method was employed in the study of lump solutions of a generalized \((3+1)\)-dimensional shallow water-like equation \([52]\) and lump and interaction solutions of \((3+1)\)-dimensional Jimbo-Miwa equation \([54]\).

The aim of this work is to determine novel exact periodic solutions of the YTSF equation using the Hirota bilinear method. This paper is structured as follows. Using the Hirota bilinear method, we discuss new cross-kink waves solutions of the nonlinear YTSF equation. After that we provide graphical illustrations of solutions of the model under consideration. Finally in Section 3, the SIVP technique is applied to determine solitary, bright, dark and singular wave solutions. Our conclusions are given in Section 4.

2. Cross-Kink Wave Solutions of YTSF Equation

Taking into account the three wave hypothesis \([8]\), we are looking for cross-kink wave solutions \(u\) of the Eq. \((1.2)\) in the form

\[
u = 2\frac{\partial}{\partial x} \ln(f) = \frac{2}{f} \left( a_1 \Omega_1 H_1 - a_2 \Omega_1 H_2 + a_3 H_3 \Omega_4 + a_4 H_6 \Omega_7 \right),
\]

where \(f = a_1 H_1 + a_2 H_2 + a_3 H_3 + a_4 H_4\) with

\[
H_1 = \exp(\Omega_1 x + \Omega_2 y + \Omega_3 t), \quad H_2 = \exp(-\Omega_1 x - \Omega_2 y - \Omega_3 t),
\]
\[
H_3 = \sin(\Omega_4 x + \Omega_5 y + \Omega_6 t), \quad H_4 = \sinh(\Omega_7 x + \Omega_8 y + \Omega_9 t),
\]
\[
H_5 = \cos(\Omega_4 x + \Omega_5 y + \Omega_6 t), \quad H_6 = \cosh(\Omega_7 x + \Omega_8 y + \Omega_9 t),
\]
and \( \Omega_i, i = 1, \ldots, 9, a_j, j = 1, \ldots, 4 \) are parameters to be determined later on. Substituting (2.1) into the Eq. (1.5) and equating the corresponding coefficients, we obtain

\[
\begin{align*}
\omega \Omega_4^4 - 6\omega \Omega_4^2 \Omega_5^2 + \omega \Omega_4^5 + 4\Omega_4 \Omega_6 - 3\Omega_5^2 - 4\Omega_7 \Omega_9 + 3\Omega_8^2 &= 0, \\
\omega \Omega_4^4 + 6\omega \Omega_4^2 \Omega_5^2 + \omega \Omega_4^5 - 4\Omega_4 \Omega_3 + 3\Omega_5^2 - 4\Omega_7 \Omega_9 + 3\Omega_8^2 &= 0, \\
2\omega \Omega_4 \Omega_7 - 2\omega \Omega_4 \Omega_5^3 + 2\Omega_4 \Omega_9 - 3\Omega_5 \Omega_8 + 2\Omega_5 \Omega_7 &= 0, \\
2\omega \Omega_4 \Omega_7 + 2\omega \Omega_4 \Omega_5^3 - 2\Omega_4 \Omega_9 + 3\Omega_5 \Omega_8 - 2\Omega_5 \Omega_7 &= 0, \\
\omega \Omega_4^4 - 6\omega \Omega_4^2 \Omega_5^2 + \omega \Omega_4^5 - 4\Omega_4 \Omega_3 + 3\Omega_5^2 + 4\Omega_4 \Omega_6 - 3\Omega_5^2 &= 0, \\
2\omega \Omega_4 \Omega_7 - 2\omega \Omega_4 \Omega_5^3 + 2\Omega_4 \Omega_9 - 3\Omega_5 \Omega_8 + 2\Omega_5 \Omega_7 &= 0,
\end{align*}
\]

The graphs presented in Fig. 1 display such periodic wave solutions including 3D plot, density plot, and 2D plot with the spaces arising at \( x = -10, x = 0, \) and \( x = 10. \)
Cross-Kink Wave Solutions and Semi-Inverse Variational Method

Figure 1: Cross-kink waves (2.3), \( a_2 = 0.5, a_3 = 1.5, a_4 = 1, \Omega_7 = 1.9, \Omega_8 = 0.2, \omega = -0.1, t = 10 \). (a) 3D plot. (b) Density plot. (c) 2D plot for \( x = -5 \) (yellow); \( x = 0 \) (orange); \( x = 5 \) (purple).

Case II.

\[
\begin{align*}
\Omega_1 &= 0, \quad \Omega_2 = \sqrt{-\omega \Omega_7 \Omega_4}, \\
\Omega_3 &= \frac{3}{2} \sqrt{-\omega \Omega_4 \Omega_8}, \quad \Omega_5 = \frac{\Omega_4 \Omega_8}{\Omega_7}, \\
\Omega_6 &= -\frac{(\omega \Omega_4^2 \Omega_7^2 - 3 \omega \Omega_4^2 \Omega_7^2 - 3 \Omega_4^2)}{4 \Omega_7^2}, \quad \Omega_9 = -\frac{3 \omega \Omega_4^2 \Omega_7^2 - \omega \Omega_7^2 - 3 \Omega_8^2}{4 \Omega_7}, \\
a_1 &= \frac{(\Omega_4^2 + \Omega_7^2)(\Omega_4^2 a_3^2 - \Omega_7^2 a_4^2)}{4 \Omega_7^2 \Omega_4^2 a_2},
\end{align*}
\] (2.4)

and \( a_2, a_3, a_4, \Omega_4, \Omega_7, \Omega_8 \) are arbitrary constants, \( \Omega_7 \neq 0 \) and \( \omega < 0 \). Substituting (2.4) into (2.1) produces the following periodic-wave solution of the Eq. (1.2):

\[
u_2 = \frac{2}{f} \left( a_3 H_5 \Omega_4 + a_4 H_6 \Omega_7 \right),
\] (2.5)

where

\[
f = \frac{(\Omega_4^2 + \Omega_7^2)(\Omega_4^2 a_3^2 - \Omega_7^2 a_4^2)}{4 \Omega_7^2 \Omega_4^2 a_2} H_1 + a_2 H_2 + a_3 H_3 + a_4 H_4,
\]
Figure 2: Cross-kink waves (2.5), $a_2 = 0.5, a_3 = 1.5, a_4 = 1, \Omega_4 = 1, \Omega_7 = 0.5, \Omega_8 = 0.2, \omega = -0.1, t = 10$.
(a) 3D plot. (b) Density plot. (c) 2D plot for $x = -5$ (yellow); $x = 0$ (orange); $x = 5$ (purple).

and

$$H_1 = e^{\sqrt{-\omega} \Omega_4 y + 3/2 \sqrt{-\omega} \Omega_4 \Omega_7 t}, \quad H_3 = \sin \left( \Omega_4 x + \frac{\Omega_4 \Omega_8}{\Omega_7} y - \frac{\left( \Omega_4^2 \Omega_7^2 - 3 \Omega_4^2 - 3 \Omega_8^2 \right) \Omega_4}{4\Omega_7^2} t \right),$$

$$H_2 = e^{-\sqrt{-\omega} \Omega_4 y - 3/2 \sqrt{-\omega} \Omega_4 \Omega_7 t}, \quad H_4 = \sinh \left( \Omega_7 x + \Omega_8 y - \frac{3 \Omega_4^2 \Omega_7^2 - \Omega_4^2 - 3 \Omega_8^2}{4\Omega_7} t \right).$$

The graphs presented in Fig. 2, display such periodic wave solution including 3D plot, density plot, and 2D plot with spaces arise at $x = -5, x = 0$, and $x = 5$.

Case III.

$$\Omega_1 = 0, \quad \Omega_2 = \sqrt{-\omega} \Omega_4 \Omega_7, \quad \Omega_3 = 0,$$

$$\Omega_5 = 0, \quad \Omega_6 = -\frac{1}{4} \Omega_4 \omega \left( \Omega_4^2 - 3 \Omega_7^2 \right), \quad \Omega_8 = 0,$$

$$\Omega_9 = -\frac{1}{4} \omega \Omega_7 \left( 3 \Omega_4^2 - \Omega_7^2 \right), \quad a_2 = 0, \quad a_4 = \frac{\Omega_4 a_3}{\Omega_7}. \quad (2.6)$$
and $a_1, a_3, \Omega_4, \Omega_7, \Omega_7 \neq 0$ with $\omega < 0$. Substituting (2.6) into (2.1) produces the following periodic-wave solution of the Eq. (1.2)

$$
u_3 = \frac{2a_3 \cos \left(\frac{1}{4}t \Omega_4 \omega \left(\Omega_2^2 - 3 \Omega_4^2\right) - x \Omega_4\right) \Omega_4}{a_1 e^{y - \omega t \Omega_4} - a_3 \sin \left(\frac{1}{4}t \Omega_4 \omega \left(\Omega_2^2 - 3 \Omega_4^2\right) - x \Omega_4\right) - \Phi} \Omega_4 + \frac{2a_3 \cos \left(\frac{1}{4}t \omega \Omega_7 \left(3 \Omega_4^2 - \Omega_7^2\right) - \Omega_7x\right)}{a_1 e^{y - \omega t \Omega_4} - a_3 \sin \left(\frac{1}{4}t \Omega_4 \omega \left(\Omega_2^2 - 3 \Omega_4^2\right) - x \Omega_4\right) - \Phi},$$

where

$$\Phi := \frac{\Omega_4 a_3}{\Omega_7} \sinh \left(\frac{1}{4}t \omega \Omega_7 \left(3 \Omega_4^2 - \Omega_7^2\right) - \Omega_7x\right).$$

**Case IV.**

$$\Omega_2 = \frac{\left(\Omega_1^2 - \Omega_4^2\right) \sqrt{\omega \left(\Omega_1^2 a_3^2 + \Omega_2^2 a_4^2 - \Omega_4^2 a_4^2\right)}}{a_3 \Omega_7},$$

$$\Omega_3 = \frac{\omega \Omega_1 \left(3 \Omega_1^2 a_3^2 + 3 \Omega_4^2 a_3^2 - 2 \Omega_1^2 \Omega_4^2 a_4^2 - 6 \Omega_1^2 \Omega_4^2 a_4^2 + 3 \Omega_2^2 a_3^2 + 3 \Omega_4^2 a_4^2\right)}{4 a_3^2 \Omega_1},$$

$$\Omega_4 = \frac{\sqrt{-\Omega_1^2 a_3^2 - \Omega_1^2 a_4^2 + \Omega_2^2 a_4^2}}{a_3},$$

$$\Omega_5 = \frac{\sqrt{-\omega \left(\Omega_1^2 - \Omega_4^2\right) \left(a_3^2 + a_4^2\right)}}{a_3 \Omega_7},$$

$$\Omega_6 = \frac{\left(3 \Omega_1^2 a_3^2 + 3 \Omega_4^2 a_3^2 - 2 \Omega_1^2 \Omega_4^2 a_4^2 - 6 \Omega_1^2 \Omega_4^2 a_4^2 + 3 \Omega_2^2 a_3^2 - \Omega_4^2 a_4^2\right) \omega}{4 a_3^2 \Omega_7},$$

$$\Omega_8 = 0, \quad a_1 = 0,$$

$$\Omega_9 = -\frac{\left(3 \Omega_1^2 a_3^2 + 3 \Omega_4^2 a_3^2 - 6 \Omega_1^2 \Omega_4^2 a_4^2 - 6 \Omega_1^2 \Omega_4^2 a_4^2 - 2 \Omega_1^2 \Omega_4^2 a_4^2 + 3 \Omega_2^2 a_3^2 + 3 \Omega_4^2 a_4^2\right) \omega}{4 a_3^2 \Omega_7},$$

and $\Omega_1, \Omega_7, a_2, a_3, a_4 = a_4$ are arbitrary constants such that $\Omega_7 \neq 0$, $\omega < 0$ and

$$\omega \left(\Omega_1^2 a_3^2 + \Omega_1^2 a_4^2 - \Omega_4^2 a_4^2\right) > 0.$$

Substituting (2.7) into (2.1) produces the following periodic-wave solution of the Eq. (1.2):

$$\nu_4 = \frac{2}{f} \left(-a_2 \Omega_1 H_2 + a_3 H_3 \Omega_4 + a_4 H_6 \Omega_7\right),$$

where

$$f = a_2 H_2 + a_3 H_3 + a_4 H_4,$$

and

$$H_2 = \exp(-\Omega_1 x - \Omega_2 y - \Omega_3 t), \quad H_3 = \sin(\Omega_4 x + \Omega_5 y + \Omega_6 t),$$

$$H_4 = \sinh(\Omega_7 x + \Omega_9 t), \quad H_5 = \cos(\Omega_4 x + \Omega_5 y + \Omega_6 t),$$

$$H_6 = \cosh(\Omega_7 x + \Omega_9 t).$$
The graphs presented in Fig. 3 display such periodic wave solution including 3D plot, density plot, and 2D plot with the spaces arising at $x = -1$, $x = 0$, and $x = 1$.

**Case V.**

\[
\begin{align*}
\Omega_2 &= \frac{(\Omega_1^2 - \Omega_7^2)}{\alpha_3^2 \Omega_7} \sqrt{\omega (\Delta_1)}, \\
\Omega_3 &= \frac{\omega \Omega_1 (3\Omega_1^4 a_3^2 + 3\Omega_4^4 a_4^2 - 2\Omega_1^2 \Omega_7^2 a_3^2 - 6\Omega_1^2 \Omega_7^2 a_4^2 + 3\Omega_4^4 a_3^2 + 3\Omega_1^4 a_4^2)}{4\Omega_7^2 a_3^2}, \\
\Omega_4 &= \frac{\sqrt{-\Delta_1}}{a_3}, \quad \Omega_5 = \frac{\sqrt{-\omega (\Omega_1^2 - \Omega_7^2)} (a_3^2 + a_4^2) \Omega_1}{a_3^2 \Omega_7}, \\
\Omega_6 &= \frac{(3\Omega_1^4 a_3^2 + 3\Omega_4^4 a_4^2 - 2\Omega_1^2 \Omega_7^2 a_3^2 - 2\Omega_1^2 \Omega_7^2 a_4^2 + 3\Omega_4^4 a_3^2 - \Omega_7^4 a_4^2)}{4a_3^4 \Omega_7^2} \omega \sqrt{-\Delta_1}, \\
\Omega_8 &= 0, \quad \Omega_9 = \frac{(3\Omega_1^4 a_3^2 + 3\Omega_4^4 a_4^2 - 6\Omega_1^2 \Omega_7^2 a_3^2 - 6\Omega_1^2 \Omega_7^2 a_4^2 - \Omega_7^4 a_3^2 + 3\Omega_1^4 a_4^2)}{4a_3^2 \Omega_7^2} \omega.
\end{align*}
\]
where \( f \) and \( a \) are arbitrary constants, \( \Omega_7 \neq 0, \omega < 0 \) and
\[
\omega \left( \Omega_1^2 a_3^2 + \Omega_1^2 a_4^2 - \Omega_7^2 a_4^2 \right) > 0.
\]
Substituting (2.9) into (2.1) produces the following periodic-wave solution of Eq. (1.2):
\[
u_5 = \frac{2}{f} \left( a_1 \Omega_1 H_1 + a_3 H_5 \Omega_4 + a_4 H_6 \Omega_7 \right),
\tag{2.10}
\]
where \( f = a_1 H_1 + a_3 H_3 + a_4 H_4 \) and
\[
H_1 = \exp(\Omega_1 x + \Omega_2 y + \Omega_3 t), \quad H_3 = \sin(\Omega_4 x + \Omega_5 y + \Omega_6 t),
\]
\[
H_4 = \sinh(\Omega_7 x + \Omega_8 t), \quad H_5 = \cos(\Omega_4 x + \Omega_5 y + \Omega_6 t),
\]
\[
H_6 = \cosh(\Omega_7 x + \Omega_8 t).
\]

**Case VI.**
\[
\Omega_2 = \left( \frac{\Omega_1^2 - \Omega_7^2}{\Omega_7} \right)^{1/2} \omega \Omega_4, \quad \Omega_3 = \frac{\Omega_1 \omega (3 \Omega_1^2 \Omega_4^2 - \Omega_1^2 \Omega_7^2 - 3 \Omega_4^2 \Omega_7^2 - 3 \Omega_4^2)}{4 \Omega_7^2},
\]
\[
\Omega_5 = \left( \frac{\Omega_1^2 \Omega_4^2 + \Omega_7^2}{\Omega_7} \right) \Omega_4, \quad \Omega_6 = -\frac{\omega \Omega_4 (3 \Omega_1^2 \Omega_4^2 + 3 \Omega_1^2 \Omega_7^2 + \Omega_4^2 \Omega_7^2 - 3 \Omega_4^2)}{4 \Omega_7^2},
\]
\[
\Omega_8 = 0, \quad \Omega_9 = \frac{\left( 3 \Omega_1^2 \Omega_4^2 + 3 \Omega_1^2 \Omega_7^2 + \Omega_4^2 \Omega_7^2 - 3 \Omega_4^2 \right) \omega}{4 \Omega_7},
\tag{2.11}
\]
\[
a_1 = -\frac{\Omega_1^2 \Omega_4^2 a_3^2 + \Omega_1^2 \Omega_4^2 a_4^2 + \Omega_1^2 \Omega_7^2 a_3^2 + \Omega_1^2 \Omega_7^2 a_4^2 + \Omega_4^2 a_3^2 + \Omega_4^2 a_4^2}{4 a_2 \left( \Omega_1^2 - \Omega_7^2 \right) \left( \Omega_1^2 + \Omega_4^2 \right)}
\]
\[
- \frac{\Omega_1^2 \Omega_4^2 a_3^2 - \Omega_4^2 a_4^2}{4 a_2 \left( \Omega_1^2 - \Omega_7^2 \right) \left( \Omega_1^2 + \Omega_4^2 \right)},
\]
and \( a_2, a_3, a_4, \Omega_1, \Omega_7 \) are arbitrary constants, \( \Omega_7 \neq 0, \omega < 0 \) and
\[
\omega \left( \Omega_1^2 a_3^2 + \Omega_1^2 a_4^2 - \Omega_7^2 a_4^2 \right) > 0.
\]
Substituting (2.11) into (2.1) produces the following periodic-wave solution of the Eq. (1.2):
\[
u_6 = \frac{2}{f} \left( a_1 \Omega_1 H_1 - a_2 \Omega_1 H_2 + a_3 H_5 \Omega_4 + a_4 H_6 \Omega_7 \right),
\tag{2.12}
\]
where \( f = a_1 H_1 + a_2 H_2 + a_3 H_3 + a_4 H_4 \) and
\[
H_1 = \exp(\Omega_1 x + \Omega_2 y + \Omega_3 t), \quad H_2 = \exp(-\Omega_1 x - \Omega_2 y - \Omega_3 t),
\]
\[
H_3 = \sin(\Omega_4 x + \Omega_5 y + \Omega_6 t), \quad H_4 = \sinh(\Omega_7 x + \Omega_8 t),
\]
\[
H_5 = \cos(\Omega_4 x + \Omega_5 y + \Omega_6 t), \quad H_6 = \cosh(\Omega_7 x + \Omega_8 t).
\]
The graphs presented in Fig. 4 display such periodic wave solution including 3D plot, density plot, and 2D plot with spaces arising at \( x = -1, x = 0, \) and \( x = 1. \)
Figure 4: Cross-kink waves (2.12), \(a_2 = 0.5, a_3 = 1, a_4 = 1, \Omega_1 = 0.5, \Omega_1 = 1, \Omega_4 = 1, \Omega_7 = 2, \omega = -0.1, t = 10\). (a) 3D plot. (b) Density plot. (c) 2D plot for \(x = -1\) (yellow); \(x = 0\) (orange); \(x = 1\) (purple).

Case VII.

\[
\begin{align*}
\Omega_1 &= \frac{2}{3} \sqrt{-\omega\Omega_6}, \\
\Omega_2 &= \frac{2}{3} \sqrt{-\omega\Omega_7}, \\
\Omega_3 &= \frac{(27\omega\Omega_4^2\Omega_8^2 - 4\Omega_6^2\Omega_8^2 + 27\Omega_6^4)\Omega_6}{54\sqrt{-\omega\Omega_6^2\Omega_7^2}}, \\
\Omega_4 &= 0, \\
\Omega_5 &= \frac{2\Omega_6\Omega_7}{3\Omega_8}, \\
\Omega_9 &= \frac{3\omega\Omega_6^4\Omega_8^2 - 4\Omega_6^2\Omega_7^2 + 9\Omega_8^4}{12\Omega_7\Omega_8^2}, \\
a_1 &= 0, \\
a_3 &= \frac{\sqrt{-9\omega\Omega_7^2\Omega_8^2 - 4\Omega_6^2a_4}}{2\Omega_6}, \\
2\Omega_6 &= a_2, a_4, \Omega_1, \Omega_7 \text{ are arbitrary constants } \Omega_7, \Omega_8 \neq 0, \omega < 0 \text{ and } 9\omega\Omega_7^2\Omega_8^2 + 4\Omega_6^2 < 0.
\end{align*}
\] (2.13)

Substituting (2.13) into (2.1) produces the following multi-wave solution of the Eq. (1.2):

\[
u_7 = \frac{2}{f} \left(-a_2\Omega_1H_2 + a_3H_5\Omega_4 + a_4H_6\Omega_7\right),
\] (2.14)
where
\[
f = a_2 H_2 + \sqrt{-9 \omega \Omega_7^2 \Omega_8^2 - 4 \Omega_6^2 a_4} H_3 + a_4 H_4,
\]
and
\[
H_2 = e^l, \quad l = -\frac{2}{3} \frac{\Omega_6}{\sqrt{-\omega \Omega_8}} x - \frac{2}{3} \frac{\Omega_6}{\sqrt{-\omega \Omega_7}} y - \frac{1}{54 \sqrt{-\omega \Omega_8^2 \Omega_7^2}} (27 \omega \Omega_7^4 \Omega_8^2 - 4 \Omega_6^2 \Omega_7^2 + 27 \Omega_8^4) \Omega_6 t,
\]
\[
H_3 = \sin \left( \frac{2}{3} \frac{\Omega_6 \Omega_7}{\Omega_8} y + \Omega_6 t \right), \quad H_5 = \cos \left( \frac{2}{3} \frac{\Omega_6 \Omega_7}{\Omega_8} y + \Omega_6 t \right),
\]
\[
H_4 = \sinh \left( \Omega_7 x + \Omega_8 y + \frac{3 \omega \Omega_7^4 \Omega_8^2 - 4 \Omega_6^2 \Omega_7^2 + 9 \Omega_8^4}{12 \Omega_7 \Omega_8^2} t \right), \quad H_6 = \cosh \left( \Omega_7 x + \Omega_8 y + \frac{3 \omega \Omega_7^4 \Omega_8^2 - 4 \Omega_6^2 \Omega_7^2 + 9 \Omega_8^4}{12 \Omega_7 \Omega_8^2} t \right).
\]

The graphs presented in Fig. 5 display such cross-kink wave solution including 3D plot, density plot, and 2D plot with spaces arising at \(x = -10, x = 0, \) and \(x = 10.\)

Figure 5: Cross-kink waves (2.14), \(a_2 = 0.5, a_4 = 0.5, \Omega_6 = 0.4, \Omega_7 = 1, \Omega_8 = -1, \omega = -0.5, t = 10.\) (a) 3D plot. (b) Density plot. (c) 2D plot for \(x = -10\) (yellow); \(x = 0\) (orange); \(x = 10\) (purple).
We obtained forty sets of solutions. The three-dimensional dynamic graphs in Figs. 1-5 are produced with Maple software. Note that exponential, cosine and hyperbolic cosine functions interact with each other and move forward.

3. Application of SIVP

Employing the wave transformation \( \xi = k(x + ay + bz - ct) \) to Eq. (1.1) once more, we arrive at the nonlinear ordinary differential equation

\[
(3a^2 - 4c)k^2u'' + bk^4u'''' + 6bk^3u'u'' = 0
\]

or

\[
(3a^2 - 4c)u'' + bk^2u'''' + 6bku'u'' = 0. \tag{3.1}
\]

Taking into account the semi-inverse variational principle \([10–12]\), we multiply (3.1) by \( u' \) and integrate the result over the real line, thus obtaining the stationary integral

\[
J = \int_{-\infty}^{\infty} \left( \frac{1}{2} (3a^2 - 4c) (u')^2 + 2kb(u')^3 - \frac{1}{2} bk^2(u')^2 + bk^2u'u'' \right) d\xi.
\]

3.1. Case I

If we use the solitary wave function

\[ u(\xi) = A \text{sech}(B\xi), \]

the stationary integral takes the form

\[ J = \frac{1}{30} A^2 B \left( -21B^2bk^2 - 12kABb + 20a - 15c \right). \]

SIVP notes that the soliton amplitude and its inverse width can be determined from the coupled system

\[
\frac{\partial J}{\partial A} = 0, \quad \frac{\partial J}{\partial B} = 0. \tag{3.2}
\]

This leads to the system of nonlinear algebraic equations

\[
\frac{1}{15} AB \left( -21B^2bk^2 - 12kABb + 20a - 15c \right) - \frac{2}{5} A^2B^2kb = 0, \tag{3.3}
\]

\[
\frac{1}{30} A^2 \left( -21B^2bk^2 - 12kABb + 20a - 15c \right) + \frac{1}{30} AB \left( -42Bbk^2 - 12Abk \right) = 0, \tag{3.4}
\]

and solving it with respect to \( A \) and \( B \) gives

\[
A = \pm \frac{7(4a - 3c)}{\sqrt{-21b(4a - 3c)}}, \quad B = \pm \frac{1}{21} \frac{\sqrt{-21b(4a - 3c)}}{bk}
\]
with the parameters satisfying the conditions

\[ k \neq 0, \quad b(4a - 3c) < 0. \]

Thus the corresponding solitary wave solution obtained from SIVP has the form

\[
u(x, y, z, t) = \pm \frac{7(4a - 3c)}{\sqrt{21b(4a - 3c)}} \text{sech} \left[ \frac{1}{21} \frac{\sqrt{-21b(4a - 3c)}}{b}(x + ay + bz - ct) \right].
\]

### 3.2. Case II

For the solitary wave function

\[ u(\xi) = A \text{sech}^2(B\xi), \]

the stationary integral takes the form

\[
J = -\frac{(240B^2bk^2 + 70kABb - 112a + 84c)A^2B}{105}.
\]

Since the soliton amplitude and its inverse width satisfy the Eqs. (3.2), we arrive at the system of nonlinear algebraic equations

\[
-\frac{2}{3}A^2B^2kb - \frac{(480B^2bk^2 + 140kABb - 224a + 168c)AB}{105} = 0,
\]

\[
-\frac{(480Bbk^2 + 70Abk)A^2B}{105} - \frac{(240B^2bk^2 + 70kABb - 112a + 84c)A^2}{105} = 0.
\]

Solving it with respect to \( A \) and \( B \) gives

\[
A = \pm \frac{192a - 144c}{5\sqrt{-21b(4a - 3c)}}, \quad B = \pm \frac{1}{30} \frac{\sqrt{-21b(4a - 3c)}}{bk}
\]

with the parameters satisfying the conditions

\[ k \neq 0, \quad b(4a - 3c) < 0. \]

The corresponding solitary wave solution has the form

\[
u(x, y, z, t) = \pm \frac{192a - 144c}{5\sqrt{-21b(4a - 3c)}} \text{sech}^2 \left[ \frac{1}{30} \frac{\sqrt{-21b(4a - 3c)}}{b}(x + ay + bz - ct) \right].
\]
3.3. Case III

For the dark soliton wave solution

\[ u(\xi) = A \tanh^2(B\xi), \]

the stationary integral takes the form

\[ J = \frac{2A^2B(-120B^2bk^2 + 35kABb + 56a - 42c)}{105}. \]

Since the soliton amplitude and its inverse width satisfy the Eqs. (3.2), we arrive at the system of nonlinear algebraic equations

\[ \frac{4AB(-120B^2bk^2 + 35kABb + 56a - 42c)}{105} + \frac{2}{3}A^2B^2kb = 0, \]
\[ \frac{2A^2(-120B^2bk^2 + 35kABb + 56a - 42c)}{105} + \frac{2A^2B(-240Bbk^2 + 35Abk)}{105} = 0. \]

Solving it with respect to \( A \) and \( B \) gives

\[ A = \mp \frac{192a - 144c}{5\sqrt{-21b(4a - 3c)}}, \quad B = \pm \frac{1}{30} \frac{\sqrt{-21b(4a - 3c)}}{bk}, \]

with the parameters satisfying the conditions

\[ k \neq 0, \quad b(4a - 3c) < 0. \]

Thus the corresponding dark wave solution has the form

\[ u(x, y, z, t) = \mp \frac{192a - 144c}{5\sqrt{-21b(4a - 3c)}} \tanh^2 \left[ \pm \frac{1}{30} \frac{\sqrt{-21b(4a - 3c)}}{b}(x + ay + bz - ct) \right]. \]

4. Conclusion

We construct periodic wave solutions of \((3 + 1)\)-dimensional potential-Yu-Toda-Sasa-Fukuyama equation. Using the bilinear form of this equation, we chose ansatz as a combination of the exponential, sine and hyperbolic sine functions. Density graphs of certain solutions in 3D and 2D situations show different cross-kink waveforms and new multi wave and cross-kink wave solutions. Moreover, we employ SIVP in order to study the solitary, bright and dark soliton wave solutions of the YTSF equation. These results can find applications in nonlinear sciences where the \((3 + 1)\)-dimensional YTSF equation are used.

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References


