Variance Swap Pricing under Hybrid Jump Model

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Abstract. This paper investigates the pricing of discrete-sampled variance swaps driven by a generalised stochastic model taking into account stochastic volatility, stochastic interest rate and jump-diffusion process. The model includes various existing models as special cases, such as the CIR model, the Heston CIR model, and the multi-factor CIR model. The integral term arising from the jump-diffusion is dealt with by employing the characteristic function and Fourier convolution. By applying a high-dimensional generalised hybrid method, a semi-analytic solution is derived. The effects of stochastic interest rate, stochastic volatility and jump rate on variance swap price are investigated. It is shown that both the stochastic volatility and the jump rate have significant effects on the fair strike price, while the effect of the stochastic interest rate is minor and can be ignored.

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1. Introduction

Variance and volatility swaps are well-known financial derivatives which allow investors to trade the realised volatility against the current implied volatility. With variance swap and volatility swap, investors can hedge and speculate risk from the asset price movement. On future realised price variance, a variance swap is a forward contract with the following payoff function

\[ \text{Payoff} = L \left( \sigma_R^2 - K \right), \]

where \( L \) denotes the notional amount of the swap per annualised volatility point, \( \sigma_R \) is the realised volatility and \( K \) is the annualised strike price [6].

A long position in a variance swap gives benefit when the realised volatility is higher than the strike price, while a short position will benefit when the realised volatility is lower.
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than the strike price. The first volatility swap was traded in 1998 and has flourished recently. Fig. 1 shows the historical trading volume. Demeterfi et al. [16] listed two main reasons to trade volatility derivatives such as the variance swap and the volatility swap. Firstly, investigators may take a long-short position in a variance swap to hedge the risk exposure of trading volatility. Secondly, the variance swap provides them with a possibility to speculate in the spread of the realised volatility and the implied volatility.

1.1. Pricing strategies of variance swap

To price the variance swap, it is very important to understand the difference between the realised volatility and the implied volatility. The implied volatility represents the market price of volatility, it can be calculated as the inverse function of the Black-Sholes pricing formula. The realised variance, denoted by $\sigma^2_R$, is calculated from the historical data of option prices via the following equation [6]

$$\sigma^2_R = \frac{AF}{N} \sum_{i=0}^{N-1} \left( \frac{S_{i+1} - S_i}{S_i} \right)^2,$$

where $S_{i+1}$ denotes the underlying stock price at the $(i+1)$-th observation time, and $N$ denotes the number of observations. Let $T$ be the life time of the contract, $AF = N/T$ is the annualised factor converting this expression to an annualised variance, which is assumed.
to be within a wide range from 5 to 252 days. The expected value of payoff function is

\[ LE^Q\left[ \sigma^2_R - K \right], \]

where \( L \) denotes the notional amount of the swap per annualised volatility point and \( K \) is the annualised strike price. The price of a variance swap is the expected present value of the future payoff, which is zero under the assumption of zero entry cost. This is similar to that for the forward contract, where the initial value of the contract is set to be zero to ensure no arbitrage. In other words, under the risk-neutral world, the variance swap pricing problem becomes calculating the expected value of the realised variance

\[ K_{var} = E^Q \left[ \sigma^2_R \right]. \quad (1.2) \]

Many attempts have been made to evaluate the expectation of (1.2) both numerically and analytically. Several approaches have been applied to investigate (1.2) analytically. Carr and Madan [10] showed that a static position of call might replicate the price of volatility product and put options. Martin [32] proposed a simple variance swap by letting the denominator of the variance payoff be a forward price geometrically increased with time, and derived the analytic solution following the work of Carr and Madan [8]. Broadie and Jain [6] investigated the closed-form approximation of the fair strike price of a continuous sampling variance swap driven by both the Merton jump and the stochastic volatility process. Swishchuk et al. [36] developed a probability approach to determine the variance swap price based on the CIR process with non-central \( \chi \) square distribution ignoring the distribution of the payoff function. Lian and Zhu [39] applied the Fourier transform to price variance swaps with discrete sampling times and derived a closed-form solution of the Heston's two-factor stochastic volatility model [27]. In 2016, Cao and Lian [7] also obtained a semi-analytic solution of the variance swap pricing problem based on the Heston-CIR hybrid model via the generalised Fourier transform, where the stochastic interest rate was investigated. More generalised stochastic volatility models are applied in pricing discrete sampled variance swap by Bernard and Cui [3]. In Filipovic and Gourier's paper, the authors introduce a class of term structure models for variance swap with the multivariate state characterizing by a quadratic diffusion function. The proposed model is proved to be a good fit to the actual data and applied in studying the optimal portfolio selection problem [22]. Recently, a variance swap with regime switching has also been studied by Elliott et al. [19]. Apart from extending the underlying driven model, some literature aims at deriving explicit solutions for more generalised variance type swaps, including the gamma swap, the corridor variance swap and the conditional variance swap [15,38]. Numerical algorithms have also been applied to study the option pricing problem of the variance swap [31,36]. In 2001, Little and Pant [31] applied the finite difference method (FDM) to solve the variance swap problem based on the constant volatility assumption [4] in which a two-dimensional (2D) problem is reduced to a one-dimensional partial differential equation system. The numerical method can be applied to deal with different models regardless of the volatility types. However, it is not applicable in high dimensional PDE due to the low efficiency.
From (1.1), the realised variance $\sigma_R^2$ varies along with different assumptions of the underlying asset process. Applying the basic Black-Scholes model is meaningless in studying volatility type derivative due to the constant volatility assumption. Thus, extended models have been applied in pricing the variance swap.

1.2. Introduction of extensive underlying asset models

Due to the well-known phenomenon of volatility skew exhibited in option pricing processes [29], existing literature appears to extend and relax the assumptions of the classical Black-Scholes model. Among all those extensions, the relaxation of fixed volatility and fixed interest rate have received much attention. There are two significant types of volatility relation models, the local volatility model and the stochastic volatility model. The time-dependent local volatility is absorbed to replace the constant volatility in Dupire’s Model [18]. In 2011, Heston [27] generalised the Black-Scholes model to a two-dimensional stochastic model by describing the volatility as a new stochastic process. However, recent empirical study shows that single-factor Heston models are overly restrictive and multi-factor stochastic models are necessary in order to obtain a more accurate result [24,27]. Heston et al. [12] investigated how much the multi-factor stochastic volatility model could improve the option pricing and analyzed the fluctuations of the slope of the volatility process. The concept of timescale was firstly proposed by Fouque et al. [13,24,34] with the volatility process being driven by a combination of fast-scale and slow-scale stochastic processes, and the concept is distinguished by the frequency of observed data in 2005 [11].

Besides the relaxation of volatility, many researchers have extensively studied models with stochastic interest rate and their applications. The stochastic interest rate was introduced by Hull and White [28], and a closed-form solution of the Black-Scholes-Hull-White model was derived by Brigo and Mercurio for European Style options [5]. However, even though the stochastic interest rate model can describe the fluctuation of the option price and enhance the long-term accuracy, it cannot simulate the skew effect or the ‘option smile’. To overcome the drawbacks, the stochastic interest rate model has been used together with the stochastic volatility model [1,25]. The Heston-Hull-White model and Heston-CIR model were studied by Grzelak and Oosterlee [26], and an affine structure solution was derived by applying the Fourier transform. Different from the literature mentioned above, the correlation effects were also considered in Grzelak and Oosterlee’s work. Recently, Kim et al. [29] studied the multi-scale volatility model and the stochastic interest rate model by applying the technique of asymptotic approximation. They derived the leading term and the first-order correction term for the European type options. However, the models above do not take the jump feature of the stock process into consideration, which is widely observed in the Financial market.

An alternative approach to capture the leptokurtic features is the jump-diffusion model [30], which is applied to sketch an unexpected abrupt change of the stock price. In 1976, Merton [33] assumed that the asset return process is generated by a Brownian motion together with a jump-diffusion process, and the jump process is driven by a compound Poisson process. In Merton’s Model, the jump size is assumed to be normally distributed. In
2002, Kou [30] assumed a double exponential distribution for the jump size for simplicity in computation. The benefit of Kou’s jump model is its operability. Many recent works [2, 9, 35] incorporate both the stochastic volatility model and the jump-diffusion model into the underlying asset process. Bates [2] introduced the stochastic volatility model with jumps (SVJ) by considering both jump-diffusion and stochastic volatility in the dynamics of the underlying stock process. In 2000, the SVJ model was then extended by Duffie et al. [17] to incorporate the jump term in the return process and the stochastic volatility process. However, according to the empirical study, the inclusion of jumps does not improve the stochastic volatility process markedly [20].

The stochastic volatility model with jump-diffusion (SVJ) was firstly applied by Broadie and Jain [6] to price the variance swap. In their work, analytical solutions of both the discrete and continuous variance swap models are obtained to describe the distribution of the stochastic return of the underlying asset price process. The effects of the stochastic volatility term and the jump-diffusion term are both shown to be significant and cannot be ignored. The approximated discrete solution was proved to be convergent to the continuous solution if the observation frequency approaches infinity. However, the disadvantage of their model is that the expression of the analytical solution is overly complicated and the error will be enlarged if the observation frequency becomes smaller. Different from their research, the model we apply in this paper is more general by further taking into consideration of the stochastic interest rate and the multi-factor stochastic volatility. To obtain the analytical solution of the discrete model, we apply the generalised Fourier transform to solve the resulted partial integral differential equation (PIDE). More importantly, the expression of our analytical solution is more straightforward and more concise.

This paper proposes a variance swap pricing model taking into account the stochastic interest rate, multi-factor stochastic volatility process, and the jump-diffusion process. A semi-closed-form solution is derived for a full correlated case. Compared to the literature mentioned above, our research has three contributions. Firstly, we consider a more general model. With proper selection of parameters, our proposed model includes various existing models as special cases. Besides, we take into consideration not only the jump-diffusion effects but also the stochastic interest rate and the multi-factor stochastic volatility process in the model. Different from Brodie and Jain’s work [6], a semi-analytic solution for the price of the discrete sampling variance swap is derived by relating the associated partial integral differential equation with the generalised Fourier transform. Furthermore, the inclusion of multi-factor processes results in a high dimensional partial differential equation (PDE). We successfully reduce the dimension of the PDE by embedding the problem under the Little and Pant framework [31]. The skew effects of the correlation between different volatility processes are also investigated. To be more specific, the payoff function of the variance swap is treated as a function of the current stock price and the previous stock price, with the former following a stochastic process, while the latter being determined at the current time. In this case, the n-dimensional PDE is reduced to an n − 1 dimensional PDE in two different periods. We then apply the generalised Fourier transform based on Cox and Ross’s work [14] to solve the first stage PDE system. In Cao’s partial correlated model [7], a semi-closed solution is derived by the assumption of an affine structure. However, when
the model is fully correlated, a non-affine item is included in our model, and in this case, we approximate the expectation of the non-affine term utilising the result in Grzelak and Oosterlee’s work [26].

The rest of the paper is organised as follows. Section 2 describes the general model and demonstrates the change of measure under the risk-neutral assumption. In Section 3, a semi-analytic solution is derived by applying the generalised Fourier transform. Numerical results are given in Section 4, followed by a conclusion in Section 5.

2. Model Setup

The price of the stock is driven by the dynamics of the jump-diffusion model

\[ dS = \mu S dt + \sigma_R(y_i)S dw_s + S d\tilde{J}^s, \] (2.1)

where \( dw_s \) is a Wiener process and \( d\tilde{J}^s = dJ^s - \lambda (E(e^\eta - 1)) \) is a compensated compound Poisson process. \( J^s = \sum_{i=1}^{N_t} (e^\eta - 1) \) is assumed to be an exponential Levy process, and \( N_t \) is a Poisson process with intensity \( \lambda \) and exponential jump size of \( e^\eta \). It should be addressed here that the return rate \( \mu \) of the stock price process is not necessarily equal to the risk-free rate \( r \) before the risk-neutral adjustment. By a careful selection of the market price of the volatility and the technique of measurement change, it transforms to \( r \) under the risk-neutral assumption.

Let \( f(y_i) \) be the volatility of the stock which is a function of the stochastic process \( y_i \) that is assumed to be driven by the following stochastic differential equation (SDE):

\[ dy_i = \alpha_i(y_i)dt + \beta_i(y_i)dw_{y_i}^i, \quad i = 1, \ldots, M, \] (2.2)

where \( M \) is the number of the stochastic volatility processes, and \( \alpha_i(y_i) \) is the drift part of the \( i \)-th stochastic volatility, \( \beta_i(y_i) \) is the volatility part of the stochastic volatility, and \( dw_{y_i}^i \) is the Wiener processes describing the random noise in volatility.

The interest rate is driven by the following stochastic process

\[ dr = m(r)dt + n(r)dw_r, \] (2.3)

where \( m(r) \) is the drift part of the interest rate process, \( n(r) \) is the volatility part of the stochastic interest rate, and \( dw_r \) is the Wiener processes describing the random noise in interest rate. In addition, \( (w_s, w_{y_1}, \ldots, w_{y_n}, w_r) \) are correlated via the following correlation matrix

\[ \text{Cor} = \begin{bmatrix} 1 & \rho_{y_1} & \cdots & \rho_{y_n} & \rho_{sr} \\ \rho_{y_1} & 1 & \cdots & \rho_{y_1y_n} & \rho_{y_1r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{y_n} & \rho_{y_ny_1} & \cdots & 1 & \rho_{y_nr} \\ \rho_{sr} & \rho_{y_1r} & \cdots & \rho_{y_nr} & 1 \end{bmatrix} \] (2.4)

with

\[ |\rho_{y_i}| < 1, \quad |\rho_{sr}| < 1, \quad |\rho_{y_1y_i}| < 1, \quad |\rho_{y_1r}| < 1, \quad |\rho_{y_nr}| < 1, \quad |\rho_{sr}| < 1, \]

whereas \( \text{Cor} \) is a positive definite matrix.
2.1. Model dynamics under the T-forward measure

In this subsection, we first conduct a measure change from the real-world probability measure \( \mathbb{P} \) to the equivalent measure of \( \mathbb{Q} \). Since the interest rate is no longer constant, a further measure change should then be implemented from \( \mathbb{Q} \) to the T-forward measure \( \mathbb{Q}^T \).

Let \( \Upsilon = \begin{bmatrix} d w_s \\ d w_y \\ d w_r \end{bmatrix} \) with \( w_y = [w_{y_1}, w_{y_2}, \ldots, w_{y_n}]^T \). By implementing the numeraire change from measure \( \mathbb{P} \) to measure \( \mathbb{Q} \), we obtain the risk neutral vector satisfying

\[
\begin{bmatrix} d \hat{w}_s \\ d \hat{w}_y \\ d \hat{w}_r \end{bmatrix} = \begin{bmatrix} d w_s \\ d w_y \\ d w_r \end{bmatrix} + \begin{bmatrix} (\mu - r)/f(y_1) \\ \Gamma(t) \\ \gamma_r(t) \end{bmatrix} dt,
\]
or

\[
\hat{\Upsilon} = \Upsilon + \Pi dt,
\]
where \( \Gamma(t) = [\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)] \) is the market price of risk from stochastic volatility and \( \gamma_r(t) \) is the market price of risk from the stochastic interest rate.

Under the risk-neutral measure and the above adjustment, Eqs. (2.1)-(2.3) can be rewritten in matrix form by

\[
D = U^P dt + \Sigma \Upsilon = U^Q dt + \Sigma \hat{\Upsilon},
\]
where \( D = [dS/S, dy, dr]^T \). \( U^P = [\mu, a(y), m(r)]^T \) is the drift part under the measure \( \mathbb{P} \). \( U^Q = U^P - \Sigma \Pi \) is the drift under the risk neutral measure \( \mathbb{Q} \) and \( \Sigma \) is an \( n \times n \) matrix representing the volatility part,

\[
\Sigma = \begin{bmatrix} f(y_1) & * & * \\ * & B(y) & * \\ * & * & n(r) \end{bmatrix}
\]

with

\[
B(y) = \begin{bmatrix} \beta_1(y_1) & * & * \\ \vdots & \ddots & \vdots \\ * & * & \beta_n(y_n) \end{bmatrix}.
\]

Now we change the numeraires form \( \mathbb{Q} \) to the T-forward measure \( \mathbb{Q}^T \). Considering \( \Upsilon^* = [d w_s^*, d w_y^*, d w_r^*]^T \), we denote the orthogonal vector such that

\[
\hat{\Upsilon} = L \Upsilon^*,
\]
in which \( L \) is a lower triangle matrix and \( Cor = LL^T \):

\[
L = \begin{bmatrix} 1 & * & \cdots & * & \cdots & * \\ \rho_{s y_1} & \tilde{\rho}_{y_1} & \cdots & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \rho_{s y_1} & \tilde{\rho}_{y_1 y_1} & \cdots & \tilde{\rho}_{y_i} & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho_{sr} & \tilde{\rho}_{y_1 r} & \cdots & \tilde{\rho}_{y_i r} & \cdots & \tilde{\rho}_r \end{bmatrix},
\]
where

\[
\tilde{\rho}_{yi} = \left(1 - \rho_{yi}^2\right)^{1/2}, \quad \tilde{\rho}_{yi} = \left(1 - \rho_{yi}^2 - \sum_{k=1}^{i-1} \tilde{\rho}_{yk}^2\right)^{1/2},
\]

\[
\tilde{\rho}_i = \left(1 - \rho_{si}^2 - \sum_{k=1}^{n} \tilde{\rho}_{yi}^2\right)^{1/2},
\]

\[
\tilde{\rho}_{yi} = \frac{\rho_{yi} - \rho_{yi}^2 \rho_{yi} - \sum_{k=1}^{i-1} \tilde{\rho}_{yk} \tilde{\rho}_{yk}}{\tilde{\rho}_{yi}}, \quad i > j,
\]

\[
\tilde{\rho}_{yi} = \frac{\rho_{yi} - \rho_{yi}^2 \rho_{yi} - \sum_{k=1}^{i-1} \tilde{\rho}_{yk} \rho_{yi} \rho_{yi}}{\tilde{\rho}_i}.
\]

It is noted that the numeraire under \( \mathbb{Q} \) is \( e^{\int_0^t r(s)ds} \) and the numeraire under the \( T \)-forward measure \( \mathbb{Q}^T \) is \( P(t, T) = A(t, T)e^{-B(t, T)n(r)} \). \( A(t, T) \) and \( B(t, T) \) can be solved from the following PDE:

\[
P_t + m(r)P_r + \frac{1}{2}n^2(r)P_{rr} - rP = 0,
\]

where \( m(r) \) and \( n(r) \) are shown in (2.3). For more details, we refer the reader to [5]. Thus, the drift part \( U^T \) can be obtained by [7]

\[
U^{Q^T} = U^Q + \Sigma LL^T \Sigma Q^T
\]

with \( \Sigma Q^T = [0, 0, \ldots, -B(t, T)n(r)]^T \). Therefore, the SDE can be rewritten by the following forms under the measure \( Q^T \),

\[
dS = (r - \rho_{si}B(t, T)n(r) + \lambda(1 - E(e^n)))Sdt + f(y_i)Sdw^*_i + SdJ^i,
\]

\[
dy_i = (\alpha_i(y_i) - \lambda_i(y_i, r)\beta_i(y_i))dt + \beta_i(y_i)dw^*_i,
\]

\[
dr = (m(r) - (\gamma_i(r) + B(t, T)n(r))n(r))dt + n(r)dw^*_r
\]

with \( \lambda_i(y_i, r) = \gamma_i(y_i) + \rho_{yi}B(t, T)n(r), i = 1, \ldots, n \) and \( \gamma_i(r) \) denoting the market price of risk (risk premium). \( [dw^*_i, dw^*_j, dw^*_r] \) is mutually independent under the measure of \( Q^T \).

### 2.2. Formulation of the two-stage processes

The variance swap value can be evaluated as the expected present value of the payoff function under the \( T \)-forward measure \( Q^T \) [6], namely

\[
\tilde{V}(t) = E^{Q^T} \left[ e^{-\int_0^t r(s)ds} L \left( \sigma_R^2 - K \right) \big| \mathcal{F}_t \right],
\]

where \( \mathcal{F}_t \) is the filtration up to time \( t \). The value of \( \tilde{V}(t) \) is zero at the initial time \( t = 0 \) under the assumption of zero entry cost [39]. Therefore, the variance swap pricing problem
becomes the calculation of the expected value of the realised variance or fair strike price  

\[ K = E^Q \left[ \sigma^2 \right] \]

in the risk neutral world.

For the reason that our general model results in a high dimensional problem, we apply the dimension reduction technique by introducing a new variable \( I_t \), driven by the underlying process \[31\],

\[ I_t = \int_0^t \delta(t_{i-1} - \tau)S_{\tau}d\tau, \tag{2.6} \]

where \( \delta \) is the Dirac delta function, and \( S_{\tau} \) is driven by \[2.1\]. \( I_t \) is only related with the value of \( S_t \) at time \( t_{i-1} \). Note that \( I_t = 0 \) if \( t < t_{i-1} \), and \( I_t = S_{t_{i-1}} \) if \( t \geq t_{i-1} \).

The fair strike price \( K_{\text{var}} \) can be calculated by taking the expectation of the payoff function, namely

\[ K_{\text{var}} = E^Q \left[ \left( \frac{S_t}{I_t} - 1 \right)^2 \right] = E^Q \left[ E^Q \left[ \left( \frac{S_t}{I_t} - 1 \right)^2 \right] | \mathcal{F}_{T-\Delta t} \right] | \mathcal{F}_t \] \tag{2.7}

by tower rule.

According to the Feynman-Kac theorem, the inner expectation and outer expectation can be calculated separately by the two-stage processes as shown in Propositions 2.1 and 2.2.

**Proposition 2.1.** The inner expectation of \( (2.7) \) can be solved from the following PIDE

\[ U_t + \mathcal{L}U + \lambda \int_R \left[ (U(Se^\eta) - U(S)) \right] \Gamma(d\eta) = 0 \tag{2.8} \]

with the continuous operator

\[ \mathcal{L} = \left[ r - \rho_s B(t, T)n(r) + \lambda (1 - E(e^\eta)) \right] S \frac{\partial}{\partial S} + \frac{1}{2} f^2(y_t)S^2 \frac{\partial^2}{\partial S^2} \]

\[ + \left[ m(r) - (\gamma_t(r) + B(t, T)n(r))n(r) \right] \frac{\partial}{\partial r} + \frac{1}{2} n^2(r) \frac{\partial^2}{\partial r^2} + \rho_s f(y_t)n(r)S \frac{\partial^2}{\partial S \partial r} \]

\[ + \sum_{i=1}^n \sum_{j=1}^n \left( (\alpha_i(y_t) - \lambda(y_i, r)\beta_j(y_j)) \partial_{y_i} + \frac{1}{2} \rho_{y_i, y_j} \beta_j(y_j) \beta_j(y_j) \partial_{y_i, y_j} \right) \]

\[ + \frac{1}{2} \rho_{sy_i, y_j} \beta_j(y_j)S \partial_{y_i, y_j} + \frac{1}{2} \rho_{sy_i, y_j} \beta_j(y_j)S \partial_{y_i, y_j} \] \tag{2.9}

subject to the terminal condition

\[ U(t, x, y, z) = \left( \frac{e^x}{T} - 1 \right)^2, \quad T - \Delta t < t \leq T \tag{2.10} \]

with \( y = [y_i]_{i=1}^n \).

**Remark 2.1.** We use Proposition 2.1 in the first stage process, from which the inner expectation in \( (2.7) \) can be obtained by solving PIDE \( (2.8) \). The \( \Delta t \) in \( (2.10) \) denotes the time step size. We assume that \( T = \bar{t} \Delta t, \bar{t} = 1, \ldots, \bar{N}, \bar{N} = AF \times T \) and \( AF \) ranges from 1 to 252 days. As there is no evidence showing the existence of a correlation between the stochastic interest process and the stochastic volatility process, we assume that \( \rho_{y_i, r} = 0 \) in \( (2.4) \).
Proposition 2.2. The outer expectation of (2.7) can be solved from the following PIDE

\[ U_t + \mathcal{L}U + \lambda \int_R \left[ u(Se^v) - u(x) \right] \Gamma(d\eta) = 0 \quad (2.11) \]

subject to the terminal condition

\[ \lim_{t \uparrow T-\Delta t} U(t, S, y, r) = \lim_{t \uparrow T-\Delta t} U(t, S, y, r), \quad 0 \leq t < T - \Delta t, \quad (2.12) \]

and the continuous operator \( \mathcal{L} \) is the same as (2.9).

Remark 2.2. We use Proposition 2.2 in the second stage process which calculates the outer expectation of (2.7). The discontinuity generated at \( t = T - \Delta t \) due to \( I_t \) in (2.6) can be dealt with by taking the limit as in (2.12). In this case, the solution of the first stage process can be set as the initial condition of the second stage process.

3. Algorithm to Price Variance Swap

In this section, we apply the generalised Fourier transform to derive a semi-analytic solution of the problem. The drifts in (2.2) are assumed to be

\[ \alpha_i(y_i) = a_i(m_i - y_i), \]

where \( a_i \) and \( m_i \) denote the mean reversion rate and the value for the \( i \)-th stochastic volatility process, and the volatility of volatility are assumed to be \( \beta_i(y_i) = b_i \sqrt{y_i} \).

The stochastic interest rate process is assumed to be a CIR process with

\[ m = k(\theta - r(t)), \]

where \( k \) and \( \theta \) denote respectively the mean reversion rate and the value of the interest process, and the volatility of the interest rate process is assumed to be \( n(r) = \eta \sqrt{r(t)} \). Thus, using the result in [26], \( B(t, T) \) has a specific form

\[ B(t, T) = \frac{2 \left( e^{(T-t)\sqrt{k^2+2\eta^2}} - 1 \right)}{2 \sqrt{k^2+2\eta^2} + (k+\sqrt{k^2+2\eta^2}) \left( e^{(T-t)\sqrt{k^2+2\eta^2}} - 1 \right)}. \]

Substituting \( y_i(y_i) = \lambda_i \sqrt{y_i}/b_i \) and \( y_r(r) = \lambda_r \sqrt{r}/\eta \) in (2.5), we have

\[ dS = (r - \rho_s B(t, T)n(r) + \lambda(1 - E(e^n)))dt + \sum_{i=1}^n \sqrt{y_i} Sdw_s^i + SdJ^i, \quad (3.2) \]

\[ dy_i = a_i^*(m_i^* - y_i)dt + b_i \sqrt{y_i} dw_{y_i}^i, \quad i = 1, \ldots, n, \quad (3.3) \]

\[ dr = (k^*(\theta^* - r) - B(t, T)\eta r)dt + \eta \sqrt{r} dw_r, \quad (3.4) \]

where \( a_i^* = a_i + \lambda_i, m_i^* = a_i m_i/(a_i + \lambda_i), i = 1, \ldots, n \) and \( k^* = k + \lambda_r, \theta^* = k \theta/(r + \lambda_r) \). When \( n = 1 \) and \( \lambda = 0 \), the model reduces to the model of Cao and Lian [7].
3.1. Stage I calculation: inner expectation

The inner expectation can be obtained by solving the PIDE shown in Proposition 2.1. Let \( \tau = T - \Delta t \), \( x = \ln S \), and combining Eqs. (3.2)-(3.4) with Eq. (2.8), we obtain the PIDE

\[
U_t - \mathcal{L}U - \lambda \mathcal{I} [u(x + \eta) - u(x)] = 0
\]

with the continuous operator

\[
\mathcal{L} = \left[ r - \rho_r B(t, T)n(r) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{y_i} \sqrt{y_j} \eta \left( 1 - E^{Q^r}(e^z) \right) \right] \frac{\partial}{\partial x} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{y_i} \sqrt{y_j} \frac{\partial^2}{\partial x^2} + \left[ k^*(\theta^* - r) - B(t, 0) \eta^2 r \right] \frac{\partial}{\partial r} + \frac{1}{2} \eta^2 r \frac{\partial^2}{\partial r^2} + \rho_{xx} \eta f(y) \frac{\partial^2}{\partial x \partial r} + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_i^* (m_i^* - y_i) \right] \partial_{y_i} + \frac{1}{2} \rho_{yy} b_i^2 \sqrt{y_i} \sqrt{y_j} \partial_{y_{ij}} + \frac{1}{2} \rho_{xy} b_i \sqrt{y_i} \sqrt{y_j} x \partial_{y_{ij}} + \frac{1}{2} \sum_{i=1}^{n} \rho_{y_i} \eta \sqrt{y_i} r \partial_{y_{ir}}
\]

subject to the initial condition

\[
U(0, x, y, z) = \left( \frac{e^x}{T} - 1 \right)^2, \quad 0 \leq \tau < \Delta t.
\]

Let

\[
V(\tau, w, y, z) = \mathcal{F}(U) = \int_{R} U(\tau, x, y, z) e^{-jwx} dx
\]

be the Fourier transform of \( U(\tau, w, y, z) \), and let \( j \) denote the imaginary number and \( j^2 = -1 \), we then obtain

\[
V_t = \left[ r - \rho_t B(t, T)n(r) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{y_i} \sqrt{y_j} \eta \left( 1 - E^{Q^r}(e^z) \right) \right] (-jw) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{y_i} \sqrt{y_j} (-jw)^2 + \left[ k^*(\theta^* - r) - B(t, 0) \eta^2 r \right] V_t + \frac{1}{2} \eta^2 r V_{rr} + \rho_{xx} \eta f(y) (-jw) V_t + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_i^* (m_i^* - y_i) \right] V_{y_i} + \frac{1}{2} \rho_{yy} b_i b_j \sqrt{y_i} \sqrt{y_j} V_{y_{ij}}
\]
V(0, w, y, r) = \mathbb{F}(U(0, w, y, r)) = \mathbb{F}\left(\left(\frac{e^x}{I} - 1\right)^2\right).

Here we use the result that
\[
\mathbb{F}\left(\int_R u(x + \eta) \Gamma(d\eta)\right) = \phi_\eta(w)V,
\]
where \(\phi_\eta(w) = \int_R e^{jw\eta} \Gamma(d\eta)\) denotes the characteristic function of the underlying process of the jump size, and \(\Gamma(\cdot)\) denotes the probability measure of \(\eta\). The form of \(\Gamma(\cdot)\) depends on the underlying distribution of \(\eta\). Table 1 presents the commonly used jump model including Merton’s jump model [33] and Kou’s double exponential model [30]. Merton’s model assumes that the jump size follows a normal distribution, while the jump size of Kou’s model is assumed to be a double exponential distribution. The Fourier transform of the jump model is specified in Appendix A.

<table>
<thead>
<tr>
<th>Model</th>
<th>(\Gamma(d\eta))</th>
<th>(\phi_\eta(w))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merton</td>
<td>(\exp(-(\eta - \mu)^2)/\sqrt{2\pi}\delta)</td>
<td>(\exp\left(j\mu w - \frac{w^2}{2}\delta^2\right))</td>
</tr>
<tr>
<td>Kou</td>
<td>(p\lambda_1 \exp(-\lambda_1 \eta)\delta_{\eta&gt;0} + (1-p)\lambda_2 \exp(\lambda_2 \eta)\delta_{\eta&lt;0}d\eta)</td>
<td>(\frac{p\lambda_1}{\lambda_1 - jw} + \frac{(1-p)\lambda_2}{\lambda_2 + jw})</td>
</tr>
</tbody>
</table>

By assuming that the solution has an affine structure and following the procedure of Heston [27], the solution is found to have the form
\[
V(\tau, w, y, r) = \exp\left(C(w, \tau) + D(w, \tau)y^T + R(w, \tau)r\right)V_0
\]
with \(D = [D_i]_{i=1}^n\), and \(V_0 = V(0, w, y, r)\).

Then by substituting Eq. (3.6) into Eq. (3.5), we obtain the following ODEs:
\[
\frac{\partial D_i}{\partial \tau} = -\frac{1}{2}(jw + w^2) + \frac{1}{2}b_i^2 D_i^2 + (\rho_{xy}\sigma_i b_i jw - \alpha_i^* m_i^*) D_i, \quad i = 1, \ldots, n,
\]
\[
\frac{\partial R}{\partial \tau} = wj + \frac{1}{2}\eta^2 R^2 - (k^* + B(T - \tau, 0)\eta^2)R,
\]
\[
\frac{\partial C}{\partial \tau} = \sum_{i=1}^n \alpha_i^* m_i^* D_i + k^* \theta^* R + \lambda(1 - E(e^x))(jw) - \lambda
\]
\[ + \rho_{xy} \eta \sqrt{\tau} (-jw) \sum_{i=1}^{2} \sqrt{y_i R} + \lambda \phi_0 (w) + \frac{1}{2} \sum_{i=1}^{n} \rho_{xy r} b_i \eta \sqrt{\gamma_i R V_{y i r}} \]

\[ + \sum_{i=1}^{n} \sum_{j \neq i} \left\{ - \sqrt{y_i y_j} \left( jw + \frac{1}{2} w^2 \right) + b_i \rho_{xy} \sqrt{y_i y_j} (jw) D_i \right\} + \frac{1}{2} \rho_{xy} b_i b_j \sqrt{y_i y_j} \right\} \]

subject to initial conditions \( C(w, 0) = 0, D_i(w, 0) = 0, \) and \( R(w, 0) = 0. \) The \( F \) can be solved numerically by using MATLAB, while \( D_i \) can be solved analytically to yield

\[ D_i(\tau) = \frac{A_i + B_i}{b_i^{\frac{3}{2}}} \frac{1 - e^{R_i \tau}}{1 - g_i e^{R_i \tau}}, \]  

where \( A_i = -a_i^* (jw \rho_{xy} b_i^* - m_i^*), B_i = \sqrt{A_i^2 + b_i^{* 2} a_i^* (w^2 + jw)}, \) and \( g_i = (A_i + B_i)/(A_i - B_i). \)

The detail of \( C \) is given in the next subsection.

In order to obtain the solution of \( U(\tau, x, y, r) \), we perform the inverse Fourier transform and obtain

\[ U(\tau, x, y, r) = F^{-1} [V(\tau, w, y, r)] = F^{-1} \left[ e^{C(w, \tau) + D(w, \tau)y^2 + R(w, \tau)r} \right] U_0 \]

with \( U_0 = U(0, x, y, r) \) satisfying

\[ U_0 = F^{-1}(V_0) = F^{-1} \left\{ \left. \left[ \frac{e^x}{T} - 1 \right] \right| \right\}. \]

Based on the generalised Fourier transform

\[ F \left[ e^{imx} \right] = \delta_m (w) \]

with \( \delta_m (w) \) satisfying

\[ \int \delta_m g(x) dx = g(m), \]

we obtain

\[ U(x, y, \tau) = \int e^{C(w, \tau) + D(w, \tau)y^2 + R(w, \tau)r} \left[ \frac{\delta_{-2j}(w)}{I^2} - \frac{\delta_{-j}(w)}{I} + \delta_0 (w) \right] e^{jwx} dw \]

\[ = \frac{e^{2x}}{I^2} \frac{\hat{C}(\tau) + \hat{D}(\tau)y^2 + \hat{R}(\tau)r}{e^{\hat{C}(\tau) + \hat{D}(\tau) + \hat{R}(\tau)r}} + 1, \]

where \( \hat{C}(\tau), \hat{D}(\tau) \) and \( \hat{R}(\tau) \) denote respectively \( C(-2j, \tau), D(-2j, \tau), \) and \( R(-2j, \tau) \), whereas \( \hat{C}(\tau) \) and \( \hat{R}(\tau) \) are notations of \( C(-i, \tau) \) and \( D(-i, \tau). \)
3.2. Stage II calculation: outer expectation

According to Proposition 2.2, the second stage process can be calculated by solving the PIDE (2.11) subject to the terminal condition (2.12). By taking the limit, we have \( I = e^x \). Thus, the initial condition at time \( \Delta t \) reduces to

\[
F(\Delta t, y, r) = e^{C(\Delta t) + \hat{h}(\Delta t)y^2 + \hat{h}^2(\Delta t)r^2} - 2e^{C(\Delta t) + \hat{h}^2(\Delta t)r^2} + 1. \quad (3.9)
\]

However, in this paper, we obtain the outer expectation of (2.7) by taking the expectation value of \( F(t, y, r) \) in (3.9),

\[
E^Q(F(\tau, y, r)) = e^{C(\Delta t)} \prod_{i=1}^n g_i \hat{h} - 2e^{C(\Delta t)} \hat{h} + 1, \quad (3.10)
\]

where \( g_i = g_i(-2j, \tau, y_i) = E^Q(e^{\hat{h}(\Delta t)y_i}), \hat{h} = h(-2j, \tau, r) = E^Q(e^{\hat{h}(\Delta t)r}), \) and \( \hat{h} = h(-j, \tau, r) = E^Q(e^{\hat{h}(\Delta t)r}) \). According to the Feynman-Kac formula, \( f_i \) can be obtained by solving the following PDE

\[
\frac{\partial g_i}{\partial \tau} = \frac{1}{2} b_i^2 \frac{\partial^2 g_i}{\partial y_i^2} + \alpha_i^* (m_i - y_i) \frac{\partial g_i}{\partial y}, \quad (3.11)
\]

\[
g_i(y_i, \Delta t) = e^{h_i(\Delta t)y_i}. \quad (3.12)
\]

Similarly, we assume that the solution of (3.11) has an affine form of

\[
g_i = e^{L_i + H_i y_i}. \quad (3.13)
\]

Substituting (3.13) into (3.11), we obtain the following ODEs

\[
\frac{\partial L_i}{\partial t} = \alpha_i^* m_i^* H_i,
\]

\[
\frac{\partial H_i}{\partial t} = -\alpha_i^* + \frac{1}{2} b_i^2 y_i^2 H_i^2
\]

subject to the initial conditions \( L_i(\hat{D}_i(\Delta t), \Delta t) = 0 \) and \( H_i(\hat{D}_i(\Delta t), \Delta t) = \hat{D}_i(\Delta t) \). By simple derivation, we obtain

\[
H_i = \frac{2\alpha_i^*}{b_i^2} \frac{e^{-\alpha_i^* \tau}}{e^{-\alpha_i^* \tau} - c_0},
\]

\[
L_i = \frac{-2\alpha_i^* m_i^*}{b_i^2} \ln \left( 1 - \frac{e^{-\alpha_i^* \tau}}{c_0} \right)
\]

with \( c_0 = 1 - 2\alpha_i^*/(b_i^2 \hat{D}_i(\Delta t)) \).

Similarly, \( \hat{h} = h(-2j, \Delta t) = E^Q(e^{\hat{h}(\Delta t)}) \) and \( \hat{h} = h(-j, \Delta t) = E^Q(e^{\hat{h}(\Delta t)}) \). According to the Feynman-Kac theorem, \( h \) can be solved from the following PDE,

\[
\frac{\partial h}{\partial \tau} = \frac{1}{2} \eta^2 \frac{\partial^2 h}{\partial r^2} - \left\{ k^*(\theta^* - r) - B(T - \tau, 0) \eta^2 r \right\} \frac{\partial h}{\partial r},
\]

\[
h(w, \Delta t, r) = e^{\hat{h}(w, \Delta t)r},
\]
which can be calculated based on the assumption of $h = e^{M+N}$. By simple derivation, we obtain

$$N = \frac{2k^*}{\eta^2} \frac{e^{-\tau}}{e^{-\theta^*\tau} - c_1},$$

$$M = \frac{-2\theta^*m^*}{\eta^2} \ln \left( 1 - \frac{e^{-m^*\tau}}{c_1} \right)$$

with $c_1 = 1 - 2\theta^*/(\eta^2 R(w, \Delta t))$.

According to (3.7), $C$ satisfies the ODE

$$\frac{\partial C}{\partial \tau} = \sum_{i=1}^{n} \alpha_i^* m^*_i D_i + k^* \theta^* E + \lambda(1 - E(e^{\tau})) (j w) - \lambda + \lambda \phi_\eta(w)$$

$$+ \sum_{i=1}^{n} \sum_{j \neq i}^{n} E^t_0 \left\{ -\sqrt{y_i y_j} \left( j w + \frac{1}{2} w^2 \right) + b_j \rho_{\eta y_j} \sqrt{y_i y_j} (j w) D_i + \rho_{y_i} \eta \sqrt{r} (-j w) \sqrt{y_i R} \right\}$$

$$= \sum_{i=1}^{n} \alpha_i^* m^*_i D_i + k^* \theta^* R + \lambda(1 - E(e^{\tau})) (j w) - \lambda + \lambda \phi_\eta(w) + \sum_{i=1}^{n} \rho_{y_i} \eta b_i E^t_0 \left\{ \sqrt{r y_i} \right\} D_i R$$

$$+ \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left\{ -\left( j w + \frac{1}{2} w^2 \right) E^Q \left( \sqrt{y_i y_j} \right) + (j w) b_i \rho_{\eta y_j} E^Q \left( \sqrt{y_i y_j} \right) D_i \right\}$$

$$+ \rho_{y_i} \eta (j w) E^Q \left( \sqrt{r y_i} \right) R \right\},$$

which can be solved numerically by MATLAB. Based on the independence property, we obtain

$$E^Q \left( \sqrt{y_i y_j} \right) = E^Q \left( \sqrt{y_i} \right) E^Q \left( \sqrt{y_j} \right) + \text{Cov} \left( \sqrt{y_i}, \sqrt{y_j} \right)$$

with $E^Q \left( \sqrt{y_i(t)} \right)$ and $\text{Cov}(y_i, y_j)$ determined by

$$E^Q \left( \sqrt{y_i(t)} \right) = \sqrt{c_i(d_i + \lambda_i - 1) + \frac{c_i d_i}{2(d_i + \lambda_i)}},$$

$$\text{Cov} \left( \sqrt{y_i(t)}, \sqrt{y_j(t)} \right) = \rho_{y_i y_j} \left( c_i - \frac{c_i d_i}{2(d_i + \lambda_i)} \right) \left( c_j - \frac{c_j d_j}{2(d_j + \lambda_j)} \right),$$

where

$$c_i(t) = \frac{b^2_i}{4a_i} (1 - e^{-a_i \tau}), \quad d_i(t) = \frac{4a^*_i m^*_i}{b^2_i}, \quad \lambda_i(t) = \frac{4m^*_i y_i(0)}{b^2_i} e^{-a_i \tau} \frac{1 - e^{-c_i \tau}}{c_1}.$$
where $E_Q^T(\sqrt{r(t)})$ can be calculated by (3.35), and $E_Q^T(\sqrt{y(t)})$ can be determined by

$$E_Q^T(\sqrt{r(t)}) = \sqrt{c_i'(d_i' + \lambda_i' - 1) + \frac{c_i'd_i'}{2(d_i' + \lambda_i')}},$$

$$\text{Cov}(\sqrt{r(t)}, \sqrt{y_i(t)}) = \rho_{ry_i} \left( c_i' - \frac{c_i'd_i'}{2(d_i' + \lambda_i')} \right) \left( c_i - \frac{c_i'd_i}{2(d_i + \lambda_i)} \right)$$

with

$$c_i'(t) = \frac{\eta^2}{4k^*} (1 - e^{-k^*t}), \quad d_i'(t) = \frac{4k^*\theta^*}{\eta^2}, \quad \lambda_i'(t) = \frac{4\theta^*r(0)e^{-k^*t}}{\eta^2 1 - e^{-k^*t}}.$$ 

### 4. Numerical Examples and Analysis

In this section, we study the special case of our hybrid model analytically by utilising the semi-analytical solution derived in (3.10). For the jump-diffusion process, we first study the one-factor hybrid model, and then the multi-factor hybrid model. Both Merton jump and Kou jump have been taken into consideration. Monte Carlo simulation is used as a benchmark to validate our model.

#### 4.1. Numerical results of the multi factor hybrid model

For $\lambda = 0$, and $i = 0$, our model reduces to the basic model studied in Little and Pant [31]. For $\lambda = 0$, and $i = 1$, it reduces to the one-factor stochastic volatility model in [39]. He, compared to the aforementioned work, our model is more general and realistic by considering both the jump process and stochastic interest rate. In this section, we compare the one-factor stochastic volatility model with different jump size distributions. The parameters we use in our examples are shown in Table 2.

<table>
<thead>
<tr>
<th>Model</th>
<th>Stochastic interest rate</th>
<th>Stochastic volatility</th>
<th>Merton jump diffusion</th>
<th>Kou jump diffusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>$k$ $\theta$ $\eta$ $\rho_{sy}$</td>
<td>$a$ $m$ $b$ $\rho_{vr}$</td>
<td>$\mu$ $\delta$ $\lambda_1$ $\lambda_2$ $p$</td>
<td></td>
</tr>
<tr>
<td>Values</td>
<td>1.2 0.05 0.01 −0.4</td>
<td>2 0.05 0.05 −0.9</td>
<td>0.05 0.086</td>
<td>40 12 0.3</td>
</tr>
</tbody>
</table>

The jump-diffusion model parameters were also adopted by [21]. The parameters of the stochastic interest rate process and the stochastic volatility process satisfying Feller’s condition were also adopted by [7]. The correlation of the stochastic volatility process and the stock process $\rho_{sy_i}$ is assumed to be $-0.4$ due to the leverage effect. Fig. 2 shows the fair
strike price with different volatility values of interest rate, and Fig. 3 displays the fair strike price with different values of long term interest rate $\theta$. It is found that the effect of stochastic interest rate exists, but it is very small and can be ignored. The correlation between the asset price and the volatility is negatively correlated due to the leverage effect, and in market practice, the volatility and the interest rate also show the negative correlation. To show the effects of all the parameters on variance swap pricing, in Table 3, we present the result of the sensitivity study. The sensitivity study investigates the relative change of the fair strike price with a percentage change of the parameters. From the sensitivity study, we can draw the conclusion that the effects of the correlation rates are very small.

To show the validity of our discrete pricing formula used in the single factor hybrid model with jump diffusion ($\lambda = 0.1$), we compare our result with Broadie and Jain’s ap-

![Figure 2: The effects of volatility of the CIR process.](image1)

![Figure 3: The effects of the long-term interest rate.](image2)

Table 3: Sensitivity Study.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0.00011</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.00218</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>2.03645E-07</td>
</tr>
<tr>
<td>$a$</td>
<td>0.01797</td>
</tr>
<tr>
<td>$m$</td>
<td>0.57341</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>4.1912E-05</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>-0.00183</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-0.02524</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.40499</td>
</tr>
<tr>
<td>$\delta$</td>
<td>-0.61375</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.0096</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>0.002617</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>2.47E-05</td>
</tr>
</tbody>
</table>
proximation [6]. In their paper, the stochastic interest rate was not considered, and the approximated formula was derived from the probabilistic approach.

From Fig. 4, it can be noted that the discrete Kvar derived from both models convergent to the continuous counterpart. However, a small discrepancy can be viewed in the work of Broadie and Jain [6] when the sampling frequency is low. This is mainly caused by the effect of ignoring the stochastic interest rate and the inaccuracy of the error term.

The effects of jump-diffusion process are compared in Figs. 5 and 6. Fig. 5 is the fair strike price calculated from the Kou model, with the jump size being driven by the double exponential distribution. Fig. 6 is the fair strike price calculated from the Merton type jump model, with the jump size satisfying the normal distribution. From Figs. 5 and 6, we conclude that the fair strike price increases with the jump intensity. Since the discrete sampled fair strike price will converge towards the continuous situation, the continuous sampled fair strike price will also increase as the jump intensity increases. Detailed proof of the continuously sampled situation is given in Appendix C.
4.2. Numerical results of multi-factor hybrid model

In this section, we extend our hybrid model to a multi-factor case. Two models are studied to show the modification effects of the multi-factor stochastic volatility on the pricing of the stochastic volatility.

4.2.1. The Fouque multi-scale volatility model

An example of the multi-factor model is the multi-scale volatility model proposed by Fouque [23]. Fouque decomposed the volatility into a two-scale process: the fast-scale process and the slow-scale process. The slow-scale process represents the variation of the asset price in the long run, which is less oscillated, while the fast-scale volatility process represents the variation of the asset price in a short run, and it is highly oscillated. To visualise the idea of different scale of the volatility, here we filter the frequency of the VIX index by applying the wavelet method, as shown in Fig. 7.

Here we model the multi-scale model as follows:

\[
\begin{align*}
    dy_1 &= \frac{1}{\xi}(m - y_1)dt + \frac{1}{\sqrt{\xi}}b\sqrt{y_1}dw_1, \\
    dy_2 &= \xi(m - y_2)dt + \sqrt{\xi}b\sqrt{y_2}dw_2,
\end{align*}
\]

where \( \xi \) denotes the scale rate, \( y_1 \) denotes the fast-scale volatility process and \( y_2 \) denotes the slow-scale volatility process. In Fig. 8, we investigate the effect of the scale-rate on the pricing of the variance swap.

Figure 7: Decomposition of the volatility by wavelet method.
4.2.2. The Heston two-factor stochastic model

As illustrated in Heston [12], one-factor stochastic models can not capture the phenomenon of option ‘smirk’, therefore at least two factors are needed to obtain a more realistic model. In this section, we compare the numerical results obtained respectively by the one-factor stochastic model and the two-factor stochastic volatility model.

We firstly implement the Monte Carlo (MC) simulation using 200,000 paths, and the initial value of the stock price is assumed to be 1. The stochastic process is discretised simply by the Euler-Maruyama scheme. As shown in Fig. 9, the results from the MC simulation are slightly different from those obtained from our semi-analytical solutions when the sample size is small, but the discrepancy disappears when the sampling frequency goes to infinity. The dashed line describes the continuous approximation of $K_{var}$ as in [6]. In practice, the jump is normally absorbed into the stock process. The parameters are calibrated by Heston et al. [12] as shown in Table 4.

<table>
<thead>
<tr>
<th>Model</th>
<th>Stochastic volatility factor $y_1$</th>
<th>Stochastic volatility factor $y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>$a_1$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>Values</td>
<td>0.1500</td>
<td>0.0059</td>
</tr>
</tbody>
</table>

We prove in Appendix C that the continuous fair strike price with Merton jump, $\mu = 0.05$ and $\sigma = 0.08$ satisfies the following equation

$$K_{var} = \lambda \left( \mu^2 + \sigma^2 \right) + \sum_{j=1}^{m} \left( y_j \frac{1 - e^{a_j T}}{a_j T} + m_j \left( 1 - \frac{1 - e^{a_j T}}{a_j T} \right) \right).$$
For the influence of jump intensity in multi-factor models, we compare the results obtained respectively by $\lambda = 0$ (Fourier2) and $\lambda = 0.1$ (Fourier1) in Fig. 10. It is noted that the inclusion of the jump-diffusion process shifts the fair strike price up both in discrete sampling model and continuous sampling model.

5. Conclusion

In this paper, we propose a generalised hybrid financial model to study the pricing of variance swaps. We consider the effect of the multi-factor stochastic volatility, stochastic interest rate and jump-diffusion process. The incorporation of multi-factors and a jump-diffusion term leads to a high dimensional PIDE. A semi-analytic solution is obtained by using the generalised Fourier transform under the two-stage framework proposed by Little and Pant [31]. With a proper selection of model parameters, the proposed model covers some existing financial models as special cases, including the CIR model, the hybrid Heston-CIR model, the multi-factor-CIR model and the jump model.

Monte Carlo simulation is used as a benchmark to show the validity of our numerical results. By applying a generalised Fourier transform, the efficiency could be enhanced. The variance swap with continuous sampling is also investigated by analysing the return. Our discrete model is proved to be convergent to the continuous case when the sampling size goes to infinity.

The effects of the stochastic interest rate and the jump are also studied in our work. As for the distribution of the jump size, both the Merton jump and the Kou jump have been taken into consideration. Comparisons have been made between different models. Our results indicate that the impact of the stochastic interest rate upon the variance swap price is not significant compared to the stochastic volatility and the jump-diffusion process. The fair strike price increases with the jump intensity regardless of the distribution of the jump size.
Appendix A

In Appendix A, we prove that the generalised Fourier transform of the integral term arising from the jump diffusion process is equivalent to the characteristic function of the underlying distribution of jump size.

\[
\mathcal{F} \left( \int_R U(x + \eta) \Gamma(d\eta) \right) = \mathcal{F} \left( \int_R U(x + \eta) p(\eta) d\eta \right) = \int_R U(x + \eta) p(\eta) e^{-iwx} d\eta dx = \int_R p(\eta) \int_R U(x + \eta) e^{-iw(\eta - y)} dy d\eta = \int_R p(\eta) e^{iwy} d\eta \int_R U(y) e^{-iw} dy = \phi_\eta(w)V.
\]

Similar results can be obtained from the Fourier convolution theorem if we let \(p(\eta) = g(-s)\).

\[
\mathcal{F} \left( \int_R U(x + \eta) \Gamma(d\eta) \right) = \mathcal{F} \left( \int_R U(x + \eta) p(\eta) d\eta \right) = -\int_R U(x - s) g(-s) e^{-iwx} dx ds = -\mathcal{F}(U(x) \otimes g(-s)) = -V(w) \mathcal{F}(g(-s)) = \phi_\eta(w)V(w).
\]

Appendix B

In this appendix, we approximate the expectation and variance of \(\sqrt{y_1y_2}\). Following the work of [26], the variable \(y_1(t)\) is approximated by the normal distribution where

\[
E(y_1(t)) = c_i(t)(d_i + \lambda_i(t)), \quad \text{Var}(y_1(t)) = c^2_i(t)(2d_i + 4\lambda_i(t))
\]

with

\[
c_i(t) = b_i^2 \left( 1 - e^{-a_i^* t} \right), \quad d_i(t) = \frac{4a_i^* m_i^*}{b_i^2}, \quad \lambda_i(t) = \frac{4m_i^* y_i(0)}{\delta_i^2} \frac{e^{-a_i^* t}}{1 - e^{-a_i^* t}}.
\]

Therefore,

\[
E\left( \sqrt{y_1(t)} \right) = \sqrt{c_i(d_i + \lambda_i - 1) + \frac{c_i d_i}{2(d_i + \lambda_i)}},
\]

\[
\text{Var} \left( \sqrt{y_1(t)} \right) = c_i - \frac{c_i d_i}{2(d_i + \lambda_i)}.
\]
From the Taylor expansion and the Merton jump assumption, we can verify that

\[
\text{Var}(\sqrt{y_i(t)}) = \frac{\text{Var}(y_i(t))}{4E(y_i(t))} = c_i - \frac{c_i d_i}{2(d_i + \lambda_i)}.
\]

This can be proved by the Taylor Series expansion. \(\text{Var}(\sqrt{y_i(t)})\) can be approximated by

\[
\text{Var}(\sqrt{y_i(t)}) \approx \frac{\text{Var}(y_i(t))}{4E(y_i(t))} = c_i - \frac{c_i d_i}{2(d_i + \lambda_i)}.
\]

For the reason that \(\text{Var}(\sqrt{y_i(t)}) = E(y_i(t)) - E^2(\sqrt{y_i(t)})\), we obtain

\[
E^2(\sqrt{y_i(t)}) = c_i(d_i + \lambda_i - 1) + \frac{c_i d_i}{2(d_i + \lambda_i)},
\]

\[
\text{Cov}(\sqrt{y_i(t)}, \sqrt{y_j(t)}) = \rho_{y_iy_j} \sqrt{\text{Var}(y_i)} \sqrt{\text{Var}(y_j)}.
\]

**Appendix C**

In order to prove (5.1), we simply assume that the stochastic volatility process is independent of each other. According to L’Hôpital’s rule, for the reason that \(\dot{C}(\Delta t) = 0\), \(\dot{D}(\Delta t) = 0\), \(\dot{E}(\Delta t) = 0\), \(\dot{F}(\Delta t) = 0\), we obtain

\[
\lim_{\Delta t \to 0} \frac{\exp(C(\Delta t) + \dot{D}(\Delta t)y + E(\Delta t)r) - 2 \exp(\dot{C}(\Delta t) + \dot{E}(\Delta t)r) + 1}{\Delta t} = \lim_{\Delta t \to 0} \left( \dot{C}'(\Delta t) + \dot{D}'(\Delta t)y + \dot{E}'(\Delta t)r - 2C'(\Delta t) + \dot{E}'(\Delta t)r \right) = \lim_{\Delta t \to 0} \left( \dot{C}'(\Delta t) - 2\dot{C}'(\Delta t) \right) + \lim_{\Delta t \to 0} \left( \dot{E}'(\Delta t) - 2\dot{E}'(\Delta t) \right) + \lim_{\Delta t \to 0} \dot{D}'(\Delta t)y. \tag{C.1}
\]

From the Taylor expansion and the Merton jump assumption, we can verify that

\[
\lim_{\Delta t \to 0} \left( \dot{C}'(\Delta t) - 2\dot{C}'(\Delta t) \right) = \lambda + \lambda \left( E(e^{2x}) - E(e^x) \right) = \lambda + \lambda (-1 + E(x^2)) = \lambda \left( E^2(x) + \text{Var}(x) \right) = \lambda \left( \mu^2 + \delta^2 \right),
\]

\[
\lim_{\Delta t \to 0} \left( \dot{E}'(\Delta t) - 2\dot{E}'(\Delta t) \right) = -2j * j - 2 * (-j) * j = 0.
\]

According to the expression of \(D_j\), we obtain

\[
\lim_{\Delta t \to 0} \dot{D}'(\Delta t)y = y.
\]

Thus, due to the property of the variance process, the limit (C.1) is equivalent to \(\lambda(\mu^2 + \sigma^2) + y\). For a continuous case, we have

\[
\text{Kvar} = \lim_{n \to \infty} \frac{AF}{N} \sum_{i=1}^{n} f_i = \lim_{\Delta t \to 0} \frac{1}{T} \sum_{i=1}^{n} \frac{1}{\Delta t} \ast \Delta t \ast f_i.
\]


\[
\begin{align*}
&= \frac{1}{T} \int_0^T \left( \lambda (\mu^2 + \sigma^2) + E(y_t) \right) dt \\
&= \frac{1}{T} \int_0^T \left( \lambda (\mu^2 + \sigma^2) + \sum_{j=1}^m \left( y_j e^{-a_j t} + m_j \left( 1 - e^{-a_j (i-1)T} \right) \right) \right) dt \\
&= \lambda (\mu^2 + \sigma^2) + \sum_{j=1}^m \left( y_j \frac{1 - e^{-a_j T}}{a_j T} + m_j \left( 1 - \frac{1 - e^{-a_j T}}{a_j T} \right) \right).
\end{align*}
\]

Though the result is in line with the result shown in [6], it is proved by the result of Fourier transform rather than by the probabilistic approach.

The above result is obtained by the assumption of the Merton jump. If the underlying process is a double exponential process instead of a normal distribution, the following result can easily be derived by substituting the density function of double exponential distribution:

\[
K_{\text{var}} = \lambda \left( \frac{p}{\lambda_1^2} + \frac{q}{\lambda_2^2} \right) (p + q + 1) - \frac{4pq}{\eta_1 \eta_2} + \sum_{j=1}^m \left( y_j \frac{1 - e^{-a_j T}}{a_j T} + m_j \left( 1 - \frac{1 - e^{-a_j T}}{a_j T} \right) \right).
\]

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References


