East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.100620.170920

Vol. **11**, No. 1, pp. 143-163
February 2021

The Riemann-Hilbert Approach and N-Soliton Solutions of a Four-Component Nonlinear Schrödinger Equation

Xin-Mei Zhou, Shou-Fu Tian*, Jin-Jie Yang and Jin-Jin Mao

School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China.

Received 10 June 2020; Accepted (in revised version) 17 September 2020.

Abstract. A four-component nonlinear Schrödinger equation associated with a 5×5 Lax pair is investigated. A spectral problem is analysed and the Jost functions are used in order to derive a Riemann-Hilbert problem connected with the equation under consideration. N-soliton solutions of the equation are obtained by solving the Riemann-Hilbert problem without reflection. For $N = 1$ and $N = 2$, the local structure and dynamic behavior of some special solutions is analysed by invoking their graphic representations.

AMS subject classifications: 35Q51, 35Q53, 35C99, 68W30, 74J35

Key words: Four-component nonlinear Schrödinger equation, Riemann-Hilbert approach, N-soliton solutions.

1. Introduction

The nonlinear Schrödinger equation (NLS) is an important integrable model. It is closely related to nonlinear problems in theoretical physics such as nonlinear optics and ion acoustic waves of plasmas. On the other hand, higher-order coupled NLS equations are used to describe the effects of cubic-quintic nonlinearity, self-deepening, and self-frequency shifting. Among numerous solutions of these equations, soliton solutions play a crucial role in some complex nonlinear phenomena. At present, there are many methods to find the solutions of nonlinear integrable models — e.g. inverse scattering transform [1], Darboux transform [14], Hirota bilinear method [6], and Lie group method [2]. In particular, let us note the inverse scattering transform method, which is especially efficient in finding soliton solutions of the corresponding initial value problems. For second-order spectral problems, the inverse scattering theory is equivalent to Riemann-Hilbert (RH) approach. On the other hand, some of the higher-order spectral problems have to be transformed into RH problem.

*Corresponding author. *Email addresses:* xmzhou@cumt.edu.cn (X.M. Zhou), sftian@cumt.edu.cn, shoufu2006@126.com (S.F. Tian), jinjieyang@cumt.edu.cn (J.J. Yang), jj-mao@cumt.edu.cn (J.J. Mao)

This approach, developed by Zakharov *et al* [34], was successively applied to various integrable systems with a single component [3–5,8–13,16–21,24–30,32,33,35–38]. However, to the best of authors' knowledge, only a few studies deal with multi-component problems.

The nonlinear Schrödinger equation has the form

$$i\frac{\partial \psi_\alpha}{\partial t} + \frac{\partial^2 \psi_\alpha}{\partial x^2} + \sum_{\beta,\gamma} \psi^*_\beta \Lambda_{\beta\gamma} \psi_\gamma \psi_\alpha = 0, \tag{1.1}$$

where Λ is an Hermitian matrix [15]. A multi-component multisoliton solution of the Eq. (1.1) has been constructed by the Hirota's bilinearisation method — cf. [7]. The well-known general two-component coupled nonlinear Schrödinger equation — cf. [22],

$$\begin{aligned} &ip_t + p_{xx} + 2\big(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*\big)p = 0,\\ &iq_t + q_{xx} + 2\big(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*\big)q = 0, \end{aligned} \tag{1.2}$$

where a, c are real constants, b is a complex constant, and "$*$" denotes the complex conjugation, is a special case of the Eq. (1.1). In physics, a and c describe the self-phase modulation and cross-phase modulation effects, and b and b^* the four-wave mixing effects.

Furthermore, the three-component nonlinear Schrödinger equations has the form

$$\begin{aligned} &iq_{1t} + q_{1xx} - 2\big[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\mathrm{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)\big]q_1 = 0,\\ &iq_{2t} + q_{2xx} - 2\big[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\mathrm{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)\big]q_2 = 0,\\ &iq_{3t} + q_{3xx} - 2\big[a|q_1|^2 + c|q_2|^2 + f|q_3|^2 + 2\mathrm{Re}(bq_1^*q_2 + dq_1^*q_3 + eq_2^*q_3)\big]q_3 = 0, \end{aligned} \tag{1.3}$$

where a, c, f are real constants, b, d, e complex constants, and "Re" denotes the real part. These equations are studied by extending the Fokas unified approach by Yan in [31]. In particular, it was shown that the Eq. (1.3) can be reduced to three-component NLS equations with various conditions on parameters a, b, c, d, e and f. More precisely,

- If $a = c = f = -1$ and $b = d = e = 0$, the Eq. (1.3) reduces to a three-component focused NLS equation.
- If $a = c = f = 1$ and $b = d = e = 0$, the Eq. (1.3) reduces to a three-component defocused NLS equation.
- If $a = -1$, $c = f = 1$ and $b = d = e = 0$ or $a = 1$, $c = f = -1$ and $b = d = e = 0$, the Eq. (1.3) reduces to a three-component mixed NLS equation.
- For other choice of parameters, the Eq. (1.3) reduces to other three-component NLS equations.

In this work, we consider a four-component nonlinear Schrödinger (FCNLS) equation — viz.

$$iq_{1t} + q_{1xx} - 2\big[a_{11}|q_1|^2 + a_{22}|q_2|^2 + a_{33}|q_3|^2 + a_{44}|q_4|^2$$

$$+2\mathrm{Re}(a_{12}q_1^*q_2+a_{13}q_1^*q_3+a_{14}q_1^*q_4+a_{23}q_2^*q_3+a_{24}q_2^*q_4+a_{34}q_3^*q_4)\big]q_1=0, \quad (1.4a)$$

$$iq_{2t}+q_{2xx}-2\big[a_{11}|q_1|^2+a_{22}|q_2|^2+a_{33}|q_3|^2+a_{44}|q_4|^2$$
$$+2\mathrm{Re}(a_{12}q_1^*q_2+a_{13}q_1^*q_3+a_{14}q_1^*q_4+a_{23}q_2^*q_3+a_{24}q_2^*q_4+a_{34}q_3^*q_4)\big]q_2=0, \quad (1.4b)$$

$$iq_{3t}+q_{3xx}-2\big[a_{11}|q_1|^2+a_{22}|q_2|^2+a_{33}|q_3|^2+a_{44}|q_4|^2$$
$$+2\mathrm{Re}(a_{12}q_1^*q_2+a_{13}q_1^*q_3+a_{14}q_1^*q_4+a_{23}q_2^*q_3+a_{24}q_2^*q_4+a_{34}q_3^*q_4)\big]q_3=0, \quad (1.4c)$$

$$iq_{4t}+q_{4xx}-2\big[a_{11}|q_1|^2+a_{22}|q_2|^2+a_{33}|q_3|^2+a_{44}|q_4|^2$$
$$+2\mathrm{Re}(a_{12}q_1^*q_2+a_{13}q_1^*q_3+a_{14}q_1^*q_4+a_{23}q_2^*q_3+a_{24}q_2^*q_4+a_{34}q_3^*q_4)\big]q_4=0, \quad (1.4d)$$

where $a_{11}, a_{22}, a_{33}, a_{44}$ are real constants and $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}$ complex constants. The FCNLS equation includes group velocity dispersion, self-phase modulation, cross-phase modulation and paired tunnel modulation. Analogously to the three-component NLS equations (1.3), the FCNLS equation (1.4) can be reduced to different four-component NLS equations with the different conditions on the parameters. The Riemann-Hilbert problem for the FCNLS equation and its multi-soliton solutions have not been yet studied, so that the construction of such a problem and finding multi-soliton solutions of the related 5×5 matrix spectral problem is of a significant interest.

The structure of this work is as follows. In Section 2, we analyse a spectral problem and properties of the Jost functions connected to the Lax pair with a 5×5 matrix spectral problem for the FCNLS equation (1.4). In Section 3, the corresponding Riemann-Hilbert problem is derived and the symmetry of the scattering matrix and temporal and spatial evolution of the scattering data are studied. The solutions of Riemann-Hilbert problem, obtained in Section 4, are used to establish the N-soliton solutions of the FCNLS equation (1.4). Besides, the propagation of one-and two-soliton solutions is discussed. Finally, some conclusions are presented in Section 5.

2. Spectral Analysis

Let us start with the Lax pair for the FCNLS equation (1.4). It can be written as

$$\Phi_x=U\Phi, \quad \Phi_t=V\Phi,$$

where Φ is a column vector function, matrices U and V have the form

$$U=\begin{pmatrix} -i\lambda & 0 & 0 & 0 & q_1\\ 0 & -i\lambda & 0 & 0 & q_2\\ 0 & 0 & -i\lambda & 0 & q_3\\ 0 & 0 & 0 & -i\lambda & q_4\\ p_1 & p_2 & p_3 & p_4 & i\lambda \end{pmatrix},$$
$$V=-2i\lambda^2\Lambda+2\lambda P+V_0, \quad V_0=-i\big(P_x+P^2\big)\Lambda,$$

λ is the spectral parameter,

$$\Lambda=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-1\end{pmatrix},\quad P=\begin{pmatrix}0&0&0&0&q_1\\0&0&0&0&q_2\\0&0&0&0&q_3\\0&0&0&0&q_4\\p_1&p_2&p_3&p_4&0\end{pmatrix},$$

and

$$\begin{aligned}p_1&=a_{11}q_1^*+a_{21}q_2^*+a_{31}q_3^*+a_{41}q_4^*,\\p_2&=a_{21}^*q_1^*+a_{22}q_2^*+a_{32}q_3^*+a_{42}q_4^*,\\p_3&=a_{31}^*q_1^*+a_{32}^*q_2^*+a_{33}q_3^*+a_{43}q_4^*,\\p_4&=a_{41}^*q_1^*+a_{42}^*q_2^*+a_{43}^*q_3^*+a_{44}q_4^*.\end{aligned}$$

It follows that

$$\begin{aligned}\Phi_x+i\lambda\Lambda\Phi&=P\Phi,\\\Phi_t+2i\lambda^2\Lambda\Phi&=Q\Phi,\end{aligned}\tag{2.1}$$

where $Q=2\lambda P+V_0$.

For $|x|\to\infty$, the Eqs. (2.1) yield

$$\Phi\propto e^{-i\lambda\Lambda x-2i\lambda^2\Lambda t}.$$

Letting

$$\mu=\Phi e^{i\lambda\Lambda x+2i\lambda^2\Lambda t}$$

gives

$$\mu\sim\mathbb{I},\quad |x|\to\infty,\tag{2.2}$$

and we arrive at the new Lax pair

$$\begin{aligned}\mu_x+i\lambda[\Lambda,\mu]&=P\mu,\\\mu_t+2i\lambda^2[\Lambda,\mu]&=Q\mu\end{aligned}\tag{2.3}$$

with the commutator $[\Lambda,\mu]:=\Lambda\mu-\mu\Lambda$. We can obtain the full differential

$$d\left(e^{i(\lambda x+2\lambda^2 t)\bar{\Lambda}}\mu\right)=e^{i(\lambda x+2\lambda^2 t)\bar{\Lambda}}\big[(Pdx+Qdt)\mu\big],\tag{2.4}$$

where $e^{\lambda\bar{\Lambda}}\mu=e^{\lambda\Lambda}\mu e^{-\lambda\Lambda}$.

In order to construct a Riemann-Hilbert problem, one has to find the solutions of the spectral problem (2.3) as $\lambda\to\infty$. For this, we represent the solution of the Eq. (2.4) in the form

$$\mu=\mu^{(0)}+\frac{\mu^{(1)}}{\lambda}+\frac{\mu^{(2)}}{\lambda^2}+o\left(\frac{1}{\lambda^3}\right),\quad \lambda\to\infty\tag{2.5}$$

with $\mu^{(0)},\mu^{(1)},\mu^{(2)}$ not related to λ. Substituting (2.5) into (2.3) and equating the coefficients at the same powers of λ yields

$$\begin{aligned} &o(1): \mu_x^{(0)} + i\left[\Lambda, \mu^{(1)}\right] = P\mu^{(0)}, \\ &o(\lambda): i\left[\Lambda, \mu^{(0)}\right] = 0 \qquad\qquad \text{for } x \text{ part}, \\ &o(\lambda): 2i\left[\Lambda, \mu^{(1)}\right] = 2P\mu^{(0)} \qquad \text{for } t \text{ part}. \end{aligned} \tag{2.6}$$

The first and third equations in (2.6) show that $\mu_x^{(0)} = 0$, so that $\mu^{(0)}$ is not related to x. Besides, (2.2) and the second equation in (2.6) imply that $\mu^{(0)}$ is a diagonal matrix. Hence,

$$\mathbb{I} = \lim_{\lambda\to\infty} \lim_{|x|\to\infty} \mu = \mu^{(0)}.$$

Now, the solutions $\mu_\pm = \mu_\pm(x,\lambda)$ of the Eq. (2.3) can be written as

$$\begin{aligned} \mu_+ &= \left([\mu_+]_1, [\mu_+]_2, [\mu_+]_3, [\mu_+]_4, [\mu_+]_5\right), \\ \mu_- &= \left([\mu_-]_1, [\mu_-]_2, [\mu_-]_3, [\mu_-]_4, [\mu_-]_5\right), \end{aligned} \tag{2.7}$$

where $[\mu_\pm]_l$, $l = 1,2,3,4,5$ denotes the l-th column of the corresponding matrix $[\mu_+]$ or $[\mu_-]$. Moreover, we note that these solutions satisfy the asymptotic conditions

$$\begin{aligned} \mu_+ &\to \mathbb{I} \quad \text{as} \quad x \to +\infty, \\ \mu_- &\to \mathbb{I} \quad \text{as} \quad x \to -\infty, \end{aligned} \tag{2.8}$$

where $\mathbb{I}$ is the 5×5 unit matrix. Let us note that the solutions $[\mu_\pm]$ for $\lambda \in \mathbb{R}$ are uniquely determined by the Volterra integral equations

$$\begin{aligned} \mu_+(x,\lambda) &= \mathbb{I} - \int_x^{+\infty} e^{-i\lambda\Lambda(x-y)} P(y)\mu_+(y,\lambda) e^{i\lambda\Lambda(x-y)} dy, \\ \mu_-(x,\lambda) &= \mathbb{I} + \int_{-\infty}^{x} e^{-i\lambda\Lambda(x-y)} P(y)\mu_-(y,\lambda) e^{i\lambda\Lambda(x-y)} dy \end{aligned}$$

with the kernel matrix

$$\begin{aligned} &e^{-i\lambda\Lambda(x-y)} P e^{i\lambda\Lambda(x-y)} \\ &:= \begin{pmatrix} 0 & 0 & 0 & 0 & q_1 e^{-2i\lambda(x-y)} \\ 0 & 0 & 0 & 0 & q_2 e^{-2i\lambda(x-y)} \\ 0 & 0 & 0 & 0 & q_3 e^{-2i\lambda(x-y)} \\ 0 & 0 & 0 & 0 & q_4 e^{-2i\lambda(x-y)} \\ p_1 e^{2i\lambda(x-y)} & p_2 e^{2i\lambda(x-y)} & p_3 e^{2i\lambda(x-y)} & p_4 e^{2i\lambda(x-y)} & 0 \end{pmatrix}. \end{aligned}$$

If $\mathrm{Re}[2i\lambda(x-y)] < 0$ and $\mathrm{Re}[-2i\lambda(x-y)] < 0$, the columns $[\mu_-]_1$, $[\mu_-]_2$, $[\mu_-]_3$, $[\mu_-]_4$ and $[\mu_+]_5$ are analytic vector-functions on $\mathbb{C}^+$, and $[\mu_+]_1$, $[\mu_+]_2$, $[\mu_+]_3$, $[\mu_+]_4$ and $[\mu_-]_5$ are analytic vectors-functions on $\mathbb{C}^-$.

Let us study the properties of $\mu_\pm$. The relation $\mathrm{tr}(P) = 0$ and Liouville's formula yield that the determinants of the matrices $\mu_\pm$ do not depend on the variable x. Therefore, the Eq. (2.8) implies

$$\det \mu_\pm = 1, \quad \lambda \in \mathbb{R}. \tag{2.9}$$

Moreover, since for $E = e^{-i\lambda\Lambda x}$ both $\mu_- E$ and $\mu_+ E$ are the solutions of the spectral problem (2.3), they are linearly interdependent and satisfy the equation

$$\mu_- E = \mu_+ E S(\lambda), \quad \lambda \in \mathbb{R}, \tag{2.10}$$

where $S(\lambda) = (s_{kj})_{5\times 5}$ is the scattering matrix. The Eqs. (2.9) and (2.10) show that

$$\det S(\lambda) = 1, \quad \lambda \in \mathbb{R}.$$

In order to write the Riemann-Hilbert problem for the Eqs. (1.4), we consider the inverse matrices

$$\mu_\pm^{-1} = \begin{pmatrix} [\mu_\pm^{-1}]^1 \\ [\mu_\pm^{-1}]^2 \\ [\mu_\pm^{-1}]^3 \\ [\mu_\pm^{-1}]^4 \\ [\mu_\pm^{-1}]^5 \end{pmatrix},$$

where $[\mu_\pm^{-1}]^l$, $l = 1,2,3,4,5$ are the rows of μ_+^- or μ_-^-.

It follows from (2.3) that $\mu_\pm^{-1}$ satisfy the following equation

$$K_x = -i\lambda[\Lambda, K] - KP. \tag{2.11}$$

The Eq. (2.11) gives that $[\mu_-^{-1}]^1$, $[\mu_-^{-1}]^2$, $[\mu_-^{-1}]^3$, $[\mu_-^{-1}]^4$, $[\mu_+^{-1}]^5$ are analytic vector-functions on $\mathbb{C}^-$ and $[\mu_+^{-1}]^1$, $[\mu_+^{-1}]^2$, $[\mu_+^{-1}]^3$, $[\mu_+^{-1}]^4$, $[\mu_-^{-1}]^5$ are analytic vector-functions on $\mathbb{C}^+$.

According to Eq. (2.10), we have

$$E^{-1}\mu_-^{-1} = R(\lambda)E^{-1}\mu_+^{-1}.$$

Theorem 2.1. *If the matrices S and R are written as*

$$S(\lambda) = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} \end{pmatrix}, \quad R(\lambda) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} \end{pmatrix},$$

then the entries $s_{11}, s_{12}, s_{13}, s_{14}, s_{21}, s_{22}, s_{23}, s_{24}, s_{31}, s_{32}, s_{33}, s_{34}, s_{41}, s_{42}, s_{43}, s_{44}$ are analytic functions in $\mathbb{C}^+$, s_{55} is analytic in $\mathbb{C}^-$, r_{55} is analytic in $\mathbb{C}^+$, and $r_{11}, r_{12}, r_{13}, r_{14}, r_{21}, r_{22}, r_{23}, r_{24}, r_{31}, r_{32}, r_{33}, r_{34}, r_{41}, r_{42}, r_{43}, r_{44}$ are analytic functions in $\mathbb{C}^-$.

Proof. It follows from (2.10) that $E^{-1}\mu_+^{-1}\mu_- E = S(\lambda)$,

$$S(\lambda) = E^{-1}\begin{pmatrix} [\mu_+^{-1}]^1[\mu_-]_1 & [\mu_+^{-1}]^1[\mu_-]_2 & [\mu_+^{-1}]^1[\mu_-]_3 & [\mu_+^{-1}]^1[\mu_-]_4 & [\mu_+^{-1}]^1[\mu_-]_5 \\ [\mu_+^{-1}]^2[\mu_-]_1 & [\mu_+^{-1}]^2[\mu_-]_2 & [\mu_+^{-1}]^2[\mu_-]_3 & [\mu_+^{-1}]^2[\mu_-]_4 & [\mu_+^{-1}]^2[\mu_-]_5 \\ [\mu_+^{-1}]^3[\mu_-]_1 & [\mu_+^{-1}]^3[\mu_-]_2 & [\mu_+^{-1}]^3[\mu_-]_3 & [\mu_+^{-1}]^3[\mu_-]_4 & [\mu_+^{-1}]^3[\mu_-]_5 \\ [\mu_+^{-1}]^4[\mu_-]_1 & [\mu_+^{-1}]^4[\mu_-]_2 & [\mu_+^{-1}]^4[\mu_-]_3 & [\mu_+^{-1}]^4[\mu_-]_4 & [\mu_+^{-1}]^4[\mu_-]_5 \\ [\mu_+^{-1}]^5[\mu_-]_1 & [\mu_+^{-1}]^5[\mu_-]_2 & [\mu_+^{-1}]^5[\mu_-]_3 & [\mu_+^{-1}]^5[\mu_-]_4 & [\mu_+^{-1}]^5[\mu_-]_5 \end{pmatrix} E,$$

and for the matrix S the claim follows from the properties of μ_+^{-1} and μ_-. The matrix $R(\lambda)$ can be treated analogously. □

3. Riemann-Hilbert Problem

In this section, we construct a respective Riemann-Hilbert problem. More precisely, considering the matrix functions

$$P_1 = P_1(x,\lambda) := \big([\mu_-]_1, [\mu_-]_2, [\mu_-]_3, [\mu_-]_4, [\mu_+]_5\big), \tag{3.1}$$

$$P_2 = P_2(x,\lambda) := \begin{pmatrix} [\mu_-^{-1}]^1 \\ [\mu_-^{-1}]^2 \\ [\mu_-^{-1}]^3 \\ [\mu_-^{-1}]^4 \\ [\mu_+^{-1}]^5 \end{pmatrix}, \tag{3.2}$$

we note that P_1 and P_2 are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, and satisfy the asymptotic conditions

$$\begin{aligned} P_1 &\to \mathbb{I} \quad \text{as} \quad \lambda \to +\infty, \\ P_2 &\to \mathbb{I} \quad \text{as} \quad \lambda \to -\infty. \end{aligned} \tag{3.3}$$

At the moment, we write P_+ for P_1 restricted to the left-hand side of the real λ-axis and P_- for P_2 restricted to the right-hand side of that axis. On the real line, these matrix-functions satisfy the equation

$$P_-(x,\lambda)P_+(x,\lambda) = G(x,\lambda), \quad \lambda \in \mathbb{R} \tag{3.4}$$

with

$$G(x,\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & r_{15}e^{-2i\lambda x} \\ 0 & 1 & 0 & 0 & r_{25}e^{-2i\lambda x} \\ 0 & 0 & 1 & 0 & r_{35}e^{-2i\lambda x} \\ 0 & 0 & 0 & 1 & r_{45}e^{-2i\lambda x} \\ s_{51}e^{2i\lambda x} & s_{52}e^{2i\lambda x} & s_{53}e^{2i\lambda x} & s_{54}e^{2i\lambda x} & 1 \end{pmatrix}.$$

According to Eq. (3.3), the canonical normalization conditions are

$$\begin{aligned} P_1 &\to \mathbb{I} \quad \text{as} \quad \lambda \to +\infty, \\ P_2 &\to \mathbb{I} \quad \text{as} \quad \lambda \to -\infty. \end{aligned}$$

Theorem 3.1.

$$\begin{aligned} \det P_1 &= r_{55} \quad \lambda \in \mathbb{C}^+, \\ \det P_2 &= s_{55} \quad \lambda \in \mathbb{C}^-. \end{aligned} \tag{3.5}$$

Proof. According to Eqs. (3.1) and (3.2), matrices P_1 and P_2 can be written in the form

$$\begin{aligned} P_1 &= \mu_- H_1 + \mu_- H_2 + \mu_- H_3 + \mu_- H_4 + \mu_+ H_5, \\ P_2 &= H_1\mu_-^{-1} + H_2\mu_-^{-1} + H_3\mu_-^{-1} + H_4\mu_-^{-1} + H_5\mu_+^{-1}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &H_1 = \operatorname{dig}(1,0,0,0,0), \quad H_2 = \operatorname{dig}(0,1,0,0,0), \quad H_3 = \operatorname{dig}(0,0,1,0,0), \\ &H_4 = \operatorname{dig}(0,0,0,1,0), \quad H_5 = \operatorname{dig}(0,0,0,0,1). \end{aligned}$$

Hence

$$\begin{aligned}\det P_1(\lambda) &= \det(\mu_- H_1 + \mu_- H_2 + \mu_- H_3 + \mu_- H_4 + \mu_+ H_5)\\ &= \det\mu_- \cdot \det(H_1 + H_2 + H_3 + H_4 + ER(\lambda)E^{-1}H_5)\\ &= 1 \cdot \det\begin{pmatrix} 1 & 0 & 0 & 0 & r_{15}e^{-2i\lambda x}\\ 0 & 1 & 0 & 0 & r_{25}e^{-2i\lambda x}\\ 0 & 0 & 1 & 0 & r_{35}e^{-2i\lambda x}\\ 0 & 0 & 0 & 1 & r_{45}e^{-2i\lambda x}\\ 0 & 0 & 0 & 0 & r_{55}\end{pmatrix}\\ &= r_{55}.\end{aligned}$$

The second equation in (3.5) can be proven analogously. The Eq. (2.9) shows that $\det\mu_\pm = 1$, which yields $\det P_1 = r_{55}, \lambda \in \mathbb{C}^+$, and $\det P_2 = s_{55}, \lambda \in \mathbb{C}^-$. □

Let $A^\dagger$ denotes the Hermitian of the matrix A. A symmetry relation for the matrix P has the form

$$P^\dagger = -BPB^{-1}, \tag{3.7}$$

where

$$B = \begin{pmatrix} a_{11} & a_{21}^* & a_{31}^* & a_{41}^* & 0\\ a_{21} & a_{22} & a_{32}^* & a_{42}^* & 0\\ a_{31} & a_{32} & a_{33} & a_{43}^* & 0\\ a_{41} & a_{42} & a_{43} & a_{44} & 0\\ 0 & 0 & 0 & 0 & -1\end{pmatrix}.$$

It follows from (2.3) and (3.7) that

$$(\mu_\pm^\dagger(\lambda^*)B)_x = -i\lambda[\Lambda, \mu_\pm^\dagger(\lambda^*)B] - \mu_\pm^\dagger(\lambda^*)BP.$$

Hence, if $\mu_\pm^\dagger(\lambda^*)B$ and $\mu_\pm^{-1}(\lambda)$ satisfy (2.11), then $\mu_\pm^\dagger(\lambda^*)B$ is linearly related to $\mu_\pm^{-1}(\lambda)$, i.e. $\mu_\pm^\dagger(\lambda^*)B = C\mu_\pm^{-1}(\lambda)$ with C independent of x. Using the large-x boundary conditions of $\mu_\pm$, we find that $C = B$. Consequently,

$$B^{-1}\mu_\pm^\dagger(\lambda^*)B = \mu_\pm^{-1}(\lambda).$$

The scattering matrix $S(\lambda)$ satisfies the equation

$$B^{-1}S^\dagger(\lambda^*)B = S^{-1}(\lambda) = R(\lambda).$$

Therefore,

$$\begin{aligned} r_{55}(\lambda) &= s_{55}^*(\lambda^*), && \lambda \in \mathbb{C}^+,\\ -s_{51}^*(\lambda) &= a_{11}r_{15} + a_{21}^*r_{25} + a_{31}^*r_{35} + a_{41}^*r_{45}, && \lambda \in \mathbb{R},\\ -s_{52}^*(\lambda) &= a_{21}r_{15} + a_{22}r_{25} + a_{32}^*r_{35} + a_{42}^*r_{45}, && \lambda \in \mathbb{R},\\ -s_{53}^*(\lambda) &= a_{31}r_{15} + a_{32}r_{25} + a_{33}r_{35} + a_{43}^*r_{45}, && \lambda \in \mathbb{R},\\ -s_{54}^*(\lambda) &= a_{41}r_{15} + a_{42}r_{25} + a_{43}r_{35} + a_{44}r_{45}, && \lambda \in \mathbb{R}. \end{aligned} \tag{3.8}$$

Theorem 3.2.

$$P_1^{\dagger}(\lambda^*) = BP_2(\lambda)B^{-1}, \quad \lambda \in \mathbb{C}^-. \tag{3.9}$$

Proof. Taking into account the first equation in (3.6), we write

$$\begin{aligned} P_1^{\dagger}(\lambda^*) &= \big(\mu_-(\lambda^*)H_1 + \mu_-(\lambda^*)H_2 + \mu_-(\lambda^*)H_3 + \mu_-(\lambda^*)H_4 + \mu_+(\lambda^*)H_5\big)^{\dagger} \\ &= H_1\mu_-^{\dagger}(\lambda^*) + H_2\mu_-^{\dagger}(\lambda^*) + H_3\mu_-^{\dagger}(\lambda^*) + H_4\mu_-^{\dagger}(\lambda^*) + H_5\mu_+^{\dagger}(\lambda^*) \\ &= BP_2(\lambda)B^{-1}, \quad \lambda \in \mathbb{C}^-, \end{aligned}$$

so that

$$P_1^{\dagger}(\lambda^*) = BP_2(\lambda)B^{-1}, \quad \lambda \in \mathbb{C}^-. \qquad \square$$

It follows from the Eqs. (3.5) and (3.8) that $\det P_1(\lambda) = (\det P_2(\lambda^*))^*$. Therefore, if $\det P_1(\lambda) = 0$, then $\det P_2(\lambda^*) = 0$. Let us assume that $\det P_1$ has N simple zeros $\{\lambda_j\}_1^N$ in $\mathbb{C}^+$, and $\det P_2$ has N simple zeros $\{\lambda_j^*\}_1^N$ in $\mathbb{C}^-$. These zeros and nonzero vectors v_j and $\hat{v}_j$, constitute complete discrete data, satisfying the following equations:

$$\begin{aligned} &P_1(\lambda_j)v_j = 0, \\ &\hat{v}_j P_2(\lambda^*) = 0, \end{aligned} \tag{3.10}$$

where v_j and $\hat{v}_j$ are a column and a row vectors, respectively. The Eqs. (3.9) and (3.10) lead to the following relation for the eigenvectors:

$$\hat{v}_j = v_j^{\dagger}B, \quad 1 \le j \le N. \tag{3.11}$$

Now we use v_j in time-spatial revolution analysis. Take the derivatives of the first equation in (3.10) with respect to x and t, so that

$$\begin{aligned} &P_{1,x}v_j + P_1 v_{j,x} = 0, \\ &P_{1,t}v_j + P_1 v_{j,t} = 0. \end{aligned} \tag{3.12}$$

Using the relation

$$\begin{aligned} P_{1,x} &= (\mu_- H_1 + \mu_- H_2 + \mu_- H_3 + \mu_+ H_4)_x \\ &= \mu_{-,x}H_1 + \mu_{-,x}H_2 + \mu_{-,x}H_3 + \mu_{+,x}H_4, \end{aligned}$$

and the Lax pair in (2.3) gives

$$\begin{aligned} P_{1,x} &= [-i\lambda(\Lambda\mu_- - \mu_-\Lambda) + P\mu_-]H_1 + [-i\lambda(\Lambda\mu_- - \mu_-\Lambda) + P\mu_-]H_2 \\ &\quad + [-i\lambda(\Lambda\mu_- - \mu_-\Lambda) + P\mu_-]H_3 + [-i\lambda(\Lambda\mu_- - \mu_-\Lambda) + P\mu_-]H_4 \\ &\quad + [-i\lambda(\Lambda\mu_+ - \mu_+\Lambda) + P\mu_+]H_5 \\ &= -i\lambda\Lambda P_1 + i\lambda P_1\Lambda + PP_1 \\ &= -i\lambda[\Lambda, P_1] + PP_1. \end{aligned} \tag{3.13}$$

Similar arguments show that

$$P_{1,t} = -2i\lambda^2[\Lambda, P_1] + QP_1. \tag{3.14}$$

Substituting (3.13) and (3.14), respectively, into the first and second equations in (3.12) and noting that $P_1 v_j = 0$, we obtain

$$\begin{aligned} &i\lambda\Lambda v_j + v_{j,x} = 0, \\ &2i\lambda^2\Lambda v_j + v_{j,t} = 0. \end{aligned} \tag{3.15}$$

According to (3.15), we have

$$v_j = e^{-i(\lambda_j x + 2\lambda_j^2 t)\Lambda} v_{j,0},$$

where $v_{j,0}$ are constant vectors and (3.11) implies

$$\hat{v}_j = v_j^\dagger(\lambda_j)B = v_{j,0}^\dagger e^{i(\lambda_j^* x + 2\lambda_j^{*2} t)\Lambda} B.$$

4. Multi-Soliton Solutions

Expanding $P_1(\lambda)$ for large λ as

$$P_1(\lambda) = \mathbb{I} + \frac{P_1^{(1)}}{\lambda} + \frac{P_1^{(2)}}{\lambda^2} + o\left(\frac{1}{\lambda^3}\right), \quad \lambda \to \infty$$

and substituting it into (2.3) yields

$$o(1) : i\left[\Lambda, P_1^{(1)}\right] = P. \tag{4.1}$$

The Eq. (4.1) produces the following relations:

$$\begin{aligned} &q_1(x,t) = 2i\left(P_1^{(1)}\right)_{15}, \quad q_2(x,t) = 2i\left(P_1^{(1)}\right)_{25}, \\ &q_3(x,t) = 2i\left(P_1^{(1)}\right)_{35}, \quad q_4(x,t) = 2i\left(P_1^{(1)}\right)_{45}, \end{aligned} \tag{4.2}$$

where $(P_1^{(1)})_{ij}$ refers to the (i,j)-entry of the matrix $P_1^{(1)}$.

In order to obtain soliton solutions, we set $G = \mathbb{I}$ in (3.4) and provide the solutions of the corresponding Riemann-Hilbert problem (3.4), viz.

$$\begin{aligned} P_1(\lambda) &= \mathbb{I} - \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{v_k \hat{v}_j (M^{-1})_{kj}}{\lambda - \hat{\lambda}_j}, \\ P_2(\lambda) &= \mathbb{I} + \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{v_k \hat{v}_j (M^{-1})_{kj}}{\lambda - \lambda_j}, \end{aligned} \tag{4.3}$$

where M is the $N \times N$ matrix with the entries

$$M_{kj} = \frac{\hat{v}_k v_j}{\lambda_j - \hat{\lambda}_k}.$$

The Eqs. (4.3) yield

$$P_1^{(1)} = -\sum_{k=1}^{N}\sum_{j=1}^{N} v_k \hat{v}_j (M^{-1})_{kj},$$

where $(M^{-1})_{kj}$ denotes the (k,j)-entry of the inverse matrix M^{-1}. Using nonzero vectors $v_{k,0} = (\alpha_k, \beta_k, \tau_k, \zeta_k, \gamma_k)^T$ and $\theta_k = -i(\lambda_k x + 2\lambda_k^2 t)$, we generate the vectors

$$\begin{aligned}
v_k &= e^{\theta_k \Lambda} v_{k,0} = \begin{pmatrix} e^{\theta_k} & 0 & 0 & 0 & 0 \\ 0 & e^{\theta_k} & 0 & 0 & 0 \\ 0 & 0 & e^{\theta_k} & 0 & 0 \\ 0 & 0 & 0 & e^{\theta_k} & 0 \\ 0 & 0 & 0 & 0 & e^{-\theta_k} \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \\ \tau_k \\ \zeta_k \\ \gamma_k \end{pmatrix} = \begin{pmatrix} \alpha_k e^{\theta_k} \\ \beta_k e^{\theta_k} \\ \tau_k e^{\theta_k} \\ \zeta_k e^{\theta_k} \\ \gamma_k e^{-\theta_k} \end{pmatrix}, \\
\hat{v}_j &= v_j^{\dagger}(\lambda_j) B \\
&= \left(\alpha_j^* e^{\theta_j^*}, \beta_j^* e^{\theta_j^*}, \tau_j^* e^{\theta_j^*}, \zeta_j^* e^{\theta_j^*}, \gamma_j^* e^{-\theta_j^*}\right) \begin{pmatrix} a_{11} & a_{21}^* & a_{31}^* & a_{41}^* & 0 \\ a_{21} & a_{22} & a_{32}^* & a_{42}^* & 0 \\ a_{31} & a_{32} & a_{33} & a_{43}^* & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \big(a_{11}\alpha_j^* e^{\theta_j^*} + a_{21}\beta_j^* e^{\theta_j^*} + a_{31}\tau_j^* e^{\theta_j^*} + a_{41}\zeta_j^* e^{\theta_j^*}, \\
&\quad a_{21}^*\alpha_j^* e^{\theta_j^*} + a_{22}\beta_j^* e^{\theta_j^*} + a_{32}\tau_j^* e^{\theta_j^*} + a_{42}\zeta_j^* e^{\theta_j^*}, \\
&\quad a_{31}^*\alpha_j^* e^{\theta_j^*} + a_{32}^*\beta_j^* e^{\theta_j^*} + a_{33}\tau_j^* e^{\theta_j^*} + a_{43}\zeta_j^* e^{\theta_j^*}, \\
&\quad a_{41}^*\alpha_j^* e^{\theta_j^*} + a_{42}^*\beta_j^* e^{\theta_j^*} + a_{43}^*\tau_j^* e^{\theta_j^*} + a_{44}\zeta_j^* e^{\theta_j^*}, -\gamma_j^* e^{-\theta_j^*}\big).
\end{aligned}$$

Obviously

$$v_k \hat{v}_j = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{pmatrix}, \tag{4.4}$$

$$\begin{aligned}
\hat{v}_k v_j = \big(&a_{11}\alpha_k^*\alpha_j + a_{21}\beta_k^*\alpha_j + a_{31}\tau_k^*\alpha_j + a_{41}\zeta_k^*\alpha_j \\
&+ a_{21}^*\alpha_k^*\beta_j + a_{22}\beta_k^*\beta_j + a_{32}\tau_k^*\beta_j + a_{42}\zeta_k^*\beta_j \\
&+ a_{31}^*\alpha_k^*\tau_j + a_{32}^*\beta_k^*\tau_j + a_{33}\tau_k^*\tau_j + a_{43}\zeta_k^*\tau_j \\
&+ a_{41}^*\alpha_k^*\zeta_j + a_{42}^*\beta_k^*\zeta_j + a_{43}^*\tau_k^*\zeta_j + a_{44}\zeta_k^*\zeta_j\big)e^{\theta_k^* + \theta_j} \\
&- \gamma_k^*\gamma_j e^{-\theta_k^* - \theta_j},
\end{aligned}$$

$$b_{15} = -\alpha_k \gamma_j^* e^{\theta_k - \theta_j^*}, \quad b_{25} = -\beta_k \gamma_j^* e^{\theta_k - \theta_j^*}, \quad b_{35} = -\tau_k \gamma_j^* e^{\theta_k - \theta_j^*},$$
$$b_{45} = -\zeta_k \gamma_j^* e^{\theta_k - \theta_j^*}, \quad b_{55} = -\gamma_k \gamma_j^* e^{\theta_k - \theta_j^*}.$$

It should be noted that the values of b_{ij}, $i \le 5, j \le 4$ are not used in the calculation of the solution, so we will omit them for convenience.

Previous considerations show that general N-soliton solutions of the FCNLS equation (1.4) can be written in the form

$$\begin{aligned} q_1 &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj}, \quad q_2 = 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \beta_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj}, \\ q_3 &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \tau_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj}, \quad q_4 = 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \zeta_k \gamma_j^* e^{\theta_k - \theta_j^*} (M^{-1})_{kj}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} M_{kj} = \frac{1}{\lambda_j - \hat{\lambda}_k} \big[& \big(a_{11} \alpha_k^* \alpha_j + a_{21} \beta_k^* \alpha_j + a_{31} \tau_k^* \alpha_j + a_{41} \zeta_k^* \alpha_j \\ & + a_{21}^* \alpha_k^* \beta_j + a_{22} \beta_k^* \beta_j + a_{32} \tau_k^* \beta_j + a_{42} \zeta_k^* \beta_j \\ & + a_{31}^* \alpha_k^* \tau_j + a_{32}^* \beta_k^* \tau_j + a_{33} \tau_k^* \tau_j + a_{43} \zeta_k^* \tau_j \\ & + a_{41}^* \alpha_k^* \zeta_j + a_{42}^* \beta_k^* \zeta_j + a_{43}^* \tau_k^* \zeta_j + a_{44} \zeta_k^* \zeta_j \big) e^{\theta_k^* + \theta_j} \\ & - \gamma_k^* \gamma_j e^{-\theta_k^* - \theta_j} \big], \quad 1 \le k, j \le N. \end{aligned}$$

Let us introduce the matrices

$$F = \begin{pmatrix} 0 & \alpha_1 e^{\theta_1} & \alpha_2 e^{\theta_2} & \dots & \alpha_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix},$$

$$G = \begin{pmatrix} 0 & \beta_1 e^{\theta_1} & \beta_2 e^{\theta_2} & \dots & \beta_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & \tau_1 e^{\theta_1} & \tau_2 e^{\theta_2} & \dots & \tau_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix},$$

$$K=\begin{pmatrix} 0 & \zeta_1 e^{\theta_1} & \zeta_2 e^{\theta_2} & \dots & \zeta_N e^{\theta_N} \\ \gamma_1^* e^{-\theta_1^*} & M_{11} & M_{12} & \dots & M_{1N} \\ \gamma_2^* e^{-\theta_2^*} & M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_N^* e^{-\theta_N^*} & M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix}$$

needed for the representation of general N-soliton solutions. In particular, we have

$$q_1=-2i\frac{\det F}{\det M},\quad q_2=-2i\frac{\det G}{\det M},\quad q_3=-2i\frac{\det H}{\det M},\quad q_4=-2i\frac{\det K}{\det M}. \tag{4.6}$$

The rest of this section is devoted to the soliton solutions for $N=1$ and $N=2$. If $N=1$, one-soliton solutions have the form

$$\begin{aligned} q_1&=2i\frac{\alpha_1\gamma_1^* e^{\theta_1-\theta_1^*}}{M_{11}},\quad q_2=2i\frac{\beta_1\gamma_1^* e^{\theta_1-\theta_1^*}}{M_{11}},\\ q_3&=2i\frac{\tau_1\gamma_1^* e^{\theta_1-\theta_1^*}}{M_{11}},\quad q_4=2i\frac{\zeta_1\gamma_1^* e^{\theta_1-\theta_1^*}}{M_{11}}, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} M_{11}=\frac{1}{\lambda_1-\hat{\lambda}_1}\big[&\big(a_{11}|\alpha_1|^2+a_{22}|\beta_1|^2+a_{33}|\tau_1|^2+a_{44}|\zeta_1|^2\\ &+a_{21}\beta_1^*\alpha_1+a_{31}\tau_1^*\alpha_1+a_{41}\zeta_1^*\alpha_1\\ &+a_{21}^*\alpha_1^*\beta_1+a_{32}\tau_1^*\beta_1+a_{42}\zeta_1^*\beta_1\\ &+a_{31}^*\alpha_1^*\tau_1+a_{32}^*\beta_1^*\tau_1+a_{43}\zeta_1^*\tau_1\\ &+a_{41}^*\alpha_1^*\zeta_1+a_{42}^*\beta_1^*\zeta_1+a_{43}^*\tau_1^*\zeta_1\big)e^{\theta_1+\theta_1^*}-|\gamma_1|^2e^{-(\theta_1+\theta_1^*)}\big], \end{aligned}$$

and $\theta_1=-i(\lambda_1 x+2\lambda_1^2 t)$. Furthermore, fixing $\gamma_1=1$, $\lambda_1=n_1+im_1$ and setting

$$\begin{aligned} -\big(&a_{11}|\alpha_1|^2+a_{22}|\beta_1|^2+a_{33}|\tau_1|^2+a_{44}|\zeta_1|^2\\ &+a_{21}\beta_1^*\alpha_1+a_{31}\tau_1^*\alpha_1+a_{41}\zeta_1^*\alpha_1\\ &+a_{21}^*\alpha_1^*\beta_1+a_{32}\tau_1^*\beta_1+a_{42}\zeta_1^*\beta_1\\ &+a_{31}^*\alpha_1^*\tau_1+a_{32}^*\beta_1^*\tau_1+a_{43}\zeta_1^*\tau_1\\ &+a_{41}^*\alpha_1^*\zeta_1+a_{42}^*\beta_1^*\zeta_1+a_{43}^*\tau_1^*\zeta_1\big)=e^{2\xi_1}, \end{aligned} \tag{4.8}$$

we write relations (4.7) as

$$\begin{aligned} q_1&=2m_1\alpha_1\gamma_1^* e^{-\xi_1}e^{\theta_1-\theta_1^*}\operatorname{sech}\big(\theta_1^*+\theta_1+\xi_1\big),\\ q_2&=2m_1\beta_1\gamma_1^* e^{-\xi_1}e^{\theta_1-\theta_1^*}\operatorname{sech}\big(\theta_1^*+\theta_1+\xi_1\big),\\ q_3&=2m_1\tau_1\gamma_1^* e^{-\xi_1}e^{\theta_1-\theta_1^*}\operatorname{sech}\big(\theta_1^*+\theta_1+\xi_1\big),\\ q_4&=2m_1\zeta_1\gamma_1^* e^{-\xi_1}e^{\theta_1-\theta_1^*}\operatorname{sech}\big(\theta_1^*+\theta_1+\xi_1\big). \end{aligned} \tag{4.9}$$

Since $\theta_k = -i(\lambda_k x + 2\lambda_k^2 t)$, we have

$$\theta_1 - \theta_1^* = -2in_1 x - 4in_1^2 t,$$
$$\theta_1 + \theta_1^* = 2m_1 x + 4im_1^2 t + 8n_1 m_1 t,$$

and the one-soliton solutions (4.7) can be written in the form

$$\begin{aligned}
q_1 &= 2m_1\alpha_1\gamma_1^* e^{-\xi_1} e^{-2in_1 x - 4in_1^2 t}\,\mathrm{sech}\big(2m_1 x + 4im_1^2 t + 8n_1 m_1 t + \xi_1\big),\\
q_2 &= 2m_1\beta_1\gamma_1^* e^{-\xi_1} e^{-2in_1 x - 4in_1^2 t}\,\mathrm{sech}\big(2m_1 x + 4im_1^2 t + 8n_1 m_1 t + \xi_1\big),\\
q_3 &= 2m_1\tau_1\gamma_1^* e^{-\xi_1} e^{-2in_1 x - 4in_1^2 t}\,\mathrm{sech}\big(2m_1 x + 4im_1^2 t + 8n_1 m_1 t + \xi_1\big),\\
q_4 &= 2m_1\zeta_1\gamma_1^* e^{-\xi_1} e^{-2in_1 x - 4in_1^2 t}\,\mathrm{sech}\big(2m_1 x + 4im_1^2 t + 8n_1 m_1 t + \xi_1\big).
\end{aligned} \tag{4.10}$$

According to the Eq. (4.10), the solution q_1 has the peak amplitude

$$\Upsilon_1 = 2m_1|\alpha_1||\gamma_1^*|e^{-\xi_1},$$

and the velocity

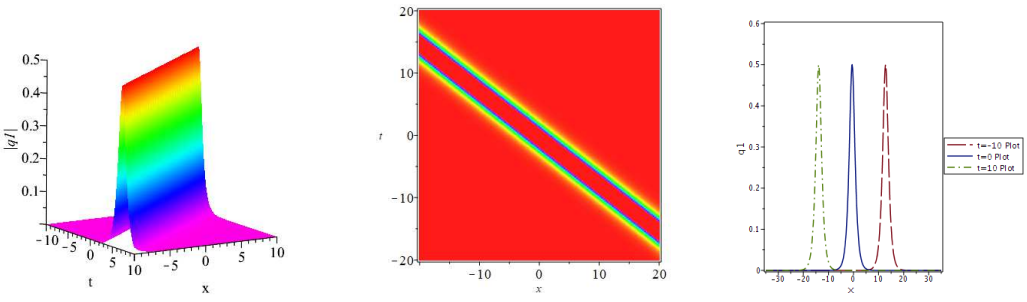
$$\varpi_1 = 2im_1 + 4n_1.$$

The peak amplitudes and the velocities of q_2, q_3 and q_4 have similar representations — viz.

$$\begin{aligned}
&\Upsilon_2 = 2m_1|\beta_1||\gamma_1^*|e^{-\xi_1}, \quad \Upsilon_3 = 2m_1|\tau_1||\gamma_1^*|e^{-\xi_1}, \quad \Upsilon_4 = 2m_1|\zeta_1||\gamma_1^*|e^{-\xi_1},\\
&\varpi_2 = 2im_1 + 4n_1, \qquad \varpi_3 = 2im_1 + 4n_1, \qquad \varpi_4 = 2im_1 + 4n_1.
\end{aligned}$$

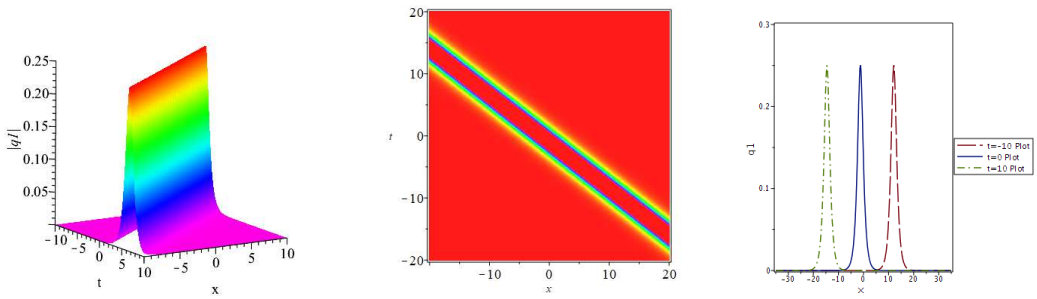
The functions $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ depend on the imaginary part of λ_1 and $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ depend on the real and the imaginary parts of λ_1. Figs. 1 and 2 show the localised structure and the dynamic behavior of one-soliton solutions. Solutions q_2, q_3 and q_4 exhibit the same behaviour and we do not show them here.

For single soliton solutions (4.10), we take $a_{ij} = -1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$. Moreover, let $\alpha_1 = \tau_1 = \beta_1 = \zeta_1 = 1/2 - \sqrt{2}/2i, \gamma_1 = 1, n_1 = 1/3, m_1 = 1/2$. The shape



(a) (b) (c)

Figure 1: One-soliton solution q_1 for $\gamma_1 = 1, n_1 = 1/3, m_1 = 1/2, a_{ii} = -1$ and $a_{ij} = 0$ if $i \neq j, i, j = 1, 2, 3, 4$. (a) Three dimensional plot at $t = 0$. (b) Density plot. (c) Wave propagation along x-axis.



(a) (b) (c)

Figure 2: One-soliton solution q_1 for $a_{ij} = -1$ $(i, j = 1, 2, 3, 4), \gamma_1 = 1, n_1 = 1/3$ and $m_1 = 1/2$. (a) Three dimensional plot at $t = 0$. (b) Density plot. (c) Wave propagation along x-axis.

of the solution q_1 is shown in Fig. 1. Fig. 2 shows solutions in the case where $a_{ij} = -1$ if $i \neq j$ and other parameters are the same as in Fig. 1. Note that now the amplitude of the soliton is smaller than in Fig. 1.

For $N = 2$, the two-soliton solutions can be written in the form

$$\begin{aligned}
q_1 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}}\big(-\alpha_1\gamma_1^* e^{\theta_1-\theta_1^*}M_{22} + \alpha_1\gamma_2^* e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \alpha_2\gamma_1^* e^{\theta_2-\theta_1^*}M_{21} - \alpha_2\gamma_2^* e^{\theta_2-\theta_2^*}M_{11}\big), \\
q_2 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}}\big(-\beta_1\gamma_1^* e^{\theta_1-\theta_1^*}M_{22} + \beta_1\gamma_2^* e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \beta_2\gamma_1^* e^{\theta_2-\theta_1^*}M_{21} - \beta_2\gamma_2^* e^{\theta_2-\theta_2^*}M_{11}\big), \\
q_3 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}}\big(-\tau_1\gamma_1^* e^{\theta_1-\theta_1^*}M_{22} + \tau_1\gamma_2^* e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \tau_2\gamma_1^* e^{\theta_2-\theta_1^*}M_{21} - \tau_2\gamma_2^* e^{\theta_2-\theta_2^*}M_{11}\big), \\
q_4 &= \frac{-2i}{M_{11}M_{22} - M_{12}M_{21}}\big(-\zeta_1\gamma_1^* e^{\theta_1-\theta_1^*}M_{22} + \zeta_1\gamma_2^* e^{\theta_1-\theta_2^*}M_{12} \\
&\quad + \zeta_2\gamma_1^* e^{\theta_2-\theta_1^*}M_{21} - \zeta_2\gamma_2^* e^{\theta_2-\theta_2^*}M_{11}\big),
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
M_{11} = \frac{1}{\lambda_1 - \lambda_1^*}\big[&\big(a_{11}|\alpha_1|^2 + a_{22}|\beta_1|^2 + a_{33}|\tau_1|^2 + a_{44}|\zeta_1|^2 \\
&+ a_{21}\beta_1^*\alpha_1 + a_{31}\tau_1^*\alpha_1 + a_{41}\zeta_1^*\alpha_1 \\
&+ a_{21}^*\alpha_1^*\beta_1 + a_{32}\tau_1^*\beta_1 + a_{42}\zeta_1^*\beta_1 \\
&+ a_{31}^*\alpha_1^*\tau_1 + a_{32}^*\beta_1^*\tau_1 + a_{43}\zeta_1^*\tau_1 \\
&+ a_{41}^*\alpha_1^*\zeta_1 + a_{42}^*\beta_1^*\zeta_1 + a_{43}^*\tau_1^*\zeta_1\big)e^{\theta_1^*+\theta_1} - |\gamma_1|^2 e^{-(\theta_1^*+\theta_1)}\big], \\
M_{12} = \frac{1}{\lambda_2 - \lambda_1^*}\big[&\big(a_{11}\alpha_1^*\alpha_2 + a_{21}\beta_1^*\alpha_2 + a_{31}\tau_1^*\alpha_2 + a_{41}\zeta_1^*\alpha_2
\end{aligned}$$

$$
\begin{aligned}
&+a_{21}^*\alpha_1^*\beta_2+a_{22}\beta_1^*\beta_2+a_{32}\tau_1^*\beta_2+a_{42}\zeta_1^*\beta_2\\
&+a_{31}^*\alpha_1^*\tau_2+a_{32}^*\beta_1^*\tau_2+a_{33}\tau_1^*\tau_2+a_{43}\zeta_1^*\tau_2\\
&+a_{41}^*\alpha_1^*\zeta_2+a_{42}^*\beta_1^*\zeta_2+a_{43}^*\tau_1^*\zeta_2+a_{44}\zeta_1^*\zeta_2\big)e^{\theta_1^*+\theta_2}-\gamma_1^*\gamma_2e^{-\theta_1^*-\theta_2}\big],\\
M_{21}=\frac{1}{\lambda_1-\lambda_2^*}\big[&\big(a_{11}\alpha_2^*\alpha_1+a_{21}\beta_2^*\alpha_1+a_{31}\tau_2^*\alpha_1+a_{41}\zeta_2^*\alpha_1\\
&+a_{21}^*\alpha_2^*\beta_1+a_{22}\beta_2^*\beta_1+a_{32}\tau_2^*\beta_1+a_{42}\zeta_2^*\beta_1\\
&+a_{31}^*\alpha_2^*\tau_1+a_{32}^*\beta_2^*\tau_1+a_{33}\tau_2^*\tau_1+a_{43}\zeta_2^*\tau_1\\
&+a_{41}^*\alpha_2^*\zeta_1+a_{42}^*\beta_2^*\zeta_1+a_{43}^*\tau_2^*\zeta_1+a_{44}\zeta_2^*\zeta_1\big)e^{\theta_2^*+\theta_1}-\gamma_2^*\gamma_1e^{-\theta_2^*-\theta_1}\big],\\
M_{22}=\frac{1}{\lambda_2-\lambda_2^*}\big[&\big(a_{11}|\alpha_2|^2+a_{22}|\beta_2|^2+a_{33}|\tau_2|^2+a_{44}|\zeta_2|^2\\
&+a_{21}\beta_2^*\alpha_2+a_{31}\tau_2^*\alpha_2+a_{41}\zeta_2^*\alpha_2\\
&+a_{21}^*\alpha_2^*\beta_2+a_{32}\tau_2^*\beta_2+a_{42}\zeta_2^*\beta_2\\
&+a_{31}^*\alpha_2^*\tau_2+a_{32}^*\beta_2^*\tau_2+a_{43}\zeta_2^*\tau_2\\
&+a_{41}^*\alpha_2^*\zeta_2+a_{42}^*\beta_2^*\zeta_2+a_{43}^*\tau_2^*\zeta_2\big)e^{\theta_2^*+\theta_2}-|\gamma_2|^2e^{-(\theta_2^*+\theta_2)}\big],
\end{aligned}
$$

and

$$
\theta_1=-i\big(\lambda_1x+2\lambda_1^2t\big),\quad \theta_2=-i\big(\lambda_2x+2\lambda_2^2t\big),\quad \lambda_1=n_1+im_1,\quad \lambda_2=n_2+im_2.
$$

Choosing $\gamma_1=\gamma_2=1,\alpha_1=\alpha_2,\beta_1=\beta_2,\tau_1=\tau_2,\zeta_1=\zeta_2$ and using the notation

$$
\begin{aligned}
e^{2\xi_1}:=-\big(&a_{11}|\alpha_1|^2+a_{22}|\beta_1|^2+a_{33}|\tau_1|^2+a_{44}|\zeta_1|^2\\
&+a_{21}\beta_1^*\alpha_1+a_{31}\tau_1^*\alpha_1+a_{41}\zeta_1^*\alpha_1\\
&+a_{21}^*\alpha_1^*\beta_1+a_{32}\tau_1^*\beta_1+a_{42}\zeta_1^*\beta_1\\
&+a_{31}^*\alpha_1^*\tau_1+a_{32}^*\beta_1^*\tau_1+a_{43}\zeta_1^*\tau_1\\
&+a_{41}^*\alpha_1^*\zeta_1+a_{42}^*\beta_1^*\zeta_1+a_{43}^*\tau_1^*\zeta_1\big),
\end{aligned}
$$

we write two-soliton solution (4.11) as

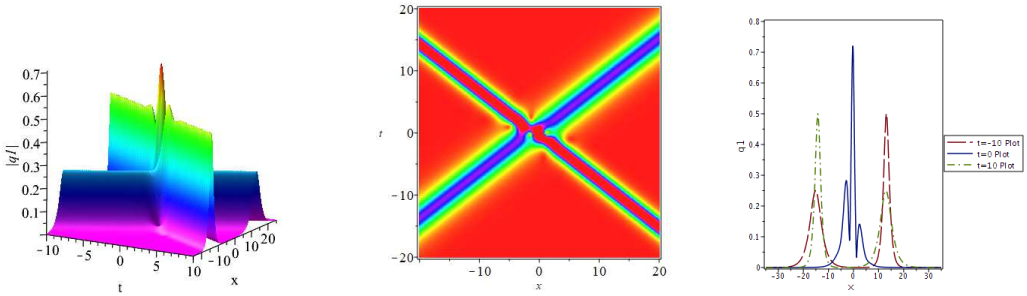
$$
\begin{aligned}
q_1&=\frac{-2i}{M_{11}M_{22}-M_{12}M_{21}}\big(-\alpha_1e^{\theta_1-\theta_1^*}M_{22}+\alpha_1e^{\theta_1-\theta_2^*}M_{12}\\
&\quad+\alpha_2e^{\theta_2-\theta_1^*}M_{21}-\alpha_2e^{\theta_2-\theta_2^*}M_{11}\big),\\
q_2&=\frac{-2i}{M_{11}M_{22}-M_{12}M_{21}}\big(-\beta_1e^{\theta_1-\theta_1^*}M_{22}+\beta_1e^{\theta_1-\theta_2^*}M_{12}\\
&\quad+\beta_2e^{\theta_2-\theta_1^*}M_{21}-\beta_2e^{\theta_2-\theta_2^*}M_{11}\big),\\
q_3&=\frac{-2i}{M_{11}M_{22}-M_{12}M_{21}}\big(-\tau_1e^{\theta_1-\theta_1^*}M_{22}+\tau_1e^{\theta_1-\theta_2^*}M_{12}\\
&\quad+\tau_2e^{\theta_2-\theta_1^*}M_{21}-\tau_2e^{\theta_2-\theta_2^*}M_{11}\big),\\
q_4&=\frac{-2i}{M_{11}M_{22}-M_{12}M_{21}}\big(-\zeta_1e^{\theta_1-\theta_1^*}M_{22}+\zeta_1e^{\theta_1-\theta_2^*}M_{12}\\
&\quad+\zeta_2e^{\theta_2-\theta_1^*}M_{21}-\zeta_2e^{\theta_2-\theta_2^*}M_{11}\big),
\end{aligned}
$$

where

$$M_{11} = \frac{-e^{\xi_1}}{im_1}\cosh\left(\theta_1^* + \theta_1 + \xi_1\right),$$
$$M_{12} = \frac{-2e^{\xi_1}}{n_2 - n_1 + i(m_1 + m_2)}\cosh\left(\theta_1^* + \theta_2 + \xi_1\right),$$
$$M_{21} = \frac{-2e^{\xi_1}}{n_1 - n_2 + i(m_1 + m_2)}\cosh\left(\theta_2^* + \theta_1 + \xi_1\right),$$
$$M_{22} = \frac{-e^{\xi_1}}{im_2}\cosh\left(\theta_2^* + \theta_2 + \xi_1\right).$$

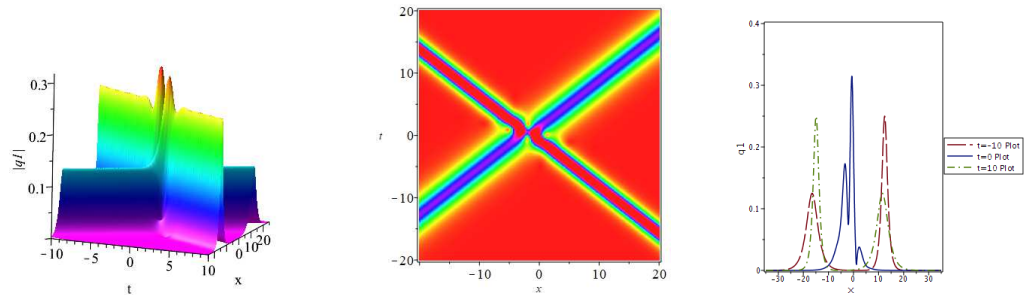
In Fig. 1 we show the elastic collision of two solitons. The solitons maintain their original shape after the collision.

The graphs in Fig. 4 are constructed with the same parameters as for the Fig. 3 with one amendment — viz. here we set $a_{ij} = -1$, $i \neq j$. Fig. 4 shows that the energy of the soliton is not transferred during the collision — i.e. this is an elastic collision. The solutions q_2, q_3 and q_4 exhibit the same behaviour. Therefore, they are not visualised here.



(a) (b) (c)

Figure 3: Two-soliton solution q_1 for $a_{ii} = -1$ and $a_{ij} = 0$ if $i \neq j, i, j = 1, 2, 3, 4, \alpha_1 = \tau_1 = \beta_1 = \zeta_1 = 1/2 - \sqrt{2}/2i, \gamma_1 = \gamma_2 = 1, \alpha_1 = \alpha_2, \beta_1 = \beta_2, \tau_1 = \tau_2, \zeta_1 = \zeta_2 n_1 = -1/3, n_2 = 1/3, m_1 = 0.25$ and $m_2 = 0.5$. (a) Three dimensional plot at $t = 0$. (b) Density plot. (c) Wave propagation along x-axis.



(a) (b) (c)

Figure 4: Two-soliton solution q_1 for $a_{ij} = -1$ $(i, j = 1, 2, 3, 4), n_1 = -1/3, n_2 = 1/3, m_1 = 0.25, m_2 = 0.5$. (a) Three dimensional plot at $t = 0$. (b) Density plot. (c) Wave propagation along x-axis.

Changing values of the parameters $\beta_1, \alpha_2, \beta_2, \zeta_2$ used for the graphs in Fig. 3 and holding other parameters unchanged, we arrive at another collision type — cf. Fig. 5. Before the collision, the energy of the right soliton is higher than of the left ones. However, after the collision the right soliton reaches the left one and the energy diminishes rapidly. Simultaneously, the soliton from the left reaches the right one and the energy increases rapidly. Thus during this collision the energy transfers from one soliton to the other. Note that similar effect is noted in [22, 23].

The graphs in Fig. 6 are constructed mainly with the same parameters as in Fig. 5, but $a_{ij} = -1, i \neq j$. The corresponding soliton collision shown in Figs. 6(a), 6(d) and 6(f) is also inelastic and an energy transfer occurs during the collision. Besides, the soliton has a smaller amplitude than the one in Fig. 5.

Remark 4.1. Considering Figs. 3-6, we note that the parameters $a_{ij}(i \neq j)$ influence the amplitude of the solitons. If $\alpha_1 = \alpha_2, \beta_1 = \beta_2, \tau_1 = \tau_2, \zeta_1 = \zeta_2$, there is no energy transfer during soliton collision and solitons hold their original shapes — i.e. we have an elastic collision. If $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2, \zeta_1 \neq \zeta_2$, an energy transfer occurs during the collision — i.e. we have an inelastic collision. In both cases, the width of the solitons remain unchanged during the collision.

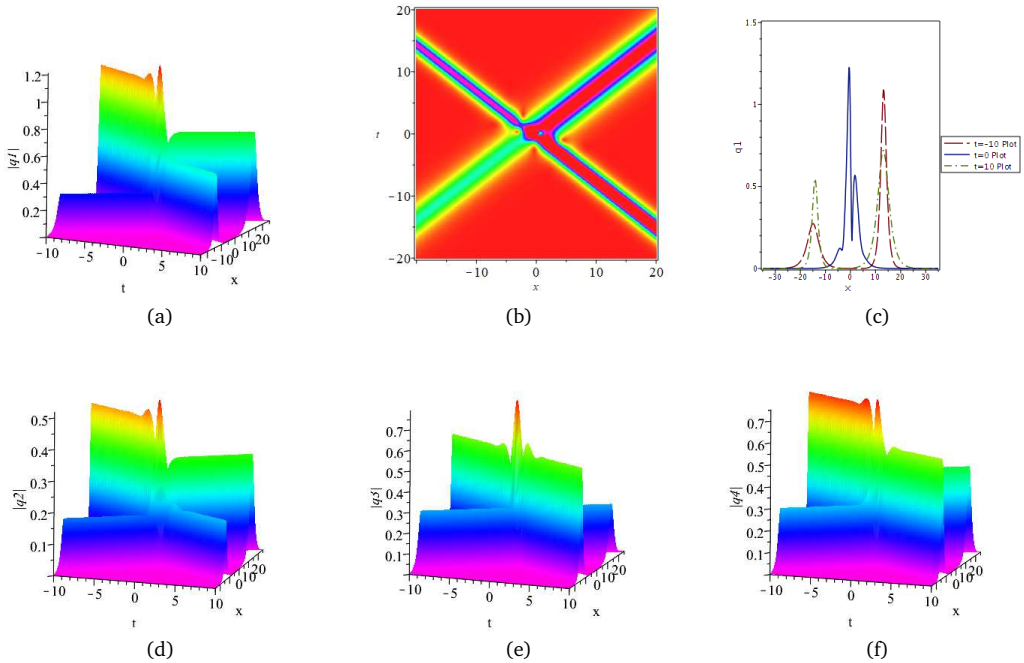


Figure 5: Two-soliton solutions for $a_{ii} = -1$ and $a_{ij} = 0$ if $i \neq j, i, j = 1, 2, 3, 4, \alpha_1 = \tau_1 = \zeta_1 = 1/2 - \sqrt{2}/2i, \beta_1 = 1/2, \gamma_1 = \gamma_2 = 1, \alpha_2 = -1/2 + \sqrt{2}/2i, \beta_2 = -0.3, \tau_2 = 1/2 - \sqrt{2}/2i, \zeta_2 = 1/2 + \sqrt{2}/2i, n_1 = -1/3, n_2 = 1/3, m_1 = 0.25, m_2 = 0.5$. (a) Three dimensional plot at $t = 0$ for q_1. (b) Density plot for q_1. (c) Wave propagation along x-axis for q_1. (d) Three dimensional plot at $t = 0$ for q_2. (e) Three dimensional plot at $t = 0$ for q_3. (f) Three dimensional plot at $t = 0$ for q_4.

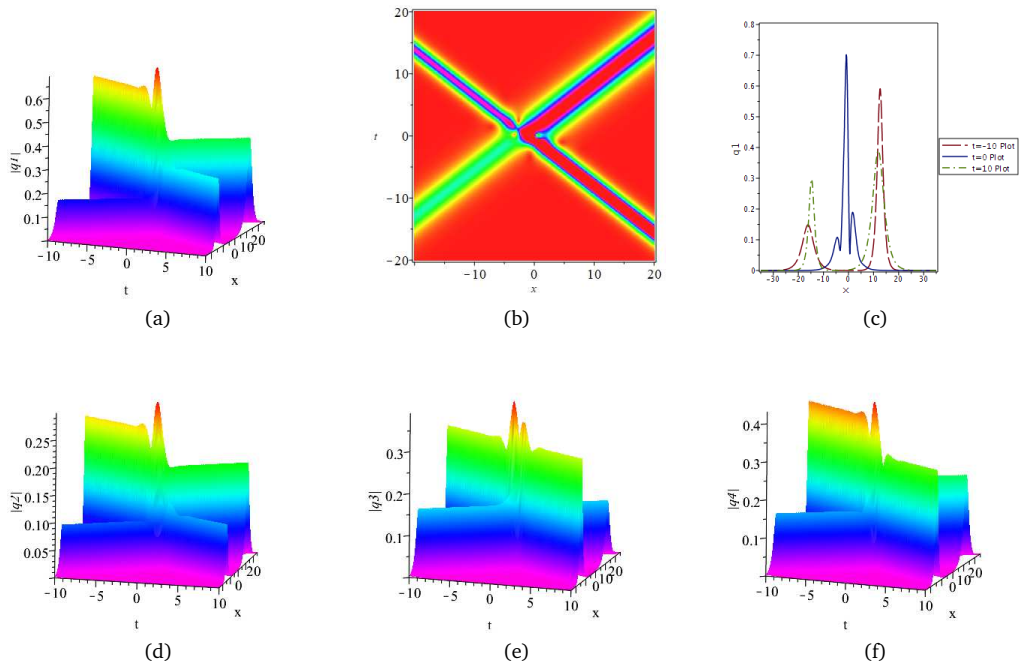


Figure 6: Two-soliton solutions for $a_{ij} = -1\,(i,j = 1,2,3,4), n_1 = -1/3, n_2 = 1/3, m_1 = 0.25, m_2 = 0.5$. (a) Three dimensional plot at $t = 0$ for q_1. (b) Density plot for q_1. (c) Wave propagation along the x-axis time for q_1. (d) Three dimensional plot at $t = 0$ for q_2. (e) Three dimensional plot at $t = 0$ for q_3. (f) Three dimensional plot at $t = 0$ for q_4.

5. Conclusions

We employ the Riemann-Hilbert approach in the study of a FCNLS equation (1.4) associated with a 5×5 Lax pair. In particular, we consider a spectral problem and use the Jost functions in order to derive a Riemann-Hilbert problem for the equation mentioned. We note a symmetry relation for the scattering matrix and study the temporal and spatial evolution of the scattering data, solve the corresponding Riemann-Hilbert problem without reflection and obtain N-soliton solutions of the FCNLS equation. Finally, we consider some solutions in the case if $N = 1$ and $N = 2$, and describe the local structure and dynamic behavior of one-and two-soliton solutions using their graphic representations.

Acknowledgments

The authors would like to thank the editors and the referees for their valuable comments and suggestions. This work was supported by the Fundamental Research Fund for the Central Universities under the Grant No. 2019ZDPY07.

References

[1] M.J. Ablowitz and P.A. Clarkson, *Solitons: Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press (1991).

[2] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag (1989).

[3] A.S. Fokas and J. Lenells, *The unified method: I Nonlinearizable problems on the half-line*, J. Phys. A **45**, 195201 (2012).

[4] X. Geng and J. Wu, *Riemann-Hilbert approachand N-soliton solutions for a generalized Sasa-Satsuma equation*, Wave Motion **60**, 62–72 (2016).

[5] B. Guo and L. Ling, *Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation*, J. Math. Phys. **53**, 133–3966 (2012).

[6] R. Hirota, *Direct Methods in Soliton Theory*, Springer (2004).

[7] T. Kanna, M. Lakshmanan, P.T. Dinda and N. Akhmediev, *Soliton collisions with shape change by intensity redistribution in mixed coupled nonlinear Schrödinger equations*, Phys. Rev. E **73**, 026604 (2006).

[8] J. Lenells, *Initial-boundary value problems for integrable evolution equations with* 3×3 *Lax pairs*, Phys. D **241**, 857 (2012).

[9] Z.Q. Li, S.F. Tian and J.J. Yang, *Riemann-Hilbert approach and soliton solutions for the higher order dispersive nonlinear Schrödinger equation with nonzero boundary conditions*, arXiv:1911.01624 (2019).

[10] Z.Q. Li, S.F. Tian, W.Q. Peng and J.J. Yang, *Inverse scattering transform and soliton classification of the higher order nonlinear Schrödinger-Maxwell-Bloch equations*, Theor. Math. Phys. **203**, 709–725 (2020).

[11] N. Liu and B. Guo, *Long-time asymptotics for the Sasa-Satsuma equation via nonlinear steepest descent method*, J. Math. Phys. **60**, 011504 (2019).

[12] W.X. Ma, *Riemann-Hilbert problems of a six-component fourth-order AKNS system and its soliton solutions*, Comput. Appl. Math. **37**, 6359–6375 (2018).

[13] W.X. Ma, *The inverse scattering transform and soliton solutions of a combined modified Korteweg-de Vries equation*, J. Math. Anal. Appl. **471**, 796–811 (2019).

[14] V.B. Matveev and M.A. Salle, *Darboux Transformation and Solitons*, Springer (1991).

[15] Y. Nogami and C.S. Warke, *Soliton solutions of multicomponent nonlinear Schrödinger equation*, Phys. Lett. A **59**, 251253 (1976).

[16] W.Q. Peng, S.F. Tian, X.B. Wang and T.T. Zhang, *Riemann-Hilbert method and multi-soliton solutions for three-component coupled nonlinear Schrödinger equations*, J. Geom. Phys. **146**, 103508 (2019).

[17] W.Q. Peng, S.F. Tian and T.T. Zhang, *Initial Value Problem for the Pair Transition Coupled Nonlinear Schrödinger Equations via the Riemann-Hilbert Method*, Complex Anal. Operator Theory **14**, 38 (2020).

[18] S.F. Tian, *Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method*, J. Differ. Equ. **262**, 506–558 (2017).

[19] D.S. Wang, B.L. Guo and X.L. Wang, *Long-time asymptotics of the focusing Kundu-Eckhaus equation with nonzero boundary conditions*, J. Differ. Equ. **266**, 5209–5253 (2019).

[20] D.S. Wang, S.W. Song, B. Xiong and W. M. Liu, *Quantized vortices in a rotating Bose-Einstein condensate with spatiotemporally modulated interaction*, Phys. Rev. A **84**, 053607 (2011).

[21] D.S. Wang and X.L. Wang, *Long-time asymptotics and the bright N-soliton solutions of the Kundu-Eckhaus equation via the Riemann-Hilbert approach*, Nonlinear Anal. Real World Appl. **41**, 334–361 (2018).

[22] D.S. Wang, D. Zhang and J. Yang, *Integrable properties of the general coupled nonlinear*

Schrödinger equations, J. Math. Phys. **51**, 023510 (2010).
[23] M. Wang, W.R. Shan, X. Lü, Y.S. Xue, Z.Q. Lin and B. Tian, *Soliton collision in a general coupled nonlinear Schrödinger system via symbolic computation*, Appl. Math. Comput. **219**, 11258–11264 (2013).
[24] L. Wen and E.G. Fan, *The Riemann-Hilbert approach to focusing Kundu-Eckhaus equation with nonzero boundary conditions*, arXiv:1910.08921 (2019).
[25] X. Wu, S.F. Tian and J.J. Yang, *Riemann-Hilbert approach and n-soliton solutions for three-component coupled Hirota equations*, East Asian J. Appl. Math. **10**, 717–731 (2020).
[26] B.Q. Xia and A.S. Fokas, *Initial-boundary value problems associated with the Ablowitz-Ladik system*, arXiv: 1703.01687 (2017).
[27] J. Xu and E.G. Fan, *Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: without solitons*, J. Differ. Equ. **259**, 1098–1148 (2015).
[28] L. Xu, D.S. Wang, X.Y. Wen and Y.L. Jiang, *Exotic localized vector waves in a two-component nonlinear wave system*, J. Nonlinear Sci. **30**, 537–564 (2020).
[29] T.Y. Xu, S.F. Tian and W.Q. Peng, *Riemann-Hilbert approach for multisoliton solutions of generalized coupled fourth-order nonlinear Schrödinger equations*, Math. Methods Appl. Sci **43**, 865–880 (2020).
[30] Z.Y. Yan, *An initial-boundary value problem for the integrable spin-1 Gross- Pitaevskii equations with a* 4×4 *Lax pair on the half-line*, Chaos **27**, 053117 (2017).
[31] Z.Y. Yan, *An initial-boundary value problem of the general three-component nonlinear Schrödinger equation with a* 4×4 *Lax pair on a finite interval*, arXiv:1704.08561 (2017).
[32] J.J. Yang and S.F. Tian, *Riemann-Hilbert problem for the modified Landau-Lifshitz equation with nonzero boundary conditions*, arXiv:1909.11263 (2019).
[33] J.J. Yang, S.F. Tian, W.Q. Peng and T.T. Zhang, *The N-coupled higher-order nonlinear Schrödinger equation: Riemann-Hilbert problem and multi-soliton solutions*, Math. Meth. Appl. Sci. **43**, 2458–2472 (2020).
[34] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaevskii, *The Theory of Solitons: The Inverse Scattering Method*, Consultants Bureau (1984).
[35] G. Zhang, S. Chen and Z. Yan, *Focusing and defocusing Hirota equations with nonzero boundary conditions: Inverse scattering transforms and soliton solutions*, Commun. Nonlinear Sci. Numer. Simul. **80**, 10492 (2020).
[36] X.E. Zhang and Y. Chen, *Inverse scattering transformation for generalized nonlinear Schrödinger equation*, Appl. Math. Lett. **98**, 306–313 (2019).
[37] Y.S. Zhang, Y. Cheng and J.S. He, *Riemann-Hilbert method and N-soliton for two-component Gerdjikov-Ivanov equation*, J. Nonlinear Math. Phys. **24**, 210–223 (2017).
[38] J. Zhu, L. Wang and X. Geng, *Riemann-Hilbert approach to TD equation with nonzero boundary condition*, Front. Math. China. **13**, 1245–1265 (2018).