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Application of the Nonlinear Steepest Descent Method to the Coupled Sasa-Satsuma Equation

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Abstract. We use spectral analysis to reduce Cauchy problem for the coupled Sasa-Satsuma equation to a 5×5 matrix Riemann-Hilbert problem. The upper and lower triangular factorisations of the jump matrix and a decomposition of the vector-valued spectral function are given. Applying various transformations related to the Riemann-Hilbert problems and suitable decompositions of the jump contours and the nonlinear steepest descent method, we establish the long-time asymptotics of the problem.

AMS subject classifications: 35Q53, 35B40, 35Q55

Key words: Coupled Sasa-Satsuma equation, nonlinear steepest descent method, long-time asymptotics.

1. Introduction

The Sasa-Satsuma equation

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u\left(|u|^2\right)_x = 0 \tag{1.1}$$

also called the higher-order nonlinear Schrödinger equation, was originally aimed to describe the propagation of pulses in optical fiber [18, 19]. It attracted a considerable attention and has been extensively studied because of significant applications. The inverse scattering method [34] and the Hirota bilinear method [12] were used to obtain N-soliton solution of this equation. On the other hand, by linearising the corresponding spectral operator it was shown that the squared eigenfunctions of the Sasa-Satsuma equation can be represented as the sums of two terms, each of which is a product of Jost and adjoint Jost functions [43]. Akhmedieva *et al.* [2] studied the rogue wave spectra of the Eq. (1.1) and its presence in the spectra of chaotic wave fields produced by the modulation instability. Ling [22] obtained high order solution formulas in the determinant form by using a generalised Darboux transformation and the formal series method. In [44], finite genus solutions

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of the Sasa-Satsuma hierarchy, associated with a 3×3 matrix spectral problem, are obtained by using asymptotic expansions of the Baker-Akhiezer function and its Riemann theta function representation [37]. The Riemann-Hilbert approach, Darboux transformation and Riccati equation are employed in investigating the integrability of multi-coupled nonlinear integrable equations and finding their exact solutions — cf. Refs. [9,11,15,20,21,27,38,41].

Let

$$\mathscr{S}(\mathbb{R})=\left\{f(x)\in C^{\infty}(\mathbb{R}):\sup_{x\in\mathbb{R}}\left|x^{\alpha}\partial^{\beta}f(x)\right|<\infty,\forall\alpha,\beta\in\mathbb{N}\right\}$$

be the Schwartz class. In this work, we use the nonlinear steepest descent method in order to study the long-time asymptotic behavior of the Cauchy problem for the coupled Sasa-Satsuma equation

$$\begin{aligned}&u_t+u_{xxx}+6\left(|u|^2+|v|^2\right)u_x+3u\left(|u|^2+|v|^2\right)_x=0,\\&v_t+v_{xxx}+6\left(|u|^2+|v|^2\right)v_x+3v\left(|u|^2+|v|^2\right)_x=0,\\&u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\end{aligned}\tag{1.2}$$

where $u(x,t)$ and $v(x,t)$ are complex-valued potentials, $u_0(x),v_0(x)\in\mathscr{S}(\mathbb{R})$ and are generic in the sense that the below defined determinant $\det a(k)$ does not vanish in the lower complex half k-plane $\mathbb{C}_-$. The coupled Sasa-Satsuma equation can describe the simultaneous propagation in birefringent or two-mode fibers [32]. In [40], multi-soliton solutions of the coupled Sasa-Satsuma equation are derived by solving a Riemann-Hilbert problem. Besides, infiniteness of conserved quantities of the Eqs. (1.2) is discussed in [33], the Painlevé property in [36], and some other characteristics in [24,28,45]. The Deift-Zhou nonlinear steepest descent method introduced in [7] is aimed to study the long-time asymptotic behavior of solutions for the mKdV equation. The method was subsequently applied to a number of integrable nonlinear evolution equations associated with numerous matrix spectral problems [4–6, 8, 10, 13, 16, 17, 23, 25, 26, 29–31, 35, 42]. However, to the best of author's knowledge, the nonlinear steepest descent method has not been used in the study of long-time asymptotics for integrable equation associated with 5×5 matrix Lax pairs and the aim of this work is to extend the Deift-Zhou method to the Eqs. (1.2) associated with such Lax pairs. The main result of this paper is the following theorem.

Theorem 1.1. *Let $(u(x,t),v(x,t))$ be the solution for the Cauchy problem of the coupled Sasa-Satsuma equation* (1.2) *with $u_0(x)$ and $v_0(x)\in\mathscr{S}(\mathbb{R})$. If $x<0$ and $|x/t|\leq C$, as $t\to\infty$, then the leading asymptotics of $(u(x,t),v(x,t))$ has the form*

$$\begin{aligned}&\big(u(x,t),v(x,t)\big)\\&=-\frac{\nu e^{\pi\nu/2}}{\sqrt{24tk_0\pi}}\Big[\delta_A^2e^{-\pi i/4}\Gamma(-i\nu)\big(\gamma_2(k_0),\gamma_4(k_0)\big)+\delta_A^{-2}e^{\pi i/4}\Gamma(i\nu)\big(\gamma_1^*(k_0),\gamma_3^*(k_0)\big)\Big]\\&\quad+\mathcal{O}\left(c(k_0)t^{-1}\log t\right),\end{aligned}$$

where C is a constant, Γ the Gamma function, $\gamma(k)=(\gamma_1(k),\gamma_2(k),\gamma_3(k),\gamma_4(k))$ the vector-

valued function defined in (2.7)*, c a rapidly decreasing function, and*

$$\begin{aligned}
&\delta_A = e^{\chi(-k_0)-8itk_0^3}(192tk_0^3)^{i\nu/2},\\
&k_0 = \sqrt{\frac{-x}{12t}}, \quad \nu = -\frac{1}{2\pi}\log\left(1+|\gamma(k_0)|^2\right),\\
&\chi(-k_0) = \frac{1}{2\pi i}\int_{-k_0}^{k_0}\log\left(\frac{1+|\gamma(\xi)|^2}{1+|\gamma(k_0)|^2}\right)\frac{d\xi}{\xi+k_0}.
\end{aligned}$$

The outline of this paper is as follows. In Section 2, the inverse scattering method is used to transform the Cauchy problem for the coupled Sasa-Satsuma equation into a matrix Riemann-Hilbert problem. In Section 3, the original matrix Riemann-Hilbert problem is reduced to a model Riemann-Hilbert problem whose solution can be represented via parabolic cylinder functions. Finally, we obtain the long-time asymptotics of the Cauchy problem for the coupled Sasa-Satsuma equation.

2. Inverse Scattering Method and Riemann-Hilbert Problems

The Eq. (1.2) is the compatibility condition for the Lax pair

$$\psi_x = (-ik\sigma + U)\psi, \tag{2.1}$$

$$\psi_t = \left(-4ik^3\sigma + Q\right)\psi, \tag{2.2}$$

where $\psi = \psi(k; x, t)$ is a 5×5 matrix, $\sigma = \mathrm{diag}\{1,1,1,1,-1\}$, and

$$U = \begin{pmatrix} 0_{4\times4} & q \\ -q^\dagger & 0 \end{pmatrix}, \quad q = (u, u^*, v, v^*)^{\mathrm{T}}, \tag{2.3}$$

$$Q = 4k^2U + 2ik\sigma\left(U_x - U^2\right) + 2U^3 - U_{xx} + [U_x, U].$$

Here, $q^\dagger$ refers to the Hermitian conjugate of a matrix q and $*$ is the complex conjugation.

Let $e^\sigma := \mathrm{diag}(e, e, e, e, e^{-1})$. Introducing the matrix eigenfunction

$$\mu(k; x, t) := \psi(k; x, t)e^{ik\sigma x + 4ik^3\sigma t},$$

we rewrite (2.1) as

$$\mu_x = -ik[\sigma, \mu] + U\mu, \tag{2.4}$$

where $[\sigma,\mu] = \sigma\mu - \mu\sigma$. Let $\mu_\pm = (\mu_{\pm L}, \mu_{\pm R})$ be the matrix Jost solutions of (2.4) with the asymptotic conditions $\mu_\pm \to I_{5\times5}$ as $x \to \pm\infty$ obtained from the Volterra integral equation

$$\mu_\pm(k; x, t) = I_{5\times5} + \int_{\pm\infty}^{x} e^{ik\sigma(\xi - x)}U(\xi, t)\mu_\pm(k; \xi, t)e^{-ik\sigma(\xi - x)}\,\mathrm{d}\xi.$$

Note that $\mu_{\pm L}$ denotes each of the first four columns of $\mu_\pm$ and $\mu_{\pm R}$ refers to the last column. It is easily seen that μ_{-L} and μ_{+R} are analytic in the upper complex half-plane $\mathbb{C}_+$ and μ_{+L}

and μ_{-R} are analytic in the lower complex half-plane $\mathbb{C}_-$. Since $\mu_\pm e^{-ik\sigma x-4ik^3\sigma t}$ are the solutions of the spectral problems (2.1) and (2.2), they are linearly related. Therefore, there is a scattering matrix $s(k)$ such that

$$\mu_- e^{-ik\sigma x-4ik^3\sigma t} = \mu_+ e^{-ik\sigma x-4ik^3\sigma t} s(k), \quad k \in \mathbb{R}. \tag{2.5}$$

Noting that the matrix U in (2.3) is traceless, one can show that $\det \mu_\pm = 1$. Combining it with (2.5) yields $\det s(k) = 1$. We also note symmetry properties of U, viz.

$$U^\dagger = -U, \quad \sigma_1 U \sigma_1 = U^*, \quad \sigma_1 = \sigma_1^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows from (2.4) that the Jost solutions $\mu_\pm$ have similar properties — viz.

$$\mu_\pm^\dagger(k^*) = \mu_\pm^{-1}(k), \quad \sigma_1 \mu_\pm^*(-k^*)\sigma_1 = \mu_\pm(k),$$

and using (2.5), we obtain

$$s^\dagger(k^*) = s^{-1}(k), \quad \sigma_1 s^*(-k^*)\sigma_1 = s(k).$$

Consequently,

$$\begin{aligned} &s_{22}^\dagger(k^*) = \det[s_{11}(k)], && s_{11}(k) = \sigma_2 s_{11}^*(-k^*)\sigma_2, \\ &s_{12}^\dagger(k^*) = -s_{21}(k)\operatorname{adj}[s_{11}(k)], && s_{21}^*(-k^*)\sigma_2 = s_{21}(k), \\ &\sigma_2 = \sigma_2^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

with $\operatorname{adj} a$ denoting the adjoint matrix to a. Therefore, the scattering matrix $s(k)$ has the form

$$s(k) = \begin{pmatrix} a(k) & -\operatorname{adj}[a^\dagger(k^*)]b^\dagger(k^*) \\ b(k) & \det[a^\dagger(k^*)] \end{pmatrix},$$

where

$$a(k) = \sigma_2 a^*(-k^*)\sigma_2, \quad b^*(-k^*)\sigma_2 = b(k).$$

If $t = 0$ and $x \to +\infty$, the Eq. (2.5) yields

$$\begin{aligned} s(k) &= \lim_{x\to+\infty} e^{ik\sigma x}\mu_-(k;x,0)e^{-ik\sigma x} \\ &= I_{5\times5} + \int_{-\infty}^{+\infty} e^{ik\sigma\xi}U(\xi,0)\mu_-(k;\xi,0)e^{-ik\sigma\xi}\, d\xi. \end{aligned}$$

It follows that

$$a(k)=I_{4\times 4}+\int_{-\infty}^{+\infty} q(\xi,0)\mu_{-21}(k;\xi,0)\,d\xi,$$
$$b(k)=-\int_{-\infty}^{+\infty} e^{-2ik\xi}q^{\dagger}(\xi,0)\mu_{-11}(k;\xi,0)\,d\xi.$$

Set

$$M(k;x,t):=\begin{cases}\big(\mu_{-L}(k)a^{-1}(k),\mu_{+R}(k)\big), & k\in\mathbb{C}_+,\\ \left(\mu_{+L}(k),\dfrac{\mu_{-R}(k)}{\det[a^{\dagger}(k^*)]}\right), & k\in\mathbb{C}_-.\end{cases}$$

Then formulas (2.5) and straightforward but tedious calculations show that for $k\in\mathbb{C}\backslash\mathbb{R}$, the matrix $M(k;x,t)$ is the unique solution of the Riemann-Hilbert problem

$$\begin{aligned}&M_+(k;x,t)=M_-(k;x,t)J(k;x,t), && k\in\mathbb{R},\\ &M(k;x,t)\to I_{5\times 5}, && k\to\infty,\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}&M_{\pm}(k;x,t)=\lim_{\varepsilon\to 0^+}M(k\pm i\varepsilon;x,t),\\ &J(k;x,t)=\begin{pmatrix} I_{4\times 4}+\gamma^{\dagger}(k^*)\gamma(k) & e^{-2it\theta}\gamma^{\dagger}(k^*)\\ e^{2it\theta}\gamma(k) & 1\end{pmatrix},\\ &\theta(k;x,t)=\frac{x}{t}k+4k^3,\quad \gamma(k)=b(k)a^{-1}(k).\end{aligned}\tag{2.7}$$

It is assumed that $a(k)$ is invertible and the reflection coefficient $\gamma(k)$, corresponding to the initial data $(u_0(x),v_0(x))$, belongs to the Schwartz space $\mathscr{S}(\mathbb{R})$ and satisfies the conditions

$$\gamma(k)=\gamma^*(-k^*)\sigma_2,\quad \sup_{k\in\mathbb{R}}|\gamma(k)|<\infty.$$

It is worth noting that since the jump matrix $J(k;x,t)$ is positive definite, the Riemann-Hilbert problem (2.6) is uniquely solvable — cf. Vanishing Lemma in [1].

Theorem 2.1. *If $M(k;x,t)$, $k\in\mathbb{C}\backslash\mathbb{R}$ is an analytic matrix-function satisfying the Riemann-Hilbert problem* (2.6)*, then*

$$q(x,t)=\big(u(x,t),u^*(x,t),v(x,t),v^*(x,t)\big)^{\mathrm{T}}=2i\lim_{k\to\infty}\big(kM(k;x,t)\big)_{12}\tag{2.8}$$

is the solution of the Cauchy problem for the coupled Sasa-Satsuma equation (1.2).

Proof. Substituting the asymptotic expansion of $M(k;x,t)$,

$$M(k;x,t)=I_{5\times 5}+\frac{M_1(x,t)}{k}+\frac{M_2(x,t)}{k^2}+\mathscr{O}\big(k^{-3}\big),\quad k\to\infty,$$

into (2.4) and comparing the coefficients of $\mathcal{O}(1/k)$ gives (2.8).

The symmetry relation $\sigma_1 J^*(-k^*)\sigma_1 = J(k)$ for the jump matrix $J(k;x,t)$ implies

$$\sigma_1 M^*(-k^*)\sigma_1 = M(k).$$

Therefore, the expressions of $u(x,t)$ and $u^*(x,t)$, $v(x,t)$ and $v^*(x,t)$ determined by (2.8) satisfy the corresponding conjugate relations. □

3. Long-Time Asymptotic Behavior

3.1. First transformation. Reoriented contour

In order to solve the Riemann-Hilbert problem (2.6), we first establish a suitable factorisation of the jump matrix $J(k;x,t)$. It is easily seen that $k_0 = \sqrt{-x/(12t)}$ are the stationary points of θ, i.e. $\frac{\partial \theta}{\partial k}\big|_{k=\pm k_0} = 0$. Note that for $k \in (-\infty,-k_0)\cup(k_0,+\infty)$, the jump matrix $J(k;x,t)$ admits the upper/lower triangular factorisation

$$J = \begin{pmatrix} I_{4\times 4} & e^{-2it\theta}\gamma^\dagger(k^*) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I_{4\times 4} & 0 \\ e^{2it\theta}\gamma(k) & 1 \end{pmatrix},$$

and for $k \in (-k_0,k_0)$, the lower/diagonal/upper triangular factorisation

$$J = \begin{pmatrix} I_{4\times 4} & 0 \\ \frac{e^{2it\theta}\gamma(k)}{1+\gamma(k)\gamma^\dagger(k^*)} & 1 \end{pmatrix}\begin{pmatrix} I_{4\times 4}+\gamma^\dagger(k^*)\gamma(k) & 0 \\ 0 & \frac{1}{1+\gamma(k)\gamma^\dagger(k^*)} \end{pmatrix}\begin{pmatrix} I_{4\times 4} & \frac{e^{-2it\theta}\gamma^\dagger(k^*)}{1+\gamma(k)\gamma^\dagger(k^*)} \\ 0 & 1 \end{pmatrix}.$$

Considering a 4×4 matrix-function $\delta(k)$, which satisfies the following Riemann-Hilbert problem

$$\begin{aligned} &\delta_+(k) = \delta_-(k)\big(I_{4\times 4}+\gamma^\dagger(k^*)\gamma(k)\big), && k \in (-k_0,k_0), \\ &\delta(k) \to I_{4\times 4}, && k \to \infty, \end{aligned} \tag{3.1}$$

we arrive at the scalar Riemann-Hilbert problem

$$\begin{aligned} &\det\delta_+(k) = \big(1+|\gamma(k)|^2\big)\det\delta_-(k), && k \in (-k_0,k_0), \\ &\det\delta(k) \to 1, && k \to \infty. \end{aligned} \tag{3.2}$$

Since the jump matrix $I_{4\times 4}+\gamma^\dagger(k^*)\gamma(k)$ is positive definite, the solution $\delta(k)$ is unique. By Plemelj formula [1], the Riemann-Hilbert problem (3.2) has the solution

$$\det\delta(k) = \left(\frac{k-k_0}{k+k_0}\right)^{i\nu} e^{\chi(k)},$$

where

$$\begin{aligned} \nu &= -\frac{1}{2\pi}\log\big(1+|\gamma(k_0)|^2\big), \\ \chi(k) &= \frac{1}{2\pi i}\int_{-k_0}^{k_0}\log\left(\frac{1+|\gamma(\xi)|^2}{1+|\gamma(k_0)|^2}\right)\frac{\mathrm{d}\xi}{\xi-k}. \end{aligned}$$

The uniqueness of the solution gives

$$\delta(k) = \left(\delta^{\dagger}(k^*)\right)^{-1} = \sigma_2\delta(-k^*)\sigma_2, \tag{3.3}$$

and substituting (3.3) in (3.1) yields

$$|\delta_+(k)|^2 = \begin{cases} 4+|\gamma(k)|^2, & k\in(-k_0,k_0), \\ 4, & k\in(-\infty,-k_0)\cup(k_0,+\infty), \end{cases}$$

$$|\delta_-(k)|^2 = \begin{cases} 4-\frac{3|\gamma(k)|^2}{1+|\gamma(k)|^2}, & k\in(-k_0,k_0), \\ 4, & k\in(-\infty,-k_0)\cup(k_0,+\infty), \end{cases}$$

$$|\det\delta_+(k)|^2 = \begin{cases} 1+|\gamma(k)|^2, & k\in(-k_0,k_0), \\ 1, & k\in(-\infty,-k_0)\cup(k_0,+\infty), \end{cases}$$

$$|\det\delta_-(k)|^2 = \begin{cases} \frac{1}{1+|\gamma(k)|^2}, & k\in(-k_0,k_0), \\ 1, & k\in(-\infty,-k_0)\cup(k_0,+\infty). \end{cases}$$

Hence, by the maximum principle, we have

$$|\delta(k)|\leqslant \text{const}<\infty, \quad |\det\delta(k)|\leqslant\text{const}<\infty, \quad k\in\mathbb{C}. \tag{3.4}$$

Moreover, it follows from (3.3) that

$$\left|\delta^{-1}(k)\right|\leqslant\text{const}<\infty, \quad \left|\left(\det\delta(k)\right)^{-1}\right|\leqslant\text{const}<\infty, \quad k\in\mathbb{C}.$$

We now set

$$M^{\Delta}(k;x,t) := M(k;x,t)\Delta(k), \tag{3.5}$$

where

$$\Delta(k) = \begin{pmatrix} \delta^{-1}(k) & 0 \\ 0 & \det\delta(k) \end{pmatrix}.$$

Changing the orientation for $k\in(-\infty,-k_0)\cup(k_0,+\infty)$ as is shown in Fig. 1 and conjugating the Riemann-Hilbert problem (2.6), we obtain an equivalent Riemann-Hilbert problem — viz.

$$\begin{aligned} &M_+^{\Delta}(k;x,t) = M_-^{\Delta}(k;x,t)J^{\Delta}(k;x,t), && k\in\mathbb{R}, \\ &M^{\Delta}(k;x,t)\to I_{5\times5}, && k\to\infty, \end{aligned} \tag{3.6}$$

where the jump matrix $J^{\Delta}(k;x,t)$ admits the lower-upper triangular factorisations:

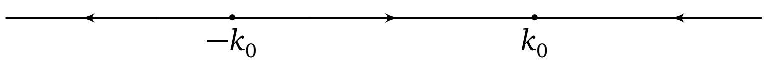

Figure 1: Reoriented contour.

$$J^{\triangle}(k;x,t)=\begin{pmatrix} I_{4\times4} & 0 \\ -\dfrac{e^{2it\theta}\gamma(k)\delta_-^{-1}(k)}{\det\delta_-(k)} & 1\end{pmatrix}\begin{pmatrix} I_{4\times4} & -e^{-2it\theta}[\det\delta_+(k)]\delta_+(k)\gamma^\dagger(k^*) \\ 0 & 1\end{pmatrix},$$
$k\in(-\infty,-k_0)\cup(k_0,+\infty)$,

$$J^{\triangle}(k;x,t)=\begin{pmatrix} I_{4\times4} & 0 \\ \dfrac{e^{2it\theta}\gamma(k)\delta_-^{-1}(k)}{[1+\gamma(k)\gamma^\dagger(k^*)]\det\delta_-(k)} & 1\end{pmatrix}\begin{pmatrix} I_{4\times4} & \dfrac{e^{-2it\theta}[\det\delta_+(k)]\delta_+(k)\gamma^\dagger(k^*)}{1+\gamma(k)\gamma^\dagger(k^*)} \\ 0 & 1\end{pmatrix},$$
$k\in(-k_0,k_0)$.

Introducing the vector-valued spectral function

$$\rho(k)=\begin{cases} -\gamma^\dagger(k^*), & k\in(-\infty,-k_0)\cup(k_0,+\infty),\\ \dfrac{\gamma^\dagger(k^*)}{1+\gamma(k)\gamma^\dagger(k^*)}, & k\in(-k_0,k_0),\end{cases}$$

we can write the above defined function $J^{\triangle}(k;x,t)$ as

$$\begin{aligned} &J^{\triangle}(k;x,t) \\ &=(b_-)^{-1}b_+=\begin{pmatrix} I_{4\times4} & 0 \\ \dfrac{e^{2it\theta}\rho^\dagger(k^*)\delta_-^{-1}(k)}{\det\delta_-(k)} & 1\end{pmatrix}\begin{pmatrix} I_{4\times4} & e^{-2it\theta}[\det\delta_+(k)]\delta_+(k)\rho(k) \\ 0 & 1\end{pmatrix}, \quad k\in\mathbb{R}.\end{aligned}\tag{3.7}$$

3.2. Second transformation. Equivalent Riemann-Hilbert problem

In this subsection, we transform the Riemann-Hilbert problem (3.6) on $\mathbb{R}$ to an equivalent Riemann-Hilbert problem on the augmented contour

$$\Sigma:=\mathbb{R}\cup L\cup L^*$$

shown in Fig. 2 with L defined by

$$L:=\left\{k=k_0+k_0\alpha e^{3\pi i/4}:-\infty<\alpha\leqslant\sqrt{2}\right\}\cup\left\{k=-k_0+k_0\alpha e^{\pi i/4}:-\infty<\alpha\leqslant\sqrt{2}\right\}.$$

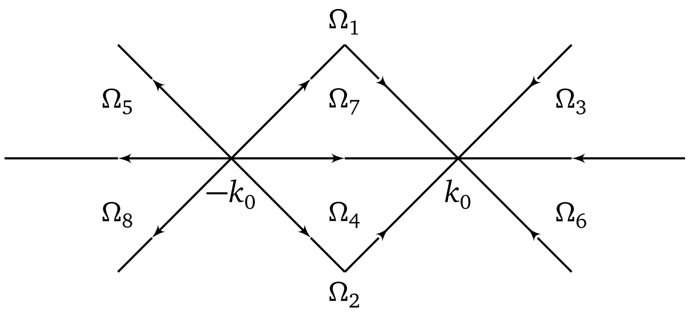

Figure 2: Oriented jump contour Σ.

Besides, we also consider the contour

$$L_\epsilon := \left\{k = k_0 + k_0\alpha e^{3\pi i/4} : \epsilon < \alpha \leqslant \sqrt{2}\right\} \cup \left\{k = -k_0 + k_0\alpha e^{\pi i/4} : \epsilon < \alpha \leqslant \sqrt{2}\right\}$$

for $0 < \epsilon < \sqrt{2}$.

In what follows, if there is a constant $C > 0$ such that $|A| \leqslant CB$, then we will write $A \lesssim B$ and if, in such a constant C depends on a parameter α, then $A \lesssim_\alpha B$.

Theorem 3.1. *The vector-valued spectral function $\rho(k)$ can be represented in the form*

$$\rho(k) = R(k) + h_1(k) + h_2(k), \quad k \in \mathbb{R},$$

where $R(k)$ is piecewise rational, $h_1(k)$ analytic on $\mathbb{R}$, and $h_2(k)$ has an analytic continuation to L. If $0 < k_0 < C$, then for any positive integer l and $t > 0$, the following estimates hold:

$$\begin{aligned}
\left|e^{-2it\theta(k)}h_1(k)\right| &\lesssim \frac{1}{\left(1+|k|^2\right)t^l}, && k \in \mathbb{R}, \\
\left|e^{-2it\theta(k)}h_2(k)\right| &\lesssim \frac{1}{\left(1+|k|^2\right)t^l}, && k \in L, \\
\left|e^{-2it\theta(k)}R(k)\right| &\lesssim e^{-16\epsilon^2 k_0^3 t}, && k \in L_\epsilon.
\end{aligned}$$

Taking the Hermitian conjugate

$$\rho^\dagger(k^*) = R^\dagger(k^*) + h_1^\dagger(k^*) + h_2^\dagger(k^*)$$

leads to the same estimates for $e^{2it\theta(k)}h_1^\dagger(k^*)$, $e^{2it\theta(k)}h_2^\dagger(k^*)$ and $e^{2it\theta(k)}R^\dagger(k^*)$ on $\mathbb{R} \cup L^*$.

Proof. It follows from [7, Proposition 1.92]. □

Factoring the matrices $b_\pm$ in (3.7) as

$$\begin{aligned}
b_+ &= b_+^o b_+^a = \left(I_{5\times5} + \omega_+^o\right)\left(I_{5\times5} + \omega_+^a\right) = \begin{pmatrix} I_{4\times4} & e^{-2it\theta}[\det\delta_+(k)]\delta_+(k)h_1(k) \\ 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} I_{4\times4} & e^{-2it\theta}[\det\delta_+(k)]\delta_+(k)[h_2(k)+R(k)] \\ 0 & 1 \end{pmatrix}, \\
b_- &= b_-^o b_-^a = \left(I_{5\times5} - \omega_-^o\right)\left(I_{5\times5} - \omega_-^a\right) = \begin{pmatrix} I_{4\times4} & 0 \\ -\dfrac{e^{2it\theta}h_1^\dagger(k^*)\delta_-^{-1}(k)}{\det\delta_-(k)} & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} I_{4\times4} & 0 \\ -\dfrac{e^{2it\theta}[h_2^\dagger(k^*) + R^\dagger(k^*)]\delta_-^{-1}(k)}{\det\delta_-(k)} & 1 \end{pmatrix},
\end{aligned}$$

we set

$$M^\sharp(k;x,t) = \begin{cases} M^\Delta(k;x,t), & k \in \Omega_1 \cup \Omega_2, \\ M^\Delta(k;x,t)(b_-^a)^{-1}, & k \in \Omega_3 \cup \Omega_4 \cup \Omega_5, \\ M^\Delta(k;x,t)(b_+^a)^{-1}, & k \in \Omega_6 \cup \Omega_7 \cup \Omega_8. \end{cases} \tag{3.8}$$

Straightforward computations show that $M^\sharp$ is the solution of the Riemann-Hilbert problem

$$\begin{aligned} &M^\sharp_+(k;x,t) = M^\sharp_-(k;x,t)J^\sharp(k;x,t), && k\in\Sigma, \\ &M^\sharp(k;x,t)\to I_{5\times 5}, && k\to\infty, \end{aligned} \tag{3.9}$$

where

$$J^\sharp(k;x,t) = \left(b^\sharp_-\right)^{-1} b^\sharp_+ = \begin{cases} I^{-1}_{5\times 5} b^a_+, & k\in L, \\ (b^a_-)^{-1} I_{5\times 5}, & k\in L^*, \\ (b^o_-)^{-1} b^o_+, & k\in\mathbb{R}. \end{cases}$$

An asymptotic condition for $M^\sharp(k;x,t)$ can be derived. For example, $(b^a_+)^{-1}$ converges to $I_{5\times 5}$ as $k\to\infty$ in $\Omega_6\cup\Omega_8$. We first consider the domain Ω_6. It follows from (3.4) that $\delta(k)$ and $\det\delta(k)$ are bounded for fixed x,t. This implies

$$\left|e^{-2it\theta}\left[\det\delta(k)\right]\delta(k)\left[h_2(k)+R(k)\right]\right| \lesssim \left|e^{-2it\theta}h_2(k)\right| + \left|e^{-2it\theta}R(k)\right|, \quad \forall k\in\Omega_6.$$

By the definition of $R(k)$, we have

$$\left|e^{-2it\theta}R(k)\right| \lesssim \frac{\left|\sum_{j=0}^m \mu_j(k-k_0)^j\right|}{|(k-i)^{m+5}|} \lesssim \frac{1}{|k-i|^5}, \quad \forall k\in\Omega_6.$$

Taking into account the convergence of $e^{-2it\theta}h_2(k)$, we finally obtain that $M^\sharp(k;x,t)\to I_{5\times 5}$ as $k\to\infty$ in Ω_6, and so on. The Riemann-Hilbert problem (3.9) is connected to singular integral equations as follows — cf. [7, P. 322] and [3]. We first define the following spaces

$$\begin{aligned} &\mathscr{L}^p(\Sigma) = \left\{ f(k) \middle| \left(\int_\Sigma |f(k)|^p |\mathrm{d}k|\right)^{1/p} < +\infty \right\}, \quad p\in\{1,2\}, \\ &\mathscr{L}^\infty(\Sigma) = \left\{ f(k) | \operatorname{ess\,sup}_{k\in\Sigma} |f(k)| < +\infty \right\}. \end{aligned}$$

Set the Cauchy operators $C_\pm$ on Σ by

$$(C_\pm f)(k) = \int_\Sigma \frac{f(\xi)}{\xi - k_\pm}\frac{\mathrm{d}\xi}{2\pi i}, \quad k\in\Sigma, \quad f\in\mathscr{L}^2(\Sigma),$$

where $C_+f\,(C_-f)$ denotes the left (right) boundary value for the oriented contour Σ in Fig. 2. For example, for $k>k_0$, we have

$$(C_+f)(k) = \lim_{\varepsilon\to 0^+}\int_\Sigma \frac{f(\xi)}{\xi-(k-i\varepsilon)}\frac{\mathrm{d}\xi}{2\pi i}.$$

Moreover, the operators $C_\pm$ are bounded from $\mathscr{L}^2(\Sigma)$ to $\mathscr{L}^2(\Sigma)$, and $C_+ - C_- = 1$. Set

$$\omega^\sharp = \omega^\sharp_+ + \omega^\sharp_-, \quad \omega^\sharp_\pm = \pm\left(b^\sharp_\pm - I_{5\times 5}\right).$$

Observe from Theorem 3.1 that, for fixed x,t, we then have

$$\omega^\sharp, \omega^\sharp_\pm \in \mathscr{L}^1(\Sigma)\cap\mathscr{L}^\infty(\Sigma). \tag{3.10}$$

Define

$$C_{\omega^\sharp} f = C_+\big(f\,\omega^\sharp_-\big) + C_-\big(f\,\omega^\sharp_+\big) \tag{3.11}$$

for a 5×5 matrix-valued function f. By property (3.10), $C_{\omega^\sharp}$ is a bounded map from $C_{\omega^\sharp}:\mathscr{L}^2(\Sigma)+\mathscr{L}^\infty(\Sigma)\to\mathscr{L}^2(\Sigma)$. If $\mu^\sharp(k;x,t)\in\mathscr{L}^2(\Sigma)+\mathscr{L}^\infty(\Sigma)$ satisfies the singular integral equation

$$\mu^\sharp = I_{5\times5} + C_{\omega^\sharp}\mu^\sharp.$$

Then

$$M^\sharp(k;x,t) = I_{5\times5} + \int_\Sigma \frac{\mu^\sharp(\xi;x,t)\omega^\sharp(\xi;x,t)}{\xi-k}\frac{\mathrm{d}\xi}{2\pi i}, \quad k\in\mathbb{C}\backslash\Sigma \tag{3.12}$$

is the solution of the Riemann-Hilbert problem (3.9).

Theorem 3.2. *The solution $q(x,t)$ for the Cauchy problem of the coupled Sasa-Satsuma equation* (1.2) *has the form*

$$\begin{aligned} q(x,t) &= \big(u(x,t),u^*(x,t),v(x,t),v^*(x,t)\big)^{\mathrm{T}} \\ &= -\frac{1}{\pi}\left(\int_\Sigma \big((1-C_{\omega^\sharp})^{-1}I_{5\times5}\big)(\xi)\omega^\sharp(\xi)\,\mathrm{d}\xi\right)_{12}. \end{aligned} \tag{3.13}$$

Proof. Using the Eqs. (2.8) and definition (3.5), (3.8) and (3.12), we obtain

$$\begin{aligned} q(x,t) &= \lim_{k\to\infty} 2i\left(kM^\sharp(k;x.t)\right)_{12} \\ &= \lim_{k\to\infty} 2i\left(kM^\Delta(k;x.t)\right)_{12} \\ &= -\frac{1}{\pi}\left(\int_\Sigma \mu^\sharp(\xi;x,t)\omega^\sharp(\xi)\,\mathrm{d}\xi\right)_{12} \\ &= -\frac{1}{\pi}\left(\int_\Sigma \big((1-C_{\omega^\sharp})^{-1}I_{5\times5}\big)(\xi)\omega^\sharp(\xi)\,\mathrm{d}\xi\right)_{12}. \end{aligned}$$

□

3.3. Third transformation. Truncated contour

Here we reduce the Riemann-Hilbert problem (3.9) on the contour Σ to a Riemann-Hilbert problem on the truncated contour Σ', where $\Sigma'=\Sigma\backslash(\mathbb{R}\cup L_\epsilon\cup L_\epsilon^*)$ shown in Fig. 3. Consider a function ω^e and write it in the form

$$\omega^e = \omega^a + \omega^b + \omega^c,$$

where $\omega^a = \omega^\sharp|_{\mathbb{R}}$ is supported on $\mathbb{R}$ and is composed of terms of type $h_1(k)$ and $h_1^\dagger(k^*)$ described in Theorem 3.1, the function $\omega^b = \omega^\sharp|_{L\cup L^*}$ is supported on $L\cup L^*$ and is composed

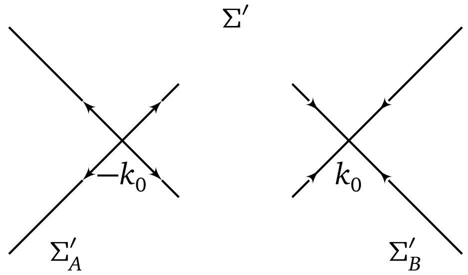

Figure 3: The oriented contour $\Sigma' = \Sigma'_A \cup \Sigma'_B$.

of terms of type $h_2(k)$ and $h_2^\dagger(k^*)$, and $\omega^c = \omega^\sharp|_{L_\epsilon \cup L_\epsilon^*}$ is supported on $L_\epsilon \cup L_\epsilon^*$ and is composed of terms of type $R(k)$ and $R^\dagger(k^*)$.

Define $\omega' = \omega^\sharp - \omega^e$. It is easily seen that ω'=0 on $\Sigma \backslash \Sigma'$. Hence, ω' is supported on contour Σ' and is composed of terms of type $R(k)$ and $R^\dagger(k^*)$.

Lemma 3.1. *If ϵ is sufficiently small, $0 < k_0 < C$ and $\tau = tk_0^3$, then*

$$\|\omega^a\|_{\mathscr{L}^1(\mathbb{R})\cap\mathscr{L}^2(\mathbb{R})\cap\mathscr{L}^\infty(\mathbb{R})} \lesssim t^{-l}, \tag{3.14}$$

$$\|\omega^b\|_{\mathscr{L}^1(L\cup L^*)\cap\mathscr{L}^2(L\cup L^*)\cap\mathscr{L}^\infty(L\cup L^*)} \lesssim t^{-l}, \tag{3.15}$$

$$\|\omega^c\|_{\mathscr{L}^1(L_\epsilon\cup L_\epsilon^*)\cap\mathscr{L}^2(L_\epsilon\cup L_\epsilon^*)\cap\mathscr{L}^\infty(L_\epsilon\cup L_\epsilon^*)} \lesssim e^{-16\epsilon^2\tau}, \tag{3.16}$$

$$\|\omega'\|_{\mathscr{L}^2(\Sigma)} \lesssim \tau^{-1/4}, \quad \|\omega'\|_{\mathscr{L}^1(\Sigma)} \lesssim \tau^{-1/2}. \tag{3.17}$$

Proof. The proof of estimates (3.14)-(3.16) follows from Theorem 3.1. Indeed, for

$$\left\{k = k_0 + k_0\alpha e^{3\pi i/4} : -\infty < \alpha < \sqrt{2}\right\}$$

the term $R(k)$ can be directly estimated as follows:

$$|R(k)| \lesssim \left(1 + |k|^5\right)^{-1}.$$

Besides, since on this contour one has Re $(i\theta) \geqslant 8k_0^3\alpha^2$, the estimates (3.4) yield

$$\left|e^{-2it\theta(k)}[\det\delta(k)]\delta(k)R(k)\right| \lesssim e^{-16k_0^3\alpha^2 t}\left(1 + |k|^5\right)^{-1}.$$

Analogously, considering $R^\dagger(k^*)$ on L^*, we obtain

$$\left|e^{-2it\theta(k)}[\det\delta(k)]^{-1}R^\dagger(k^*)\delta^{-1}(k)\right| \lesssim e^{-16k_0^3\alpha^2 t}\left(1 + |k|^5\right)^{-1},$$

and direct calculations lead to (3.17). □

Lemma 3.2. *As $t \to \infty$, for $0 < k_0 < C$, the operator $(1 - C_{\omega'})^{-1}$: $\mathscr{L}^2(\Sigma) \to \mathscr{L}^2(\Sigma)$ exists and is uniformly bounded — i.e.*

$$\left\|(1 - C_{\omega'})^{-1}\right\|_{\mathscr{L}^2(\Sigma)} \lesssim 1.$$

The operator $(1 - C_{\omega^\sharp})^{-1}$: $\mathscr{L}^2(\Sigma) \to \mathscr{L}^2(\Sigma)$ also exists and is uniformly bounded — i.e.

$$\left\|(1 - C_{\omega^\sharp})^{-1}\right\|_{\mathscr{L}^2(\Sigma)} \lesssim 1.$$

Proof. The conclusion follows [7, Proposition 2.23 and Corollary 2.25]. □

Theorem 3.3. *We have*

$$\begin{aligned}&\int_{\Sigma}\big((1-C_{\omega^\sharp})^{-1}I_{5\times 5}\big)(\xi)\omega^\sharp(\xi)\,\mathrm{d}\xi\\ &=\int_{\Sigma}\big((1-C_{\omega'})^{-1}I_{5\times 5}\big)(\xi)\omega'(\xi)\,\mathrm{d}\xi+\mathscr{O}\big(\tau^{-l}\big),\quad \tau\to\infty.\end{aligned}\tag{3.18}$$

Proof. Taking into account the second resolvent identity in [14], we write

$$\begin{aligned}&\big((1-C_{\omega^\sharp})^{-1}I_{5\times 5}\big)\,\omega^\sharp\\ &=\big((1-C_{\omega'})^{-1}I_{5\times 5}\big)\,\omega'+\omega^e+\big((1-C_{\omega'})^{-1}(C_{\omega^e}I_{5\times 5})\big)\,\omega^\sharp\\ &\quad+\big((1-C_{\omega'})^{-1}(C_{\omega'}I_{5\times 5})\big)\,\omega^e+\big((1-C_{\omega'})^{-1}C_{\omega^e}(1-C_{\omega^\sharp})^{-1}\big)(C_{\omega^\sharp}I_{5\times 5})\omega^\sharp.\end{aligned}\tag{3.19}$$

Consequently, Lemma 3.1 and Proposition 3.2 lead to the inequalities

$$\begin{aligned}&\|\omega^e\|_{\mathscr{L}^1(\Sigma)}\leqslant\|\omega^a\|_{\mathscr{L}^1(\mathbb{R})}+\|\omega^b\|_{\mathscr{L}^1(L\cup L^*)}+\|\omega^c\|_{\mathscr{L}^1(L_\epsilon\cup L_\epsilon^*)}\lesssim\tau^{-l},\\ &\big\|\big((1-C_{\omega'})^{-1}(C_{\omega^e}I_{5\times 5})\big)\,\omega^\sharp\big\|_{\mathscr{L}^1(\Sigma)}\\ &\leqslant\big\|(1-C_{\omega'})^{-1}\big\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^e}I_{5\times 5}\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)}\lesssim\|\omega^e\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)}\lesssim\tau^{-l-1/4},\\ &\big\|\big((1-C_{\omega'})^{-1}(C_{\omega'}I_{5\times 5})\big)\,\omega^e\big\|_{\mathscr{L}^1(\Sigma)}\\ &\leqslant\big\|\big(1-C_{\omega'}^{-1}\big)\big\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega'}I_{5\times 5}\|_{\mathscr{L}^2(\Sigma)}\|\omega^e\|_{\mathscr{L}^2(\Sigma)}\lesssim\|\omega'\|_{\mathscr{L}^2(\Sigma)}\|\omega^e\|_{\mathscr{L}^2(\Sigma)}\lesssim\tau^{-l-1/4},\\ &\big\|\big((1-C_{\omega'})^{-1}C_{\omega^e}(1-C_{\omega^\sharp})^{-1}\big)(C_{\omega^\sharp}I_{5\times 5})\omega^\sharp\big\|_{\mathscr{L}^1(\Sigma)}\\ &\leqslant\big\|(1-C_{\omega'})^{-1}\big\|_{\mathscr{L}^2(\Sigma)}\big\|(1-C_{\omega^\sharp})^{-1}\big\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^e}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^\sharp}I_{5\times 5}\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)}\\ &\lesssim\|\omega^e\|_{\mathscr{L}^\infty(\Sigma)}\|\omega^\sharp\|^2_{\mathscr{L}^2(\Sigma)}\lesssim\tau^{-l-1/2}.\end{aligned}$$

Substituting these estimates into (3.19) gives (3.18). □

Remark 3.1. Noting that $\omega'(k)=0$ for $k\in\Sigma\backslash\Sigma'$, we denote $C_{\omega'}|_{\mathscr{L}^2(\Sigma')}$ as the restriction of $C_{\omega'}$ to $\mathscr{L}^2(\Sigma')$. For convenience, we rewrite $C_{\omega'}|_{\mathscr{L}^2(\Sigma')}$ as $C_{\omega'}$, then

$$\int_{\Sigma}\big((1-C_{\omega'})^{-1}I_{5\times 5}\big)(\xi)\omega'(\xi)\,\mathrm{d}\xi=\int_{\Sigma'}\big((1-C_{\omega'})^{-1}I_{5\times 5}\big)(\xi)\omega'(\xi)\,\mathrm{d}\xi.$$

Lemma 3.3. *We have*

$$\begin{aligned}q(x,t)&=\big(u(x,t),u^*(x,t),v(x,t),v^*(x,t)\big)^{\mathrm{T}}\\ &=-\frac{1}{\pi}\left(\int_{\Sigma'}\big((1-C_{\omega'})^{-1}I_{5\times 5}\big)(\xi)\omega'(\xi)\,\mathrm{d}\xi\right)_{12}+\mathscr{O}\big(\tau^{-l}\big),\quad\tau\to\infty.\end{aligned}\tag{3.20}$$

Proof. It follows directly from (3.13) and (3.18). □

Consider the contour

$$\Sigma' = L' \cup (L')^*,$$

where

$$L' = L \backslash L_\epsilon,$$

and let $\mu' = (1 - C_{\omega'})^{-1} I_{5\times 5}$ on Σ'. Similar to (3.12), we note that

$$M'(k;x,t) = I_{5\times 5} + \int_{\Sigma'} \frac{\mu'(\xi;x,t)\omega'(\xi;x,t)}{\xi - k} \frac{\mathrm{d}\xi}{2\pi i}$$

solves the Riemann-Hilbert problem

$$\begin{aligned} &M'_+(k;x,t) = M'_-(k;x,t)J'(k;x,t), && k \in \Sigma', \\ &M'(k;x,t) \to I_{5\times 5}, && k \to \infty, \end{aligned}$$

where

$$\begin{aligned} &J' = \left(b'_-\right)^{-1} b'_+ = \left(I_{5\times 5} - \omega'_-\right)^{-1} \left(I_{5\times 5} + \omega'_+\right), \\ &\omega' = \omega'_+ + \omega'_-, \\ &b'_+ = \begin{pmatrix} I_{4\times 4} & e^{-2it\theta}[\det \delta(k)]\delta(k)R(k) \\ 0 & 1 \end{pmatrix}, \quad b'_- = I_{5\times 5} \quad \text{on } L', \\ &b'_+ = I_{5\times 5}, \quad b'_- = \begin{pmatrix} I_{4\times 4} & 0 \\ -\dfrac{e^{2it\theta} R^\dagger(k^*)\delta^{-1}(k)}{\det \delta(k)} & 1 \end{pmatrix} \quad \text{on } (L')^*. \end{aligned}$$

3.4. Fourth transformation. Decomposition of truncated contour

In this subsection, we show how to separate the contributions of the two crosses in Σ' to the solution $q(x,t)$ in formula (3.20). Write $\Sigma' = \Sigma'_A \cup \Sigma'_B$, where Σ'_A and Σ'_B are the disjoint crosses

$$\begin{aligned} &\Sigma'_A = \left\{k = -k_0 + hk_0 e^{\pi i/4} | -\infty < h \leqslant \epsilon\right\} \cup \left\{k = -k_0 + hk_0 e^{-\pi i/4} | -\infty < h \leqslant \epsilon\right\}, \\ &\Sigma'_B = \left\{k = k_0 + hk_0 e^{-3\pi i/4} | -\infty < h \leqslant \epsilon\right\} \cup \left\{k = k_0 + hk_0 e^{3\pi i/4} | -\infty < h \leqslant \epsilon\right\}. \end{aligned}$$

Set

$$\omega'_\pm = \omega'_{A\pm} + \omega'_{B\pm},$$

where

$$\begin{aligned} &\omega'_{A\pm}(k) = 0, \quad k \in \Sigma'_B, \\ &\omega'_{B\pm}(k) = 0, \quad k \in \Sigma'_A, \end{aligned}$$

and define the operators $C_{\omega'_A}$ and $C_{\omega'_B} : \mathscr{L}^2(\Sigma') + \mathscr{L}^\infty(\Sigma') \to \mathscr{L}^2(\Sigma')$ similar to (3.11). Noting that $C_{\omega'} = C_{\omega'_A} + C_{\omega'_B}$, we arrive at the following lemma.

Lemma 3.4.

$$\begin{aligned}
&\left\|C_{\omega_B'}C_{\omega_A'}\right\|_{\mathscr{L}^2(\Sigma')}=\left\|C_{\omega_A'}C_{\omega_B'}\right\|_{\mathscr{L}^2(\Sigma')}\lesssim_{k_0}\tau^{-1/2},\\
&\left\|C_{\omega_B'}C_{\omega_A'}\right\|_{\mathscr{L}^\infty(\Sigma')\to\mathscr{L}^2(\Sigma')}=\left\|C_{\omega_A'}C_{\omega_B'}\right\|_{\mathscr{L}^\infty(\Sigma')\to\mathscr{L}^2(\Sigma')}\lesssim_{k_0}\tau^{-3/4}.
\end{aligned}$$

Proof. See Lemma 3.5 in Ref. [7]. □

Theorem 3.4. *We have*

$$\begin{aligned}
q(x,t)&=\big(u(x,t),u^*(x,t),v(x,t),v^*(x,t)\big)^{\mathrm{T}}\\
&=-\frac{1}{\pi}\left(\int_{\Sigma'}\left(\left(1-C_{\omega'}\right)^{-1}I_{5\times5}\right)(\xi)\omega'(\xi)\,d\xi\right)_{12}\\
&=-\frac{1}{\pi}\left(\int_{\Sigma_A'}\left(\left(1-C_{\omega_A'}\right)^{-1}I_{5\times5}\right)(\xi)\omega_A'(\xi)\,d\xi\right)_{12}\\
&\quad-\frac{1}{\pi}\left(\int_{\Sigma_B'}\left(\left(1-C_{\omega_B'}\right)^{-1}I_{5\times5}\right)(\xi)\omega_B'(\xi)\,d\xi\right)_{12}+\mathscr{O}\left(\frac{c(k_0)}{\tau}\right),\quad\tau\to\infty. \qquad (3.21)
\end{aligned}$$

Proof. The representation

$$\begin{aligned}
&\left(1-C_{\omega_A'}-C_{\omega_B'}\right)\left(1+C_{\omega_A'}\left(1-C_{\omega_A'}\right)^{-1}+C_{\omega_B'}\left(1-C_{\omega_B'}\right)^{-1}\right)\\
&=1-C_{\omega_B'}C_{\omega_A'}\left(1-C_{\omega_A'}\right)^{-1}-C_{\omega_A'}C_{\omega_B'}\left(1-C_{\omega_B'}\right)^{-1}
\end{aligned}$$

yields

$$\begin{aligned}
\left(1-C_{\omega'}\right)^{-1}=1&+C_{\omega_A'}\left(1-C_{\omega_A'}\right)^{-1}+C_{\omega_B'}\left(1-C_{\omega_B'}\right)^{-1}\\
&+\left[1+C_{\omega_A'}\left(1-C_{\omega_A'}\right)^{-1}+C_{\omega_B'}\left(1-C_{\omega_B'}\right)^{-1}\right]\\
&\times\left[1-C_{\omega_B'}C_{\omega_A'}\left(1-C_{\omega_A'}\right)^{-1}-C_{\omega_A'}C_{\omega_B'}\left(1-C_{\omega_B'}\right)^{-1}\right]^{-1}\\
&\times\left[C_{\omega_B'}C_{\omega_A'}\left(1-C_{\omega_A'}\right)^{-1}+C_{\omega_A'}C_{\omega_B'}\left(1-C_{\omega_B'}\right)^{-1}\right].
\end{aligned}$$

This representation, combined with Lemmas 3.1, 3.4 and Proposition 3.2, gives (3.21). □

3.5. Rescaling and further reduction of Riemann-Hilbert problems

Let us transform contours Σ_A' and Σ_B' to the crosses located at the origin. For this, we first extend the contours Σ_A' and Σ_B' as

$$\begin{aligned}
\hat{\Sigma}_A'&=\left\{-k_0+k_0\alpha e^{\pm\pi i/4}|\alpha\in\mathbb{R}\right\},\\
\hat{\Sigma}_B'&=\left\{k_0+k_0\alpha e^{\pm\pi i/4}|\alpha\in\mathbb{R}\right\},
\end{aligned}$$

and define $\hat{\omega}'_{A\pm}$ and $\hat{\omega}'_{B\pm}$ on Σ'_A and Σ'_B, respectively, by

$$\hat{\omega}'_{A\pm}=\begin{cases}\omega'_{A\pm}, & k\in\Sigma'_A,\\ 0, & k\in\hat{\Sigma}'_A\backslash\Sigma'_A,\end{cases}\qquad \hat{\omega}'_{B\pm}=\begin{cases}\omega'_{B\pm}, & k\in\Sigma'_B,\\ 0, & k\in\hat{\Sigma}'_B\backslash\Sigma'_B.\end{cases}$$

Let Σ_A and Σ_B denote the contours $\{k=k_0\alpha e^{\pm\pi i/4}|\alpha\in\mathbb{R}\}$ in Fig. 4 oriented outward as in Σ'_A, $\hat{\Sigma}'_A$, and inward as in Σ'_B, $\hat{\Sigma}'_B$, respectively. Introduce the scaling operators

$$N_A:\mathscr{L}^2(\hat{\Sigma}'_A)\to\mathscr{L}^2(\Sigma_A),\quad f(k)\mapsto(N_Af)(k)=f\left(-k_0+\frac{k}{\sqrt{48tk_0}}\right),$$
$$N_B:\mathscr{L}^2(\hat{\Sigma}'_B)\to\mathscr{L}^2(\Sigma_B),\quad f(k)\mapsto(N_Bf)(k)=f\left(k_0+\frac{k}{\sqrt{48tk_0}}\right),$$

and define

$$\omega_A=N_A\hat{\omega}'_A,\quad \omega_B=N_B\hat{\omega}'_B.$$

Direct calculations show that

$$C_{\hat{\omega}'_A}=N_A^{-1}C_{\omega_A}N_A,\quad C_{\hat{\omega}'_B}=N_B^{-1}C_{\omega_B}N_B,$$

where $C_{\omega_A}:\mathscr{L}^2(\Sigma_A)\to\mathscr{L}^2(\Sigma_A)$ and $C_{\omega_B}:\mathscr{L}^2(\Sigma_B)\to\mathscr{L}^2(\Sigma_B)$ are bounded operators. On the contour

$$L_A=\left\{k=\alpha k_0\sqrt{48tk_0}e^{-3\pi i/4}:-\epsilon<\alpha<+\infty\right\}$$

we have

$$\omega_A=\omega_{A+}=\begin{pmatrix}0 & (N_As_1)(k)\\ 0 & 0\end{pmatrix}.$$

On the other hand, on the contour

$$L_A^*=\left\{k=\alpha k_0\sqrt{48tk_0}e^{3\pi i/4}:-\epsilon<\alpha<+\infty\right\}$$

we have

$$\omega_A=\omega_{A-}=\begin{pmatrix}0 & 0\\ (N_As_2)(k) & 0\end{pmatrix},$$

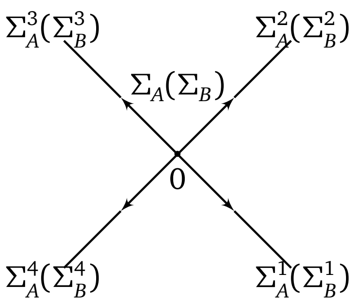

Figure 4: The oriented contour Σ_A or Σ_B(reoriented).

where

$$s_1(k) = e^{-2it\theta}[\det\delta(k)]\delta(k)R(k),$$
$$s_2(k) = \frac{e^{2it\theta}R^{\dagger}(k^*)\delta^{-1}(k)}{\det\delta(k)}.$$

Lemma 3.5. *As $t\to\infty$, and $k\in L_A$, for an arbitrary positive integer l,*

$$\left|(N_A\tilde{\delta})(k)\right| \lesssim t^{-l}, \tag{3.22}$$

where $\tilde{\delta}(k) = e^{-2it\theta}[\delta(k) - \det\delta(k)I_{4\times4}]R(k)$.

Proof. It follows from (3.1) and (3.2) that $\tilde{\delta}$ satisfies the following Riemann-Hilbert problem

$$\begin{aligned} &\tilde{\delta}_+(k) = \tilde{\delta}_-(k)\left(1+|\gamma(k)|^2\right) + e^{-2it\theta}f(k), && k\in(-k_0,k_0),\\ &\tilde{\delta}(k)\to 0, && k\to\infty, \end{aligned}$$

where $f(k) = [\delta_-(\gamma^{\dagger}\gamma - |\gamma|^2 I_{4\times4})R](k)$. The solution of this problem has the form

$$\tilde{\delta}(k) = X(k)\int_{-k_0}^{k_0} \frac{e^{-2it\theta(\xi)}f(\xi)}{X_+(\xi)(\xi-k)}\frac{d\xi}{2\pi i},$$
$$X(k) = \exp\left\{\frac{1}{2\pi i}\int_{-k_0}^{k_0}\frac{\log\left(1+|\gamma(\xi)|^2\right)}{\xi-k}d\xi\right\}.$$

Note that

$$\begin{aligned} \gamma^{\dagger}\gamma R - |\gamma|^2 R &= \gamma^{\dagger}\gamma(R-\rho) - |\gamma|^2(R-\rho)\\ &= \left(|\gamma|^2 I_{4\times4} - \gamma^{\dagger}\gamma\right)(h_1+h_2), \end{aligned}$$

and $(|\gamma|^2 I_{4\times4} - \gamma^{\dagger}\gamma)$ consists of the components of γ. Therefore, following Theorem 3.1, we represent $f(k)$ as the sum $f(k) = f_1(k) + f_2(k)$, where $f_2(k)$ has an analytic continuation to L_t, cf. Fig. 5, where

$$\begin{aligned} L_t := &\left\{k = k_0 + k_0\alpha e^{3\pi i/4} : 0\leqslant\alpha\leqslant\sqrt{2}\left(1-\frac{1}{2t}\right)\right\}\\ &\cup\left\{k = \frac{k_0}{t} - k_0 + k_0\alpha e^{\pi i/4} : 0\leqslant\alpha\leqslant\sqrt{2}\left(1-\frac{1}{2t}\right)\right\}. \end{aligned}$$

Then for $l\geqslant 2$, we have

$$\left|e^{-2it\theta(k)}f_1(k)\right| \lesssim \frac{1}{\left(1+|k|^2\right)t^l}, \quad k\in\mathbb{R},$$
$$\left|e^{-2it\theta(k)}f_2(k)\right| \lesssim \frac{1}{\left(1+|k|^2\right)t^l}, \quad k\in L_t.$$

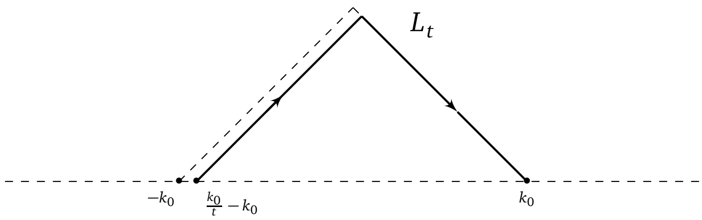

Figure 5: The contour L_t.

Since $k \in L_A$, it follows that

$$\begin{aligned}(N_A\tilde{\delta})(k) =& X\left(\frac{k}{\sqrt{48tk_0}}-k_0\right)\int_{-k_0}^{\frac{k_0}{t}-k_0}\frac{e^{-2it\theta(\xi)}f(\xi)}{X_+(\xi)(\xi+k_0-k/\sqrt{48tk_0})}\frac{d\xi}{2\pi i}\\ &+X\left(\frac{k}{\sqrt{48tk_0}}-k_0\right)\int_{\frac{k_0}{t}-k_0}^{k_0}\frac{e^{-2it\theta(\xi)}f_1(\xi)}{X_+(\xi)(\xi+k_0-k/\sqrt{48tk_0})}\frac{d\xi}{2\pi i}\\ &+X\left(\frac{k}{\sqrt{48tk_0}}-k_0\right)\int_{\frac{k_0}{t}-k_0}^{k_0}\frac{e^{-2it\theta(\xi)}f_2(\xi)}{X_+(\xi)(\xi+k_0-k/\sqrt{48tk_0})}\frac{d\xi}{2\pi i}\\ &:= A_1+A_2+A_3,\end{aligned}$$

and

$$\begin{aligned}|A_1| &\lesssim \int_{-k_0}^{\frac{k_0}{t}-k_0}\frac{\left|e^{-2it\theta(\xi)}f(\xi)\right|}{\left|\xi+k_0-k/\sqrt{48tk_0}\right|}d\xi \lesssim t^{-l-1},\\ |A_2| &\lesssim \int_{\frac{k_0}{t}-k_0}^{k_0}\frac{\left|e^{-2it\theta(\xi)}f_1(\xi)\right|}{\left|\xi+k_0-k/\sqrt{48tk_0}\right|}d\xi \leqslant t^{-l}\frac{\sqrt{2}t}{k_0}\left(2k_0-\frac{k_0}{t}\right)\lesssim t^{-l+1}.\end{aligned}$$

By using the Cauchy's theorem, we can evaluate the integral A_3 over the contour L_t instead of the interval $(k_0/t-k_0, k_0)$ and obtain $|A_3| \lesssim t^{-l+1}$. Finally, we have (3.22). □

Note. Similarly, if $t \to \infty$ and $k \in L_A^*$, then

$$\left|(N_A\hat{\delta})(k)\right| \lesssim t^{-l},$$

where

$$\hat{\delta}(k) = e^{2it\theta(k)}R^\dagger(k^*)\left[\delta^{-1}(k)-[\det\delta(k)]^{-1}I_{4\times 4}\right].$$

Set

$$\begin{aligned}J^{A^0} &= (I_{5\times 5}-\omega_{A^0-})^{-1}(I_{5\times 5}+\omega_{A^0+}),\\ \delta_A &= e^{\chi(-k_0)-8i\tau}(192\tau)^{i\nu/2},\end{aligned}$$

$$\omega_{A^0}=\omega_{A^0+}=\begin{cases}\begin{pmatrix}0_{4\times4} & -(\delta_A)^2(-k)^{-2\nu i}e^{ik^2/2}\gamma^\dagger(-k_0)\\ 0 & 0\end{pmatrix}, & k\in\Sigma_A^4,\\ \begin{pmatrix}0_{4\times4} & (\delta_A)^2(-k)^{-2\nu i}e^{ik^2/2}\frac{\gamma^\dagger(-k_0)}{1+|\gamma(-k_0)|^2}\\ 0 & 0\end{pmatrix}, & k\in\Sigma_A^2,\end{cases}\tag{3.23}$$

$$\omega_{A^0}=\omega_{A^0-}=\begin{cases}\begin{pmatrix}0_{4\times4} & 0\\ (\delta_A)^{-2}(-k)^{2\nu i}e^{-ik^2/2}\gamma(-k_0) & 0\end{pmatrix}, & k\in\Sigma_A^3,\\ \begin{pmatrix}0_{4\times4} & 0\\ -(\delta_A)^{-2}(-k)^{2\nu i}e^{-ik^2/2}\frac{\gamma(-k_0)}{1+|\gamma(-k_0)|^2} & 0\end{pmatrix}, & k\in\Sigma_A^1.\end{cases}\tag{3.24}$$

Lemma 3.5 and [7, Lemma 3.35] give

$$\|\omega_A-\omega_{A^0}\|_{\mathscr{L}^\infty(\Sigma_A)\cap\mathscr{L}^1(\Sigma_A)\cap\mathscr{L}^2(\Sigma_A)}\lesssim_{k_0} t^{-1/2}\log t,$$

and

$$\begin{aligned}&\int_{\Sigma_A'}\Big(\big(1-C_{\omega_A'}\big)^{-1}I_{5\times5}\Big)(\xi)\omega_A'(\xi)\,d\xi\\ &=\int_{\hat\Sigma_A'}\Big(\big(1-C_{\hat\omega_A'}\big)^{-1}I_{5\times5}\Big)(\xi)\hat\omega_A'(\xi)\,d\xi\\ &=\int_{\hat\Sigma_A'}\Big(N_A^{-1}\big(1-C_{\omega_A}\big)^{-1}N_AI_{5\times5}\Big)(\xi)\hat\omega_A'(\xi)\,d\xi\\ &=\int_{\hat\Sigma_A'}\Big(\big(1-C_{\omega_A}\big)^{-1}I_{5\times5}\Big)\Big((\xi+k_0)\sqrt{48tk_0}\Big)N_A\hat\omega_A'\Big((\xi+k_0)\sqrt{48tk_0}\Big)\,d\xi\\ &=\frac{1}{\sqrt{48tk_0}}\int_{\Sigma_A}\Big(\big(1-C_{\omega_A}\big)^{-1}I_{5\times5}\Big)(\xi)\omega_A(\xi)\,d\xi\\ &=\frac{1}{\sqrt{48tk_0}}\int_{\Sigma_A}\Big(\big(1-C_{\omega_{A^0}}\big)^{-1}I_{5\times5}\Big)(\xi)\omega_{A^0}(\xi)\,d\xi+\mathscr{O}\big(c(k_0)t^{-1}\log t\big).\end{aligned}\tag{3.25}$$

The integral $\int_{\Sigma_B'}((1-C_{\omega_B'})^{-1}I_{5\times5})(\xi)\omega_B'(\xi)\,d\xi$ is evaluated analogously.

Theorem 3.5. *We have*

$$\begin{aligned}q(x,t)=&-\frac{1}{\sqrt{48tk_0}}\left(\int_{\Sigma_A}\Big(\big(1-C_{\omega_{A^0}}\big)^{-1}I_{5\times5}\Big)(\xi)\omega_{A^0}(\xi)\frac{d\xi}{\pi}\right)_{12}\\ &-\frac{1}{\sqrt{48tk_0}}\left(\int_{\Sigma_B}\Big(\big(1-C_{\omega_{B^0}}\big)^{-1}I_{5\times5}\Big)(\xi)\omega_{B^0}(\xi)\frac{d\xi}{\pi}\right)_{12}\\ &+\mathscr{O}\big(c(k_0)t^{-1}\log t\big),\quad \tau\to\infty.\end{aligned}\tag{3.26}$$

Proof. The relation (3.26) follows from Theorem 3.4 and the Eq. (3.25). □

Let $k \in \mathbb{C} \backslash \Sigma_A$. Considering function

$$M^{A^0}(k;x,t) = I_{5\times5} + \int_{\Sigma_A} \frac{((1-C_{\omega_{A^0}})^{-1}I_{5\times5})(\xi)\omega_{A^0}(\xi)}{\xi - k}\frac{d\xi}{2\pi i}, \tag{3.27}$$

we note that it satisfies the Riemann-Hilbert problem

$$\begin{aligned} &M_+^{A^0}(k;x,t) = M_-^{A^0}(k;x,t)J^{A^0}(k;x,t), && k\in\Sigma_A, \\ &M^{A^0}(k;x,t) \to I_{5\times5}, && k\to\infty. \end{aligned} \tag{3.28}$$

In particular,

$$M^{A^0}(k) = I_{5\times5} + \frac{M_1^{A^0}}{k} + \mathcal{O}\left(k^{-2}\right), \quad k\to\infty. \tag{3.29}$$

It follows from (3.27) and (3.29) that

$$M_1^{A^0} = -\int_{\Sigma_A} \left(\left(1-C_{\omega_{A^0}}\right)^{-1} I_{5\times5}\right)(\xi)\omega_{A^0}(\xi)\frac{d\xi}{2\pi i}. \tag{3.30}$$

There is a similar Riemann-Hilbert problem with B^0 on Σ_B, viz.

$$\begin{aligned} &M_+^{B^0}(k;x,t) = M_-^{B^0}(k;x,t)J^{B^0}(k;x,t), && k\in\Sigma_B, \\ &M^{B^0}(k;x,t) \to I_{5\times5}, && k\to\infty \end{aligned}$$

with

$$\begin{aligned} &J^{B^0} = (I_{5\times5} - \omega_{B^0-})^{-1}(I_{5\times5} + \omega_{B^0+}), \\ &\delta_B = e^{\chi(k_0)+8i\tau}(192\tau)^{-i\nu/2}, \end{aligned}$$

where

$$\omega_{B^0} = \omega_{B^0+} = \begin{cases} \begin{pmatrix} 0_{4\times4} & -(\delta_B)^2(k)^{2\nu i}e^{-ik^2/2}\gamma^\dagger(k_0) \\ 0 & 0 \end{pmatrix}, & k\in\Sigma_B^1, \\ \begin{pmatrix} 0_{4\times4} & (\delta_B)^2(k)^{2\nu i}e^{-ik^2/2}\frac{\gamma^\dagger(k_0)}{1+|\gamma(k_0)|^2} \\ 0 & 0 \end{pmatrix}, & k\in\Sigma_B^3, \end{cases} \tag{3.31}$$

$$\omega_{B^0} = \omega_{B^0-} = \begin{cases} \begin{pmatrix} 0_{4\times4} & 0 \\ (\delta_B)^{-2}(k)^{-2\nu i}e^{ik^2/2}\gamma(k_0) & 0 \end{pmatrix}, & k\in\Sigma_B^2, \\ \begin{pmatrix} 0_{4\times4} & 0 \\ -(\delta_B)^{-2}(k)^{-2\nu i}e^{ik^2/2}\frac{\gamma(k_0)}{1+|\gamma(k_0)|^2} & 0 \end{pmatrix}, & k\in\Sigma_B^4. \end{cases} \tag{3.32}$$

Combining (3.23)-(3.24) with (3.31)-(3.32) yields

$$J^{A^0}(k) = \sigma_1\left(J^{B^0}\right)^*(-k^*)\sigma_1.$$

By uniqueness,

$$M^{A^0}(k) = \sigma_1 \big(M^{B^0}\big)^*(-k^*)\sigma_1$$

and, consequently,

$$M_1^{A^0} = -\sigma_1 \big(M_1^{B^0}\big)^* \sigma_1.$$

Therefore, taking into account (3.26) and (3.30), we obtain

$$q(x,t) = \frac{i}{\sqrt{12tk_0}} \Big(M_1^{A^0} - \sigma_1 \big(M_1^{A^0}\big)^* \sigma_1\Big)_{12} + \mathcal{O}\big(c(k_0)t^{-1}\log t\big). \tag{3.33}$$

3.6. Solving a model problem

Here, we are going to compute $(M_1^{A^0})_{12}$. Define the matrix function

$$\Psi(k) = H(k)(-k)^{-i\nu\sigma} e^{1/4ik^2\sigma}, \quad H(k) = (\delta_A)^{-\sigma} M^{A^0}(k)(\delta_A)^{\sigma}. \tag{3.34}$$

It follows from (3.28) that

$$\Psi_+(k) = \Psi_-(k)v(-k_0), \quad k \in \Sigma_A, \tag{3.35}$$

where

$$v(-k_0) = (-k)^{i\nu\sigma} e^{-1/4ik^2\sigma} (\delta_A)^{-\sigma} J^{A^0} (\delta_A)^{\sigma} e^{1/4ik^2\sigma} (-k)^{-i\nu\sigma}.$$

Noting that on the rays $\Sigma_A^1, \Sigma_A^2, \Sigma_A^3, \Sigma_A^4$ the jump matrix $v(-k_0)$ does not depend on k, we write

$$\frac{d\Psi_+(k)}{dk} = \frac{d\Psi_-(k)}{dk} v(-k_0). \tag{3.36}$$

The Eqs. (3.35) and (3.36) yield

$$\frac{d\Psi_+(k)}{dk}\Psi_+^{-1}(k) = \frac{d\Psi_-(k)}{dk}\Psi_-^{-1}(k),$$

so that $(d\Psi(k)/dk)\Psi^{-1}(k)$ has no jump across Σ_A and is entire function. Besides, taking into account (3.34), we obtain

$$\begin{aligned}\frac{d\Psi(k)}{dk}\Psi^{-1}(k) &= \frac{dH(k)}{dk}H^{-1}(k) + \frac{1}{2}ikH(k)\sigma H^{-1}(k) - \frac{i\nu}{k}H(k)\sigma H^{-1}(k) \\ &= \mathcal{O}\left(\frac{1}{k}\right) + \frac{1}{2}ik\sigma - \frac{1}{2}i\delta_A^{-\sigma}\left[\sigma, M_1^{A^0}\right]\delta_A^{\sigma}.\end{aligned}$$

The Liouville's theorem yields

$$\frac{d\Psi(k)}{dk} - \frac{1}{2}ik\sigma\Psi(k) = \beta\Psi(k), \tag{3.37}$$

where

$$\beta = -\frac{1}{2}i\delta_A^{-\sigma}\left[\sigma, M_1^{A^0}\right]\delta_A^{\sigma} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.$$

In particular

$$\left(M_1^{A^0}\right)_{12} = i\delta_A^2\beta_{12}. \tag{3.38}$$

Writing

$$\Psi(k) = \begin{pmatrix} \Psi_{11}(k) & \Psi_{12}(k) \\ \Psi_{21}(k) & \Psi_{22}(k) \end{pmatrix}$$

and using (3.37) gives

$$\begin{aligned} &\frac{d^2\beta_{21}\Psi_{11}(k)}{dk^2} = \left(\beta_{12}\beta_{21} + \frac{i}{2} - \frac{k^2}{4}\right)\beta_{21}\Psi_{11}(k), \\ &\Psi_{21}(k) = \frac{1}{\beta_{21}\beta_{12}}\left(\frac{d\beta_{21}\Psi_{11}(k)}{dk} - \frac{i}{2}k\beta_{21}\Psi_{11}(k)\right), \end{aligned} \tag{3.39}$$

$$\begin{aligned} &\frac{d^2\Psi_{22}(k)}{dk^2} = \left(\beta_{21}\beta_{12} - \frac{i}{2} - \frac{k^2}{4}\right)\Psi_{22}(k), \\ &\beta_{21}\Psi_{12}(k) = \frac{d\Psi_{22}(k)}{dk} + \frac{i}{2}k\Psi_{22}(k). \end{aligned} \tag{3.40}$$

It is well-known that the Weber equation

$$\frac{d^2g(\zeta)}{d\zeta^2} + \left(a + \frac{1}{2} - \frac{\zeta^2}{4}\right)g(\zeta) = 0$$

has the solution

$$g(\zeta) = c_1D_a(\zeta) + c_2D_a(-\zeta).$$

Here, D_a is the standard parabolic-cylinder function. We recall that it satisfies the following equations

$$\begin{aligned} &\frac{dD_a(\zeta)}{d\zeta} + \frac{\zeta}{2}D_a(\zeta) - aD_{a-1}(\zeta) = 0, \\ &D_a(\pm\zeta) = \frac{\Gamma(a+1)e^{i\pi a/2}}{\sqrt{2\pi}}D_{-a-1}(\pm i\zeta) + \frac{\Gamma(a+1)e^{-i\pi a/2}}{\sqrt{2\pi}}D_{-a-1}(\mp i\zeta), \end{aligned} \tag{3.41}$$

where Γ denotes the Gamma function. Using [39, pp. 347-349] we write

$$D_a(\zeta) = \begin{cases} \zeta^a e^{-\zeta^2/4}\left(1 + \mathcal{O}(\zeta^{-2})\right), & |\arg\zeta| < \dfrac{3\pi}{4}, \\ \zeta^a e^{-\zeta^2/4}\left(1 + \mathcal{O}(\zeta^{-2})\right) & \\ \quad -\dfrac{\sqrt{2\pi}}{\Gamma(-a)}e^{a\pi i + \zeta^2/4}\zeta^{-a-1}\left(1 + \mathcal{O}(\zeta^{-2})\right), & \dfrac{\pi}{4} < \arg\zeta < \dfrac{5\pi}{4}, \\ \zeta^a e^{-\zeta^2/4}\left(1 + \mathcal{O}(\zeta^{-2})\right) & \\ \quad -\dfrac{\sqrt{2\pi}}{\Gamma(-a)}e^{-a\pi i + \zeta^2/4}\zeta^{-a-1}\left(1 + \mathcal{O}(\zeta^{-2})\right), & -\dfrac{5\pi}{4} < \arg\zeta < -\dfrac{\pi}{4} \end{cases} \tag{3.42}$$

as $\zeta \to \infty$. Set $a = -i\beta_{21}\beta_{12}$,

$$\begin{aligned} \beta_{21}\Psi_{11}(k) &= c_1 D_a\left(e^{3\pi i/4}k\right) + c_2 D_a\left(e^{-\pi i/4}k\right), \\ \Psi_{22}(k) &= c_3 D_{-a}\left(e^{-3\pi i/4}k\right) + c_4 D_{-a}\left(e^{\pi i/4}k\right). \end{aligned}$$

If $\arg k \in (-\pi, -3\pi/4) \cup (3\pi/4, \pi)$ and $k \to \infty$, then

$$\Psi_{11}(k)(-k)^{i\nu}e^{-ik^2/4} \to I_{4\times 4}, \quad \Psi_{22}(k)(-k)^{-i\nu}e^{ik^2/4} \to 1.$$

It follows from (3.42) that

$$\begin{aligned} \beta_{21}\Psi_{11}(k) &= \beta_{21}e^{\pi\nu/4}D_a\left(e^{3\pi i/4}k\right), \quad \nu = \beta_{21}\beta_{12}, \\ \Psi_{22}(k) &= e^{\pi\nu/4}D_{-a}\left(e^{-3\pi i/4}k\right). \end{aligned}$$

This and the Eqs. (3.39), (3.40) imply

$$\begin{aligned} \Psi_{21}(k) &= \beta_{21}e^{\pi(\nu+i)/4}D_{a-1}\left(e^{3\pi i/4}k\right), \\ \beta_{21}\Psi_{12}(k) &= a e^{\pi(\nu+i)/4}D_{-a-1}\left(e^{-3\pi i/4}k\right). \end{aligned}$$

Analogously, if $\arg k \in (\pi/4, 3\pi/4)$ and $k \to \infty$, then

$$\Psi_{11}(k)(-k)^{i\nu}e^{-ik^2/4} \to I_{4\times 4}, \quad \Psi_{22}(k)(-k)^{-i\nu}e^{ik^2/4} \to 1,$$

which implies

$$\begin{aligned} \beta_{21}\Psi_{11}(k) &= \beta_{21}e^{-3\pi\nu/4}D_a\left(e^{-\pi i/4}k\right), \\ \Psi_{22}(k) &= e^{\pi\nu/4}D_{-a}\left(e^{-3\pi i/4}k\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} \Psi_{21}(k) &= \beta_{21}e^{-3\pi(\nu+i)/4}D_{a-1}\left(e^{-\pi i/4}k\right), \\ \beta_{21}\Psi_{12}(k) &= a e^{\pi(\nu+i)/4}D_{-a-1}\left(e^{-3\pi i/4}k\right). \end{aligned}$$

In particular, if $\arg k = 3\pi/4$, then

$$\Psi_+(k) = \Psi_-(k)\begin{pmatrix} I_{4\times 4} & 0 \\ -\gamma(-k_0) & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} &\beta_{21}e^{\pi(\nu+i)/4}D_{a-1}\left(e^{3\pi i/4}k\right) \\ &= \beta_{21}e^{-3\pi(\nu+i)/4}D_{a-1}\left(e^{-\pi i/4}k\right) - \gamma(-k_0)e^{\pi\nu/4}D_{-a}\left(e^{-3\pi i/4}k\right). \end{aligned} \tag{3.43}$$

We derive by (3.41) that

$$\begin{aligned} &D_{-a}\left(e^{-3\pi i/4}k\right) \\ &= \frac{\Gamma(-a+1)e^{-i\pi a/2}}{\sqrt{2\pi}}D_{a-1}\left(e^{-\pi i/4}k\right) + \frac{\Gamma(-a+1)e^{i\pi a/2}}{\sqrt{2\pi}}D_{a-1}\left(e^{3\pi i/4}k\right). \end{aligned} \tag{3.44}$$

Substituting (3.44) into (3.43) and separating the coefficients at the independent functions leads to the following equation:

$$\beta_{21} = \frac{e^{\pi\nu/2+3\pi i/4}\Gamma(-a+1)}{\sqrt{2\pi}}\gamma(-k_0) = \frac{e^{\pi\nu/2-3\pi i/4}\nu\Gamma(i\nu)}{\sqrt{2\pi}}\gamma(-k_0).$$

Note that $(M^{A^0})^{-1}(k)$ and $(M^{A^0})^{\dagger}(k^*)$ satisfy the same Riemann-Hilbert problem and the uniqueness of the solution gives

$$\left(M^{A^0}\right)^{-1}(k) = \left(M^{A^0}\right)^{\dagger}(k^*).$$

Therefore,

$$\beta_{12} = -\beta_{21}^{\dagger} = \frac{e^{\pi\nu/2-\pi i/4}\nu\Gamma(-i\nu)}{\sqrt{2\pi}}\sigma_2\gamma^{\mathrm{T}}(k_0). \tag{3.45}$$

Theorem 1.1 now follows from (3.33), (3.38) and (3.45).

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