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An hp-Version of C^0-Continuous Petrov-Galerkin Time-Stepping Method for Second-Order Volterra Integro-Differential Equations with Weakly Singular Kernels

Shuangshuang Li, Lina Wang and Lijun Yi*

Department of Mathematics, Shanghai Normal University,
Shanghai 200234, China.

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Abstract. An hp-version of C^0-CPG time-stepping method for second-order Volterra integro-differential equations with weakly singular kernels is studied. In contrast to the methods reducing second-order problems to first-order systems, here the CG and DG methodologies are combined to directly discretise the second-order derivative. An a priori error estimate in the H^1-norm, fully explicit with respect to the local discretisation and regularity parameters, is derived. It is shown that for analytic solutions with start-up singularities, exponential rates of convergence can be achieved by using geometrically refined time steps and linearly increasing approximation orders. Theoretical results are illustrated by numerical examples.

AMS subject classifications: 65R20, 65M60, 65M15

Key words: hp-version, second-order Volterra integro-differential equation, weakly singular kernel, continuous Petrov-Galerkin method, exponential convergence.

1. Introduction

Let T and $\alpha \in [0,1)$ be real numbers, $I := (0,T]$ and $D := \{(t,s) : 0 \le s \le t \le T\}$. In this work, we study numerical methods for the following linear second-order Volterra integro-differential equations (VIDEs):

$$\begin{aligned} &u''(t) = p(t)u'(t) + q(t)u(t) + f(t) + \int_0^t (t-s)^{-\alpha} K(t,s)u(s)ds, \quad t \in I, \\ &u(0) = u_0, \quad u'(0) = u_1. \end{aligned} \tag{1.1}$$

*Corresponding author. *Email addresses:* `1771027043@qq.com` (S. Li), `1842988390@qq.com` (L. Wang), `ylj5152@shnu.edu.cn` (L. Yi)

The functions $p, q, f : I \to \mathbb{R}$ and $K(t,s)\colon D \to \mathbb{R}$ are assumed to be continuous on their respective domains. If $0 < \alpha < 1$, the Eq. (1.1) is referred to as the weakly singular VIDE. Hereafter, sometimes we will write $\dot{u}$ and $\ddot{u}$ for u' and u'', respectively.

Equations of the form (1.1) arise in various areas of physics and engineering — cf. [9] and the references therein. In the last few decades, the numerical analysis of VIDEs attracted considerable attention and the list of numerical methods developed includes collocation [5,14,20,28], Runge-Kutta [4,38], continuous and discontinuous Galerkin methods [21,23]. We also refer the reader to the monographs [6,22]. However, to the best of authors' knowledge, most of the methods deal with VIDEs of the first-order. For the second-order VIDEs, numerical approaches are not well studied and are mainly restricted to collocation methods [1,7,29,32].

It is well-known that the solutions of integral and integro-differential equations of Volterra type with weakly singular kernels are generally not smooth at the initial point [6]. Such singular behavior may result in a low convergence rate of the corresponding numerical method even if high order polynomials are used. In order to overcome the problems generated by the solution singularities, a number of special approaches such as collocation method with graded meshes [5], nonpolynomial spline collocation method [3], and hybrid collocation method [11] were developed. These methods are mainly based on the h-version approach with diminishing time steps and polynomials of a fixed order. Therefore, the best possible convergence order can be only algebraic. In contrast, the p- and hp-version approaches employ approximation polynomials of various order. In particular, since hp-version methods allow locally varying mesh sizes and approximation orders, smooth solutions with possible local singularities can be approximated with a high algebraic order or even with exponential convergence rate [26].

In recent years, p- and hp-versions of Galerkin finite element methods are widely used in approximations of VIDEs. For example, an hp-version of the discontinuous Galerkin time-stepping method for the first-order VIDEs and parabolic VIDEs is, respectively, studied in [8,24], an hp-version of the continuous Petrov-Galerkin method for the first-order linear and nonlinear VIDEs is considered in [35–37], and hp-versions of the discontinuous and continuous Galerkin methods for nonlinear initial value problems are discussed in [25, 33,34]. Some other high-order methods such as spectral Galerkin and collocation methods have been also applied to Volterra type equations — cf. Refs. [10,12,13,15,17,19,27,31,32]. However, the hp-methods for the second-order VIDEs are not well studied and so far, to the best of our knowledge, the problem mentioned has been studied only in [18] for the equations with smooth kernels.

The present work extends the approach of [18] to the second-order VIDE (1.1) with weakly singular kernels. In the method under consideration, the trial spaces consist of C^0-continuous piecewise polynomials, whereas test spaces use discontinuous piecewise polynomials. At each time step, the formulation can be decoupled into local problems, so that the method can be viewed as a time-stepping scheme. Such a C^0-CPG time-stepping method has been before employed in time discretisation of the second-order linear evolution problems [16,30], but the error analysis is based on the traditional h-version approach. We provide certain local time steps conditions, which ensure the well-posedness of the hp-

version C^0-CPG formulation. We also establish a priori error bound in the H^1-norm, fully explicit in local time steps, local approximation orders, and the local regularity of exact solutions. Besides, we prove that for analytic solutions with start-up singularities, the C^0-CPG method based on a special hp-discretisation can achieve the exponential convergence.

The outline of the paper is as follows. In Section 2, we introduce an hp-version of the C^0-CPG method for (1.1) and show the existence and uniqueness of discrete solutions. Section 3 deals with a priori error estimates for the hp-version of the C^0-CPG method. Numerical examples presented in Section 4 are aimed to verify the theoretical results. Finally, some concluding remarks are given in Section 5.

2. An hp-Version of C^0-CPG Time-Stepping Method

Let $\mathscr{T}_h$ be the partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$$

of the interval $(0,T)$ into subintervals $I_n := (t_{n-1}, t_n)$, $n = 1,\ldots,N$, $k_n := t_n - t_{n-1}$ the length of the subinterval I_n and $k := \max\limits_{1\le n\le N} k_n$. For each subinterval I_n we assign a local approximation order $r_n \ge 1$, store all the orders in the polynomial degree vector $\mathbf{r} = (r_1, r_2, \cdots, r_N)$, and define hp-version trial $S^{\mathbf{r},1}(\mathscr{T}_h)$ and test $S^{\mathbf{r}-1,0}(\mathscr{T}_h)$ spaces as follows:

$$\begin{aligned} S^{\mathbf{r},1}(\mathscr{T}_h) &= \left\{u \in H^1(I) : u|_{I_n} \in P_{r_n}(I_n), 1 \le n \le N\right\}, \\ S^{\mathbf{r}-1,0}(\mathscr{T}_h) &= \left\{u \in L^2(I) : u|_{I_n} \in P_{r_n-1}(I_n), 1 \le n \le N\right\}, \end{aligned}$$

where $P_j(I_n)$, $j = r_n, r_n - 1$ denotes the set of all polynomials of degree at most j on I_n. It is clear that the functions in $S^{\mathbf{r}-1,0}(\mathscr{T}_h)$ can be discontinuous at the interior points of the partition $\mathscr{T}_h$.

Let $\varphi : (0,T) \to \mathbb{R}$ be a piecewise continuous function with respect to the partition $\mathscr{T}_h$ and let φ_n^- and φ_n^+ be the left and right limits of φ at the nodes $\{t_n\}_{n=0}^N$, i.e.

$$\begin{aligned} \varphi_n^- &= \lim_{s\to 0, s>0} \varphi(t_n - s), \quad 1 \le n \le N, \\ \varphi_n^+ &= \lim_{s\to 0, s>0} \varphi(t_n + s), \quad 0 \le n \le N-1. \end{aligned}$$

The jumps at the interior nodes are defined by $[\varphi]_n = \varphi_n^+ - \varphi_n^-$ for $1 \le n \le N-1$.

The hp-version of C^0-CPG method for the Eq. (1.1) consists in finding $U \in S^{\mathbf{r},1}(\mathscr{T}_h)$ such that the equation

$$\begin{aligned} &\sum_{n=1}^N \int_{I_n} \ddot{U}\varphi dt + \sum_{n=1}^N \left[\dot{U}\right]_{n-1} \varphi_{n-1}^+ \\ = &\sum_{n=1}^N \int_{I_n} \left(p\dot{U} + qU + f\right)\varphi dt + \sum_{n=1}^N \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} K(t,s) U(s) ds\right) \varphi dt, \\ &U(0) = u_0 \end{aligned} \tag{2.1}$$

is satisfied for all $\varphi \in S^{\mathbf{r-1},0}(\mathscr{T}_h)$. Note that $\dot{U}_0^- := u_1$.

Let us show how the scheme (2.1) is derived. Setting $v := u'$, we write the first equation in (1.1) as

$$v'(t) = p(t)v(t) + q(t)u(t) + f(t) + \int_0^t (t-s)^{-\alpha} K(t,s)u(s)ds, \quad t \in I.$$

If $U \in S^{\mathbf{r},1}(\mathscr{T}_h)$ is an approximation of u and V an approximation of v, then the standard discontinuous Galerkin method for the first-order VIDEs [8] allows to determine $V \in S^{\mathbf{r-1},0}(\mathscr{T}_h)$ by solving the variational equation

$$\begin{aligned}&\sum_{n=1}^{N}\int_{I_n} V'\varphi dt + \sum_{n=1}^{N}[V]_{n-1}\varphi_{n-1}^{+}\\ &= \sum_{n=1}^{N}\int_{I_n}(pV + qU + f)\varphi dt + \sum_{n=1}^{N}\int_{I_n}\left(\int_0^t (t-s)^{-\alpha}K(t,s)U(s)ds\right)\varphi dt \end{aligned} \tag{2.2}$$

valid for all $\varphi \in S^{\mathbf{r-1},0}(\mathscr{T}_h)$. In order to enforce the relation $v = u'$, an additional variational equation should be added in the usual way. However, here we set $V = U'$ in (2.2), thus obtaining (2.1).

Since the test functions are discontinuous, the problem (2.1) can be split into local problems on intervals I_n, $1 \le n \le N$. Therefore, the C^0-CPG method (2.1) can be interpreted as a time stepping scheme — i.e. if U is given on the time intervals $I_m, 1 \le m \le n-1$, the term $U|_{I_n} \in P_{r_n}(I_n)$ is determined by solving the equation

$$\begin{aligned}&\int_{I_n} \ddot{U}\varphi dt + \dot{U}_{n-1}^{+}\varphi_{n-1}^{+}\\ &= \dot{U}_{n-1}^{-}\varphi_{n-1}^{+} + \int_{I_n}\left(p\dot{U} + qU + f\right)\varphi dt + \int_{I_n}\left(\int_0^t (t-s)^{-\alpha}K(t,s)U(s)ds\right)\varphi dt,\\ &U|_{I_n}(t_{n-1}) = U|_{I_{n-1}}(t_{n-1})\end{aligned} \tag{2.3}$$

valid for all $\varphi \in P_{r_n-1}(I_n)$. Here, we set $U|_{I_1}(t_0) = u_0$ and $\dot{U}_0^- = u_1$.

In order to show the well-posedness of the discrete solutions defined by (2.1), we need a few technical results.

Lemma 2.1 (cf. Yi & Guo [37, Lemma 2.2]). *Let $\{t_n\}_{n=0}^N$ be a partition $\mathscr{T}_h$. If $\alpha < 1$ and $g \in L^2(t_{n-1}, t_n)$, then*

$$\int_{t_{n-1}}^{t_n}\left(\int_{t_{n-1}}^{t}(t-s)^{-\alpha}g(s)ds\right)^2 dt \le \frac{k_n^{2(1-\alpha)}}{(1-\alpha)^2}\int_{t_{n-1}}^{t_n} g^2(s)ds \tag{2.4}$$

and if $g \in L^2(0, t_n)$, then

$$\int_0^{t_n}\left(\int_0^t (t-s)^{-\alpha}g(s)ds\right)^2 dt \le \frac{t_n^{2(1-\alpha)}}{(1-\alpha)^2}\int_0^{t_n} g^2(s)ds. \tag{2.5}$$

Lemma 2.2 (cf. Schötzau & Schwab [25, Lemma 2.4]). *On each interval I_n, the inequality*

$$\int_{I_n} |\varphi|^2 dt \le \frac{1}{k_n}\left(\int_{I_n} \varphi dt\right)^2 + \frac{1}{2}\int_{I_n} (t_n - t)(t - t_{n-1})|\dot{\varphi}|^2 dt$$

holds for all $\varphi \in P_{r_n}(I_n)$, $r_n \ge 0$.

Lemma 2.3 (The Poincaré-Friedrichs inequality — cf. Braess [2]). *Let $J = (a, b) \subset \mathbb{R}$ and $h := b - a$. If $u \in H^1(J)$ and $u(a) = 0$, then it satisfies the inequality*

$$\|u\|_{L^2(J)} \le h\|\dot{u}\|_{L^2(J)}.$$

Now we can address the existence and uniqueness of the discrete solutions. Setting

$$\bar{p} := \max_{t\in I}|p(t)|, \quad \bar{q} := \max_{t\in I}|q(t)|, \quad \bar{K} := \max_{(t,s)\in D}|K(t,s)|, \tag{2.6}$$

we define the constants C_n, $1 \le n \le N$ by

$$C_n = \begin{cases} \bar{p} + \bar{q}k_n + \dfrac{\bar{K}k_n^{2-\alpha}}{1-\alpha}, & \text{if } r_n = 1, \\ \sqrt{\dfrac{5}{2}}\left(\bar{p} + \bar{q}k_n + \dfrac{\bar{K}k_n^{2-\alpha}}{1-\alpha}\right), & \text{if } r_n > 1. \end{cases} \tag{2.7}$$

Theorem 2.1. *If the partition $\mathscr{T}_h$ satisfies the inequalities*

$$C_n k_n < 1, \quad 1 \le n \le N \tag{2.8}$$

with the constants $C_n > 0$ defined in (2.7)*, then the discrete problem* (2.1) *has a unique solution $U \in S^{\mathbf{r},1}(\mathscr{T}_h)$.*

Proof. Since the hp-version of the C^0-CPG method (2.1) is a time-stepping scheme, it suffices to prove the existence and uniqueness of the discrete solution of the Eq. (2.3) for $n = 1$. For $n \ge 2$ the proof is analogous. Thus let us first show the uniqueness of the solution of (2.3) for $n = 1$. Assume that there are two solutions U_1 and U_2 of (2.3) on I_1. The difference $E = U_1 - U_2$ satisfies the equation

$$\begin{aligned} &\int_{I_1} \ddot{E}\varphi dt + \dot{E}_0^+\varphi_0^+ \\ &= \dot{E}_0^-\varphi_0^+ + \int_{I_1} p\dot{E}\varphi dt + \int_{I_1} qE\varphi dt + \int_{I_1}\left(\int_0^t (t-s)^{-\alpha}K(t,s)E(s)ds\right)\varphi dt \end{aligned} \tag{2.9}$$

for all $\varphi \in P_{r_1-1}(I_1)$. Using integration by parts in (2.9) gives

$$\begin{aligned} &-\int_{I_1} \dot{E}\dot{\varphi} dt + \dot{E}_1^-\varphi_1^- \\ &= \dot{E}_0^-\varphi_0^+ + \int_{I_1} p\dot{E}\varphi dt + \int_{I_1} qE\varphi dt + \int_{I_1}\left(\int_0^t (t-s)^{-\alpha}K(t,s)E(s)ds\right)\varphi dt \end{aligned} \tag{2.10}$$

for all $\varphi \in P_{r_1-1}(I_1)$. We next consider the following two cases.

Case I. $r_1 = 1$. It is clear that $\ddot{E} = 0$ and $\dot{E}_0^- = 0$. Choosing $\varphi = \dot{E}$ in (2.9) and using (2.6) along with the Cauchy-Schwarz inequality and the inequality (2.4) gives

$$
\begin{aligned}
&\|\dot{E}\|^2_{L^\infty(I_1)} = |\dot{E}_0^+|^2 \\
=& \int_{I_1} p|\dot{E}|^2 dt + \int_{I_1} qE\dot{E}dt + \int_{I_1}\left(\int_0^t (t-s)^{-\alpha}K(t,s)E(s)ds\right)\dot{E}dt \\
\le& \bar{p}\|\dot{E}\|_{L^\infty(I_1)}\int_{I_1}|\dot{E}|dt + \bar{q}\|\dot{E}\|_{L^\infty(I_1)}\int_{I_1}|E|dt + \bar{K}\|\dot{E}\|_{L^\infty(I_1)}\int_{I_1}\left(\int_0^t (t-s)^{-\alpha}|E(s)|ds\right)dt \\
\le& \bar{p}k_1^{1/2}\|\dot{E}\|_{L^\infty(I_1)}\|\dot{E}\|_{L^2(I_1)} + \bar{q}k_1^{1/2}\|\dot{E}\|_{L^\infty(I_1)}\|E\|_{L^2(I_1)} + \bar{K}k_1^{1/2}\|\dot{E}\|_{L^\infty(I_1)}\frac{k_1^{1-\alpha}}{1-\alpha}\|E\|_{L^2(I_1)}.
\end{aligned}
$$

It follows that

$$
\|\dot{E}\|_{L^\infty(I_1)} \le \bar{p}k_1^{1/2}\|\dot{E}\|_{L^2(I_1)} + \bar{q}k_1^{1/2}\|E\|_{L^2(I_1)} + \bar{K}k_1^{1/2}\frac{k_1^{1-\alpha}}{1-\alpha}\|E\|_{L^2(I_1)} \tag{2.11}
$$

and since $\|\dot{E}\|_{L^\infty(I_1)} = k_1^{-1/2}\|\dot{E}\|_{L^2(I_1)}$, the inequality (2.11) can be written as

$$
(1-\bar{p}k_1)\,\|\dot{E}\|_{L^2(I_1)} \le \bar{q}k_1\|E\|_{L^2(I_1)} + \bar{K}\frac{k_1^{2-\alpha}}{1-\alpha}\|E\|_{L^2(I_1)}. \tag{2.12}
$$

Taking into account that $E(0) = 0$ and applying Lemma 2.3 to (2.12), we obtain

$$
\left(1-\bar{p}k_1-\bar{q}k_1^2-\frac{\bar{K}k_1^{3-\alpha}}{1-\alpha}\right)\|E\|_{L^2(I_1)} \le 0
$$

for $\bar{p}k_1 < 1$. Therefore, if

$$
\left(\bar{p}+\bar{q}k_1+\frac{\bar{K}k_1^{2-\alpha}}{1-\alpha}\right)k_1 < 1,
$$

then $E = 0$.

Case II. $r_1 > 1$. Choosing $\varphi = (t-t_0)\ddot{E}$ in (2.9) yields $\varphi_0^+ = 0$. Recalling the definitions (2.6) and using the Cauchy-Schwarz inequality along with the inequality (2.4) yields

$$
\begin{aligned}
&\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt \\
\le& \bar{p}\left(\int_{I_1}|\dot{E}(t)^2(t-t_0)|dt\right)^{1/2}\left(\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt\right)^{1/2} \\
&+\bar{q}\left(\int_{I_1}|E(t)^2(t-t_0)|dt\right)^{1/2}\left(\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt\right)^{1/2} \\
&+\bar{K}\left(\int_{I_1}\left(\int_0^t (t-s)^{-\alpha}|E(s)|ds\right)^2(t-t_0)dt\right)^{1/2}\left(\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \bar{p}k_1^{1/2}\|\dot{E}\|_{L^2(I_1)}\left(\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt\right)^{1/2}+\bar{q}k_1^{1/2}\|E\|_{L^2(I_1)}\left(\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt\right)^{1/2}\\
&\quad+\bar{K}k_1^{1/2}\frac{k_1^{1-\alpha}}{1-\alpha}\|E\|_{L^2(I_1)}\left(\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt\right)^{1/2}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_{I_1}|\ddot{E}(t)|^2(t-t_0)dt\\
&\leq 3\bar{p}^2k_1\|\dot{E}\|^2_{L^2(I_1)}+3\bar{q}^2k_1\|E\|^2_{L^2(I_1)}+3\bar{K}^2k_1\frac{k_1^{2(1-\alpha)}}{(1-\alpha)^2}\|E\|^2_{L^2(I_1)}.
\end{aligned}
\tag{2.13}
$$

Choosing $\varphi=t_1-t$ in (2.10) implies $\varphi_1^-=0$. Therefore, the definitions (2.6), the Cauchy-Schwarz inequality and the inequality (2.4) give

$$
\begin{aligned}
\int_{I_1}\dot{E}(t)dt &\leq \bar{p}\left(\int_{I_1}|\dot{E}(t)|^2dt\right)^{1/2}\left(\int_{I_1}(t_1-t)^2dt\right)^{1/2}\\
&\quad+\bar{q}\left(\int_{I_1}|E(t)|^2dt\right)^{1/2}\left(\int_{I_1}(t_1-t)^2dt\right)^{1/2}\\
&\quad+\bar{K}\left(\int_{I_1}\left(\int_0^t(t-s)^{-\alpha}|E(s)|ds\right)^2dt\right)^{1/2}\left(\int_{I_1}(t_1-t)^2dt\right)^{1/2}\\
&\leq \frac{\sqrt{3}}{3}\bar{p}k_1^{3/2}\|\dot{E}\|_{L^2(I_1)}+\frac{\sqrt{3}}{3}\bar{q}k_1^{3/2}\|E\|_{L^2(I_1)}+\frac{\sqrt{3}}{3}\bar{K}\frac{k_1^{1-\alpha}}{1-\alpha}k_1^{3/2}\|E\|_{L^2(I_1)}.
\end{aligned}
$$

Consequently, we have

$$
\left(\int_{I_1}\dot{E}(t)dt\right)^2\leq \bar{p}^2k_1^3\|\dot{E}\|^2_{L^2(I_1)}+\bar{q}^2k_1^3\|E\|^2_{L^2(I_1)}+\bar{K}^2\frac{k_1^{2(1-\alpha)}}{(1-\alpha)^2}k_1^3\|E\|^2_{L^2(I_1)}. \tag{2.14}
$$

It follows from Lemma 2.2 and the inequalities (2.13), (2.14) that

$$
\begin{aligned}
\|\dot{E}\|^2_{L^2(I_1)} &\leq \frac{1}{k_1}\left(\int_{I_1}\dot{E}dt\right)^2+\frac{k_1}{2}\int_{I_1}|\ddot{E}|^2(t-t_0)dt\\
&\leq \frac{5}{2}\bar{p}^2k_1^2\|\dot{E}\|^2_{L^2(I_1)}+\frac{5}{2}\bar{q}^2k_1^2\|E\|^2_{L^2(I_1)}+\frac{5}{2}\bar{K}^2\frac{k_1^{2(1-\alpha)}}{(1-\alpha)^2}k_1^2\|E\|^2_{L^2(I_1)},
\end{aligned}
$$

or equivalently

$$
\left(1-\frac{5}{2}\bar{p}^2k_1^2\right)\|\dot{E}\|^2_{L^2(I_1)}\leq \frac{5}{2}\bar{q}^2k_1^2\|E\|^2_{L^2(I_1)}+\frac{5}{2}\bar{K}^2\frac{k_1^{2(1-\alpha)}}{(1-\alpha)^2}k_1^2\|E\|^2_{L^2(I_1)}. \tag{2.15}
$$

Since $E(0) = 0$, the application of Lemma 2.3 to (2.15) show that if $(5/2)\bar{p}^2 k_1^2 < 1$, then

$$\left(1 - \frac{5}{2}\bar{p}^2 k_1^2 - \frac{5}{2}\bar{q}^2 k_1^4 - \frac{5}{2}\bar{K}^2 \frac{k_1^{2(3-\alpha)}}{(1-\alpha)^2}\right)\|E\|^2_{L^2(I_1)} \le 0. \tag{2.16}$$

Assuming now that $C_1 k_1 < 1$, where C_1 is the constant defined in (2.7), we obtain $E = 0$.

Thus the uniqueness of the solution is established for any $r_n \ge 1$. Since (2.3) is a linear finite dimensional problem, its solvability follows from the uniqueness property. ☐

Remark 2.1. The mesh condition (2.8) in Theorem 2.1 implies that the well-posedness of discrete solutions is completely independent of the local approximation orders $r_n, 1 \le n \le N$. It is worth noting that (2.8) is only a sufficient condition and is needed because of our proof method. It is not necessary for the well-posedness and numerical experiments show that the *hp*-version of the C^0-CPG scheme can still be convergent, even if (2.8) is not satisfied.

3. Error Analysis

In order to study the errors of the method, we introduce a projection operator that has been previously used [18] during the study of the C^0-CPG method for the second-order VIDEs with smooth kernels. Here we also assume that the partition $\mathcal{T}_h$ satisfies the condition (2.8).

Let $\Lambda = [-1, 1]$ and $r \ge 1$. For any function $u \in H^1(\Lambda)$ such that $u'(\xi)$ is continuous at $\xi = 1$, the projection operator $\Pi^r_\Lambda : H^1(\Lambda) \longrightarrow P_r(\Lambda)$ is defined by

$$\begin{aligned} &\int_\Lambda \left(u - \Pi^r_\Lambda u\right)' \varphi d\xi = 0 \quad \text{for all} \quad \varphi \in P_{r-2}(\Lambda), \\ &\Pi^r_\Lambda u(-1) = u(-1), \quad \left(\Pi^r_\Lambda u\right)'(1) = u'(1). \end{aligned} \tag{3.1}$$

Note that the first condition is not required if $r = 1$. For $r \ge 2$, we can choose $\varphi = 1$ in (3.1) and use integration by parts to obtain $\Pi^r_\Lambda u(1) = u(1)$.

For a general interval $J = (a, b)$, we set $\Pi^r_J u = [\Pi^r_\Lambda(u \circ \mathcal{M})] \circ \mathcal{M}^{-1}$ with $\mathcal{M} : \Lambda \to J$ be the linear transformation $\xi \mapsto t = (a + b + (b - a)\xi)/2$. Let $\mathcal{T}_h$ be an arbitrary partition of $(0, T)$. Then for any $u \in H^2(I)$ we can define a piecewise polynomial $\mathcal{I}u$ by

$$\mathcal{I}u|_{I_n} = \Pi^{r_n}_{I_n} u, \quad 1 \le n \le N.$$

It follows from (3.1) that

$$(\mathcal{I}u)^+_{n-1} = u^+_{n-1}, \quad (\dot{\mathcal{I}u})^-_n = \dot{u}^-_n, \quad 1 \le n \le N.$$

Moreover, if $r_n \ge 2$, then $\mathcal{I}u \in S^{r,1}(\mathcal{T}_h)$ and

$$\int_{I_n} (u - \mathcal{I}u)' \varphi dt = 0, \quad \forall\ \varphi \in P_{r_n-2}(I_n). \tag{3.2}$$

We recall the following lemma.

Lemma 3.1 (Approximation properties of $\mathscr{I}u$, cf. Li *et al.* [18]). *Let $\mathscr{T}_h$ be a partition of $(0,T)$. If $u|_{I_n} \in H^{s_{0,n}+1}(I_n)$ for $s_{0,n} \geq 1$, then for any real s_n, $0 \leq s_n \leq \min\{r_n, s_{0,n}\}$, the inequality*

$$\|u - \mathscr{I}u\|^2_{H^1(0,T)} \leq C \sum_{n=1}^{N} \left(\frac{k_n}{2}\right)^{2s_n} \frac{\Gamma(r_n+1-s_n)}{\Gamma(r_n+1+s_n)} \|u\|^2_{H^{s_n+1}(I_n)} \tag{3.3}$$

holds with a constant $C > 0$ independent of k_n, r_n and s_n.

Now we can derive abstract error estimates for the method under consideration. Let u be the exact solution of (1.1) and U be the approximate solution derived by the method (2.1). As usual, we represent the error $e := u - U$ in the form

$$e = (u - \mathscr{I}u) + (\mathscr{I}u - U) := \eta + \xi.$$

To derive the error bound, we need the discrete Gronwall inequality.

Lemma 3.2 (Discrete Gronwall inequality, — cf. e.g. [6]). *Let $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ be sequences of nonnegative real numbers and $b_1 \leq b_2 \leq \cdots \leq b_N$. If there is a constant $C \geq 0$ and weights $w_i > 0$, $1 \leq i \leq N-1$ such that*

$$a_1 \leq b_1, \quad a_n \leq b_n + C \sum_{i=1}^{n-1} w_i a_i, \quad 2 \leq n \leq N,$$

then

$$a_n \leq b_n \exp\left(C \sum_{i=1}^{n-1} w_i \right), \quad 1 \leq n \leq N.$$

Since Lemma 3.1 can be used to bound η, our task reduces to the estimate of ξ.

Lemma 3.3. *For any $1 \leq n \leq N$ and $r_n \geq 1$, the following estimates hold:*

$$|\dot{\xi}_n^-|^2 \leq C\left(\|\eta\|^2_{L^2(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)} + \|\dot{\eta}\|^2_{L^2(0,t_n)} + \|\dot{\xi}\|^2_{L^2(0,t_n)} \right), \tag{3.4}$$

$$\left(\int_{I_n} \dot{\xi} dt \right)^2 \leq C k_n^2 \left(\|\eta\|^2_{L^2(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)} + \|\dot{\eta}\|^2_{L^2(0,t_n)} + \|\dot{\xi}\|^2_{L^2(0,t_n)} \right), \tag{3.5}$$

$$\int_{I_n} (t - t_{n-1}) |\ddot{\xi}|^2 dt \leq C k_n \left(\|\eta\|^2_{L^2(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)} + \|\dot{\eta}\|^2_{L^2(0,t_n)} + \|\dot{\xi}\|^2_{L^2(0,t_n)} \right), \tag{3.6}$$

where constant $C > 0$ depends on $\bar{p}$, $\bar{q}$, $\bar{K}$ and t_n only.

Proof. The Eqs. (1.1) and (2.3) yield the local Galerkin orthogonality property, i.e.

$$\int_{I_n} \ddot{e} \varphi dt + \dot{e}^+_{n-1} \varphi^+_{n-1}$$
$$= \dot{e}^-_{n-1} \varphi^+_{n-1} + \int_{I_n} p \dot{e} \varphi dt + \int_{I_n} q e \varphi dt + \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} K(t,s) e(s) ds \right) \varphi dt \tag{3.7}$$

for all $\varphi \in P_{r_n - 1}(I_n)$.

We again consider the situations $r_n = 1$ and $r_n > 1$, separately.

Case I. $r_n = 1$. Integration by parts in the left-hand side of (3.7) gives

$$\dot{e}_n^- \varphi_n^- = \dot{e}_{n-1}^- \varphi_{n-1}^+ + \int_{I_n} p\dot{e}\varphi dt + \int_{I_n} qe\varphi dt + \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} K(t,s)e(s)ds \right) \varphi dt. \quad (3.8)$$

Choose now $\varphi = 1$ in (3.8). Since $\dot{e}_0^- = 0$ and $\dot{\eta}_n^- = 0$ for $1 \le n \le N$, we obtain

$$\begin{aligned}
\dot{\xi}_n^- &= \dot{\xi}_{n-1}^- + \int_{I_n} p\dot{e} dt + \int_{I_n} qe dt + \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} K(t,s)e(s)ds \right) dt \\
&\le \dot{\xi}_{n-1}^- + \bar{p} \int_{I_n} |\dot{e}(t)| dt + \bar{q} \int_{I_n} |e(t)| dt + \bar{K} \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} |e(s)| ds \right) dt \\
&\le \dot{\xi}_{n-1}^- + \bar{p} k_n^{1/2} \|\dot{e}\|_{L^2(I_n)} + \bar{q} k_n^{1/2} \|e\|_{L^2(I_n)} + \bar{K} k_n^{1/2} \left\| \int_0^t (t-s)^{-\alpha} |e(s)| ds \right\|_{L^2(I_n)}. \quad (3.9)
\end{aligned}$$

Here, we set $\dot{\xi}_0^- = 0$. Iterating (3.9) yields

$$\dot{\xi}_n^- \le \bar{p} \sum_{i=1}^n k_i^{1/2} \|\dot{e}\|_{L^2(I_i)} + \bar{q} \sum_{i=1}^n k_i^{1/2} \|e\|_{L^2(I_i)} + \bar{K} \sum_{i=1}^n k_i^{1/2} \left\| \int_0^t (t-s)^{-\alpha} |e(s)| ds \right\|_{L^2(I_i)}. \quad (3.10)$$

Squaring both sides of (3.10), using Cauchy-Schwarz inequality and (2.5) gives

$$\begin{aligned}
\left| \dot{\xi}_n^- \right|^2 &\le 3\bar{p}^2 \left(\sum_{i=1}^n k_i^{1/2} \|\dot{e}\|_{L^2(I_i)} \right)^2 + 3\bar{q}^2 \left(\sum_{i=1}^n k_i^{1/2} \|e\|_{L^2(I_i)} \right)^2 \\
&\quad + 3\bar{K}^2 \left(\sum_{i=1}^n k_i^{1/2} \left\| \int_0^t (t-s)^{-\alpha} |e(s)| ds \right\|_{L^2(I_i)} \right)^2 \\
&\le 3\bar{p}^2 t_n \|\dot{e}\|_{L^2(0,t_n)}^2 + 3\bar{q}^2 t_n \|e\|_{L^2(0,t_n)}^2 + 3\bar{K}^2 t_n \frac{t_n^{2(1-\alpha)}}{(1-\alpha)^2} \|e\|_{L^2(0,t_n)}^2 \\
&\le C \left(\|\eta\|_{L^2(0,t_n)}^2 + \|\xi\|_{L^2(0,t_n)}^2 + \|\dot{\eta}\|_{L^2(0,t_n)}^2 + \|\dot{\xi}\|_{L^2(0,t_n)}^2 \right),
\end{aligned}$$

and the inequality (3.4) is proven for $r_n = 1$. This yields the estimate (3.5) — i.e.

$$\left(\int_{I_n} \dot{\xi} dt \right)^2 = k_n^2 \left| \dot{\xi}_n^- \right|^2 \le C k_n^2 \left(\|\eta\|_{L^2(0,t_n)}^2 + \|\xi\|_{L^2(0,t_n)}^2 + \|\dot{\eta}\|_{L^2(0,t_n)}^2 + \|\dot{\xi}\|_{L^2(0,t_n)}^2 \right),$$

and (3.6) follows directly from the fact that $\ddot{\xi} = 0$ if $r_n = 1$.

Case II. $r_n > 1$. Since $\dot{\eta}_n^- = 0$ for any $1 \le n \le N$, we can use integration by parts in (3.7) and (3.2) to obtain

$$\begin{aligned}
&\dot{\xi}_n^- \varphi_n^- - \int_{I_n} \dot{\xi} \dot{\varphi} dt \\
&= \dot{\xi}_{n-1}^- \varphi_{n-1}^+ + \int_{I_n} p\dot{e}\varphi dt + \int_{I_n} qe\varphi dt + \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} K(t,s)e(s)ds \right) \varphi dt \quad (3.11)
\end{aligned}$$

for all $\varphi \in P_{r_n-1}(I_n)$. Integrating the result by parts yields

$$\begin{aligned}&\dot{\xi}_{n-1}^{+}\varphi_{n-1}^{+}+\int_{I_n}\ddot{\xi}\varphi dt\\ =&\dot{\xi}_{n-1}^{-}\varphi_{n-1}^{+}+\int_{I_n}p\dot{e}\varphi dt+\int_{I_n}qe\varphi dt+\int_{I_n}\left(\int_0^t(t-s)^{-\alpha}K(t,s)e(s)ds\right)\varphi dt \qquad (3.12)\end{aligned}$$

for all $\varphi \in P_{r_n-1}(I_n)$. For $\varphi = \dot{\xi}$, the Eq. (3.12) takes the form

$$\begin{aligned}&\frac{1}{2}\left|\dot{\xi}_{n}^{-}\right|^{2}+\frac{1}{2}\left|\dot{\xi}_{n-1}^{+}\right|^{2}\\ =&\dot{\xi}_{n-1}^{-}\dot{\xi}_{n-1}^{+}+\int_{I_n}p\dot{e}\dot{\xi}dt+\int_{I_n}qe\dot{\xi}dt+\int_{I_n}\left(\int_0^t(t-s)^{-\alpha}K(t,s)e(s)ds\right)\dot{\xi}dt\\ \leq&\frac{1}{2}\left|\dot{\xi}_{n-1}^{-}\right|^{2}+\frac{1}{2}\left|\dot{\xi}_{n-1}^{+}\right|^{2}+\bar{p}\int_{I_n}\left|\dot{e}\dot{\xi}\right|dt+\bar{q}\int_{I_n}\left|e\dot{\xi}\right|dt+\bar{K}\int_{I_n}\left(\int_0^t(t-s)^{-\alpha}|e(s)|ds\right)|\dot{\xi}|dt\\ \leq&\frac{1}{2}\left|\dot{\xi}_{n-1}^{-}\right|^{2}+\frac{1}{2}\left|\dot{\xi}_{n-1}^{+}\right|^{2}+\bar{p}\|\dot{e}\|_{L^2(I_n)}\|\dot{\xi}\|_{L^2(I_n)}+\bar{q}\|e\|_{L^2(I_n)}\|\dot{\xi}\|_{L^2(I_n)}\\ &+\bar{K}\left\|\int_0^t(t-s)^{-\alpha}|e(s)|ds\right\|_{L^2(I_n)}\|\dot{\xi}\|_{L^2(I_n)},\end{aligned}$$

which implies that

$$\begin{aligned}\left|\dot{\xi}_{n}^{-}\right|^{2}\leq&\left|\dot{\xi}_{n-1}^{-}\right|^{2}+2\bar{p}\|\dot{e}\|_{L^2(I_n)}\|\dot{\xi}\|_{L^2(I_n)}+2\bar{q}\|e\|_{L^2(I_n)}\|\dot{\xi}\|_{L^2(I_n)}\\ &+2\bar{K}\left\|\int_0^t(t-s)^{-\alpha}|e(s)|ds\right\|_{L^2(I_n)}\|\dot{\xi}\|_{L^2(I_n)}\\ \leq&\left|\dot{\xi}_{n-1}^{-}\right|^{2}+\bar{p}\|\dot{e}\|^2_{L^2(I_n)}+\bar{p}\|\dot{\xi}\|^2_{L^2(I_n)}+\bar{q}\|e\|^2_{L^2(I_n)}+\bar{q}\|\dot{\xi}\|^2_{L^2(I_n)}\\ &+\bar{K}\left\|\int_0^t(t-s)^{-\alpha}|e(s)|ds\right\|^2_{L^2(I_n)}+\bar{K}\|\dot{\xi}\|^2_{L^2(I_n)}. \qquad (3.13)\end{aligned}$$

Iterating (3.13) and using (2.5) gives

$$\begin{aligned}\left|\dot{\xi}_{n}^{-}\right|^{2}\leq&\bar{p}\|\dot{e}\|^2_{L^2(0,t_n)}+\bar{p}\|\dot{\xi}\|^2_{L^2(0,t_n)}+\bar{q}\|e\|^2_{L^2(0,t_n)}+\bar{q}\|\dot{\xi}\|^2_{L^2(0,t_n)}\\ &+\bar{K}\left\|\int_0^t(t-s)^{-\alpha}e(s)ds\right\|^2_{L^2(0,t_n)}+\bar{K}\|\dot{\xi}\|^2_{L^2(0,t_n)}\\ \leq&\bar{p}\|\dot{e}\|^2_{L^2(0,t_n)}+\bar{p}\|\dot{\xi}\|^2_{L^2(0,t_n)}+\bar{q}\|e\|^2_{L^2(0,t_n)}+\bar{q}\|\dot{\xi}\|^2_{L^2(0,t_n)}\\ &+\bar{K}\frac{t_n^{2(1-\alpha)}}{(1-\alpha)^2}\|e\|^2_{L^2(0,t_n)}+\bar{K}\|\dot{\xi}\|^2_{L^2(0,t_n)}\\ \leq&C\left(\|\eta\|^2_{L^2(0,t_n)}+\|\xi\|^2_{L^2(0,t_n)}+\|\dot{\eta}\|^2_{L^2(0,t_n)}+\|\dot{\xi}\|^2_{L^2(0,t_n)}\right),\end{aligned}$$

so that (3.4) is also proven for $r_n > 1$. Furthermore, choosing $\varphi = (t_{n-1} - t)$ in (3.11) and using (2.5), gives

$$
\begin{aligned}
& -k_n\dot{\xi}_n^- + \int_{I_n} \dot{\xi}\, dt \\
&= \int_{I_n} p(t)\dot{e}(t)(t_{n-1} - t)dt + \int_{I_n} q(t)e(t)(t_{n-1} - t)dt \\
&\quad + \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} K(t,s)e(s)ds \right)(t_{n-1} - t)dt \\
&\le \bar{p} \int_{I_n} |\dot{e}(t)(t_{n-1} - t)|dt + \bar{q} \int_{I_n} |e(t)(t_{n-1} - t)|dt \\
&\quad + \bar{K} \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} |e(s)|ds \right) |t_{n-1} - t|dt \\
&\le \frac{\sqrt{3}}{3} k_n^{3/2} \bar{p} \|\dot{e}\|_{L^2(I_n)} + \frac{\sqrt{3}}{3} k_n^{3/2} \bar{q} \|e\|_{L^2(I_n)} + \frac{\sqrt{3}}{3} k_n^{3/2} \bar{K} \left\| \int_0^t (t-s)^{-\alpha} |e(s)|ds \right\|_{L^2(0,t_n)} \\
&\le \frac{\sqrt{3}}{3} k_n^{3/2} \bar{p} \|\dot{e}\|_{L^2(I_n)} + \frac{\sqrt{3}}{3} k_n^{3/2} \bar{q} \|e\|_{L^2(I_n)} + \frac{\sqrt{3}}{3} k_n^{3/2} \bar{K} \frac{t_n^{1-\alpha}}{1-\alpha} \|e\|_{L^2(0,t_n)}.
\end{aligned}
$$

This and (3.4) imply the estimate (3.5). Indeed, we have

$$
\begin{aligned}
\left(\int_{I_n} \dot{\xi}\, dt \right)^2 &\le 4k_n^2 \left|\dot{\xi}_n^-\right|^2 + \frac{4}{3} k_n^3 \bar{p}^2 \|\dot{e}\|_{L^2(I_n)}^2 + \frac{4}{3} k_n^3 \bar{q}^2 \|e\|_{L^2(I_n)}^2 + \frac{4}{3} k_n^3 \bar{K}^2 \frac{t_n^{2(1-\alpha)}}{(1-\alpha)^2} \|e\|_{L^2(0,t_n)}^2 \\
&\le Ck_n^2 \left(\|\eta\|_{L^2(0,t_n)}^2 + \|\xi\|_{L^2(0,t_n)}^2 + \|\dot{\eta}\|_{L^2(0,t_n)}^2 + \|\dot{\xi}\|_{L^2(0,t_n)}^2 \right).
\end{aligned}
$$

Finally, choosing $\varphi = (t - t_{n-1})\ddot{\xi}$ in (3.12) and using (2.5), we obtain

$$
\begin{aligned}
& \int_{I_n} (t - t_{n-1})|\ddot{\xi}|^2 dt \\
&= \int_{I_n} p(t)\dot{e}(t)(t - t_{n-1})\ddot{\xi} dt + \int_{I_n} q(t)e(t)(t - t_{n-1})\ddot{\xi} dt \\
&\quad + \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} K(t,s)e(s)ds \right)(t - t_{n-1})\ddot{\xi} dt \\
&\le \bar{p} \int_{I_n} |\dot{e}(t)(t - t_{n-1})\ddot{\xi}|dt + \bar{q} \int_{I_n} |e(t)(t - t_{n-1})\ddot{\xi}|dt \\
&\quad + \bar{K} \int_{I_n} \left(\int_0^t (t-s)^{-\alpha} |e(s)|ds \right) |(t - t_{n-1})\ddot{\xi}|dt \\
&\le \bar{p} k_n^{1/2} \|\dot{e}\|_{L^2(I_n)} \left(\int_{I_n} (t - t_{n-1})|\ddot{\xi}|^2 dt \right)^{1/2} + \bar{q} k_n^{1/2} \|e\|_{L^2(I_n)} \left(\int_{I_n} (t - t_{n-1})|\ddot{\xi}|^2 dt \right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+\bar{K}k_n^{1/2}\left\|\int_0^t (t-s)^{-\alpha}|e(s)|ds\right\|_{L^2(0,t_n)}\left(\int_{I_n}(t-t_{n-1})|\ddot{\xi}|^2dt\right)^{1/2}\\
\leq\ &\bar{p}k_n^{1/2}\|\dot{e}\|_{L^2(I_n)}\left(\int_{I_n}(t-t_{n-1})|\ddot{\xi}|^2dt\right)^{1/2}+\bar{q}k_n^{1/2}\|e\|_{L^2(I_n)}\left(\int_{I_n}(t-t_{n-1})|\ddot{\xi}|^2dt\right)^{1/2}\\
&+\bar{K}k_n^{1/2}\frac{t_n^{1-\alpha}}{1-\alpha}\|e\|_{L^2(0,t_n)}\left(\int_{I_n}(t-t_{n-1})|\ddot{\xi}|^2dt\right)^{1/2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\int_{I_n}(t-t_{n-1})|\ddot{\xi}|^2dt &\leq \left(\bar{p}k_n^{1/2}\|\dot{e}\|_{L^2(I_n)}+\bar{q}k_n^{1/2}\|e\|_{L^2(I_n)}+\bar{K}k_n^{1/2}\frac{t_n^{1-\alpha}}{1-\alpha}\|e\|_{L^2(0,t_n)}\right)^2\\
&\leq Ck_n\left(\|\eta\|^2_{L^2(0,t_n)}+\|\xi\|^2_{L^2(0,t_n)}+\|\dot{\eta}\|^2_{L^2(0,t_n)}+\|\dot{\xi}\|^2_{L^2(0,t_n)}\right).
\end{aligned}
$$

This completes the proof of (3.6) for $r_n > 1$. □

We are ready to present an abstract error bound for the hp-version of C^0-CPG method (2.1).

Theorem 3.1. *If u is the solution of* (1.1) *and U its approximate solution obtained by the method* (2.1)*, then for any $r_n \geq 2$ and any sufficiently small k_n, the following estimate holds:*

$$
\|u-U\|_{H^1(0,T)} \leq C\|u-\mathscr{I}u\|_{H^1(0,T)}, \tag{3.14}
$$

where $C > 0$ is a constant, which depends on $\bar{p}$, $\bar{q}$, $\bar{K}$, and T only.

Proof. It follows from Lemma 2.2 and estimates (3.5), (3.6) that

$$
\begin{aligned}
\int_{I_n}|\dot{\xi}|^2dt &\leq \frac{1}{k_n}\left(\int_{I_n}\dot{\xi}dt\right)^2+\frac{k_n}{2}\int_{I_n}(t-t_{n-1})|\ddot{\xi}|^2dt\\
&\leq Ck_n\left(\|\eta\|^2_{L^2(0,t_n)}+\|\xi\|^2_{L^2(0,t_n)}+\|\dot{\eta}\|^2_{L^2(0,t_n)}+\|\dot{\xi}\|^2_{L^2(0,t_n)}\right)
\end{aligned}
$$

for $r_n \geq 1$. If k_n is sufficiently small, the above inequality can be rewritten as

$$
\|\dot{\xi}\|^2_{L^2(I_n)} \leq Ck_n\left(\|\eta\|^2_{H^1(0,t_n)}+\|\xi\|^2_{L^2(0,t_n)}+\|\dot{\xi}\|^2_{L^2(0,t_{n-1})}\right),
$$

or equivalently,

$$
\frac{\|\dot{\xi}\|^2_{L^2(I_n)}}{k_n} \leq C\left(\|\eta\|^2_{H^1(0,t_n)}+\|\xi\|^2_{L^2(0,t_n)}\right)+\sum_{i=1}^{n-1}k_i\frac{\|\dot{\xi}\|^2_{L^2(I_i)}}{k_i}.
$$

The discrete Gronwall inequality — cf. Lemma 3.2, implies

$$
\frac{\|\dot{\xi}\|^2_{L^2(I_n)}}{k_n} \leq C\left(\|\eta\|^2_{H^1(0,t_n)}+\|\xi\|^2_{L^2(0,t_n)}\right)\cdot\exp\left(C\sum_{i=1}^{n-1}k_i\right).
$$

Consequently, we have

$$\begin{aligned}\|\dot{\xi}\|^2_{L^2(I_n)} &\le Ck_n e^{Ct_{n-1}}\left(\|\eta\|^2_{H^1(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)}\right)\\ &\le Ck_n\left(\|\eta\|^2_{H^1(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)}\right).\end{aligned} \tag{3.15}$$

Summing the estimates (3.15) in i from 1 to n yields

$$\|\dot{\xi}\|^2_{L^2(0,t_n)} \le C\sum_{i=1}^{n} k_i\left(\|\eta\|^2_{H^1(0,t_i)} + \|\xi\|^2_{L^2(0,t_i)}\right) \le Ct_n\left(\|\eta\|^2_{H^1(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)}\right). \tag{3.16}$$

Since $\xi \in H^1(0,T)$ for $r_n \ge 2$, $1 \le n \le N$, we can write

$$\left|\xi^+_{n-1}\right|^2 = \left|\xi^-_{n-1}\right|^2 = \left(\int_0^{t_{n-1}} \dot{\xi}\,dt\right)^2 \le t_{n-1}\|\dot{\xi}\|^2_{L^2(0,t_{n-1})}, \tag{3.17}$$

and since

$$\xi(t) = \int_{t_{n-1}}^{t} \dot{\xi}\,dt + \xi^+_{n-1},$$

we obtain

$$\begin{aligned}\|\xi\|^2_{L^2(I_n)} &\le \int_{I_n}\left(\int_{t_{n-1}}^{t_n} |\dot{\xi}|dt + \left|\xi^+_{n-1}\right|\right)^2 dt = k_n\left(\int_{I_n} |\dot{\xi}|dt + \left|\xi^+_{n-1}\right|\right)^2\\ &\le 2k_n^2\|\dot{\xi}\|^2_{L^2(I_n)} + 2k_n\left|\xi^+_{n-1}\right|^2 \le 2k_n^2\|\dot{\xi}\|^2_{L^2(I_n)} + 2k_n t_{n-1}\|\dot{\xi}\|^2_{L^2(0,t_{n-1})}\\ &\le Ck_n\|\dot{\xi}\|^2_{L^2(0,t_n)}.\end{aligned} \tag{3.18}$$

Substituting (3.16) into (3.18) gives

$$\|\xi\|^2_{L^2(I_n)} \le Ck_n\left(\|\eta\|^2_{H^1(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)}\right).$$

If k_n is sufficiently small, the above inequality can be rewritten as

$$\|\xi\|^2_{L^2(I_n)} \le Ck_n\|\eta\|^2_{H^1(0,t_n)} + Ck_n\|\xi\|^2_{L^2(0,t_{n-1})},$$

or equivalently,

$$\frac{\|\xi\|^2_{L^2(I_n)}}{k_n} \le C\|\eta\|^2_{H^1(0,t_n)} + C\sum_{i=1}^{n-1} k_i\frac{\|\xi\|^2_{L^2(I_i)}}{k_i}.$$

Applying the discrete Gronwall inequality — cf. Lemma 3.2, yields

$$\frac{\|\xi\|^2_{L^2(I_n)}}{k_n} \le C\|\eta\|^2_{H^1(0,t_n)} \cdot \exp\left(C\sum_{i=1}^{n-1} k_i\right),$$

or

$$\|\xi\|^2_{L^2(I_n)} \le Ck_n e^{Ct_{n-1}}\|\eta\|^2_{H^1(0,t_n)}. \tag{3.19}$$

Summing (3.19) in i from 1 to n leads to the inequality

$$\|\xi\|^2_{L^2(0,t_n)} \le C\sum_{i=1}^{n} k_i\|\eta\|^2_{H^1(0,t_i)} \le Ct_n\|\eta\|^2_{H^1(0,t_n)}, \tag{3.20}$$

and combining estimates (3.15), (3.19) and (3.20) gives

$$\begin{aligned}\|\xi\|^2_{H^1(I_n)} &= \|\xi\|^2_{L^2(I_n)} + \|\dot{\xi}\|^2_{L^2(I_n)} \\ &\le Ck_n\left(\|\eta\|^2_{H^1(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)}\right) \le Ck_n\|\eta\|^2_{H^1(0,t_n)}.\end{aligned} \tag{3.21}$$

Summing (3.21) in n from 1 to N, we get

$$\|\xi\|^2_{H^1(0,T)} \le C\sum_{n=1}^{N} k_n\|\eta\|^2_{H^1(0,t_n)} \le CT\|\eta\|^2_{H^1(0,T)}, \tag{3.22}$$

and the estimate (3.14) follows from the triangle inequality and (3.22). □

Remark 3.1. The C^0-CPG scheme is designed for $r_n \ge 1$, but the main result of Theorem 3.1 is valid only for $r_n \ge 2$ because our proof uses the global continuity of piecewise polynomials $\mathscr{I}u$ and for $r_n = 1$ the corresponding can be discontinuous.

3.1. Convergence of the hp-version of C^0-CPG method

We now evaluate the errors for the hp-version of C^0-CPG time-stepping method on an arbitrary partition $\mathscr{T}_h$.

Theorem 3.2. *Let $\mathscr{T}_h$ be a partition of $(0,T)$, u the exact solution of (1.1), and U the approximate solution obtained by the method (2.1). If $u|_{I_n} \in H^{s_{0,n}+1}(I_n)$ for $s_{0,n} \ge 1$, then for $r_n \ge 2$ and sufficiently small k_n, the estimate*

$$\|u-U\|^2_{H^1(0,T)} \le C\sum_{n=1}^{N}\left(\frac{k_n}{2}\right)^{2s_n}\frac{\Gamma(r_n+1-s_n)}{\Gamma(r_n+1+s_n)}\|u\|^2_{H^{s_n+1}(I_n)} \tag{3.23}$$

holds for any real s_n, $0 \le s_n \le \min\{r_n, s_{0,n}\}$ with a constant $C > 0$ independent of k_n, r_n and s_n.

Moreover, if $\mathscr{T}_h$ is a quasi-uniform partition of $(0,T)$, i.e. if there is a constant $C_q > 0$ such that $k \le C_q k_n$ for all $1 \le n \le N$, $r_n \equiv r \ge 2$ and $u \in H^{s+1}(I)$ for $s \ge 1$, then

$$\|u-U\|_{H^1(0,T)} \le C\frac{k^{\min\{s,r\}}}{r^s}\|u\|_{H^{s+1}(0,T)} \tag{3.24}$$

with a constant $C > 0$ independent of k and r.

Proof. The estimate (3.23) immediately follows from Theorem 3.1 and the approximation properties of $\mathscr{I}u$, cf. Lemma 3.1. The estimate (3.24) follows from (3.23) and Stirling's formula. □

Remark 3.2. The estimate (3.23) is fully explicit with respect to the local time steps k_n, the local approximation order r_n, and the local regularity s_n of the exact solution.

Remark 3.3. The estimate (3.24) implies that the hp-version of C^0-CPG method converges if the time step k diminishes or/and if the polynomial degree r increases. In particular, (3.24) shows that the p-version approach can produce arbitrary high-order algebraic convergence rate (or spectral convergence), provided that the exact solution u is smooth enough.

3.2. Exponential convergence for singular solutions

As we know, the solution of VIDEs (1.1) with weakly singular kernels — i.e. if $0 < \alpha < 1$, are generally not smooth at $t = 0^+$ [6, 29]. Such singular behavior of u near $t = 0$ may lead to suboptimal convergence rates if we use quasi-uniform time partition.

In this section, we consider the hp-version of C^0-CPG method for problems, the solutions of which have start-up singularities as $t \to 0$. More precisely, we assume that the solution u of the problem (1.1) is analytic in $(0, T]$ and satisfies the analytic regularity

$$\left|u^{(s)}(t)\right| \le C d^s \Gamma(s+1) t^{\theta - s}, \quad t \in (0, T], \quad s \in \mathbb{N}_0, \quad \theta > 2 \tag{3.25}$$

for positive constants C and d, which may depend on u.

We show that the hp-version of C^0-CPG time-stepping method based on geometrically refined time steps and linearly increasing approximation orders, leads to exponential rates of convergence for the solutions satisfying condition (3.25).

We begin with some definitions —cf. [8, 25, 37].

Definition 3.1 (Geometric partition)**.** A geometric partition $\mathscr{T}_{M,\sigma}$ of $(0, T)$ with grading factor $\sigma \in (0, 1)$ and M levels of refinement is obtained by first quasi-uniformly partitioning $(0, T)$ into (coarse) intervals $\{J_\ell\}_{\ell=1}^{L}$, and then the first interval $J_1 = (0, T_1)$ near $t = 0$ is further subdivided into $M + 1$ subintervals $\{I_m\}_{m=1}^{M+1}$ by using the time steps

$$t_0 = 0, \quad t_m = \sigma^{M-m+1} T_1, \quad 1 \le m \le M + 1.$$

The parameter $\sigma \in (0, 1)$ is called the geometric refinement factor. Obviously, the time steps $k_m = t_m - t_{m-1}$ satisfy $k_m = \lambda t_{m-1}$ with $\lambda := (1 - \sigma)/\sigma$ for $2 \le m \le M + 1$.

Definition 3.2 (Linearly increasing approximation order)**.** Let $\mathscr{T}_{M,\sigma}$ be a geometric mesh of $(0, T)$. An approximation degree vector $\mathbf{r}$ on $\mathscr{T}_{M,\sigma}$ is called linear with slope $\nu > 0$ if $r_1 = 2$, $r_m = \max\{2, \lfloor \nu m \rfloor\}$ for $2 \le m \le M + 1$ on the geometrically refined elements $\{I_m\}_{m=1}^{M+1}$ and if $r_\ell = \max\{2, \lfloor \nu(M + 1) \rfloor\}$ on the coarse element J_ℓ for $2 \le \ell \le L$.

The following result establishes the exponential rate of convergence in terms of the degree of freedom for the hp-version of C^0-CPG time-stepping method.

Theorem 3.3. *Assume that the solution u of the Eq.* (1.1) *satisfies the condition* (3.25)*, $\mathscr{T}_{M,\sigma}$ is a geometric mesh of* $(0, T)$ *satisfying* (2.8) *and U the approximate solution of* (1.1) *obtained* *by the method* (2.1). *Then there exists a slope $\nu_0>0$ solely depending on σ and θ such that for all linear polynomial degree vectors* $\mathbf{r}$ *with a slope $\nu\ge\nu_0$, the error estimate*

$$\|u-U\|_{H^1(0,T)}\le Ce^{-b\sqrt{D}} \tag{3.26}$$

holds with constants $C,b>0$ independent of $D=\dim\big(S^{\mathbf{r}-1,0}(\mathcal{T}_{M,\sigma})\big)$.

Proof. The proof of (3.26) is similar to the corresponding proofs of [8, 25, 37], but we present it here in order to make the paper self-contained.

By Theorem 3.1, we only have to estimate the term $u-\mathscr{I}u$. For this purpose, we choose $\nu\ge1$ such that $r_m=\lfloor \nu m\rfloor\ge2$ on the geometrically refined intervals $\{I_m\}_{m=2}^{M+1}$ and $r_\ell=\lfloor\nu(M+1)\rfloor\ge2$ on the coarse intervals $\{J_\ell\}_{\ell=2}^{L}$. By Theorem 3.1, we have

$$\|u-U\|^2_{H^1(0,T)}\le C\left(\sum_{1\le m\le M+1}\|u-\mathscr{I}u\|^2_{H^1(I_m)}+\sum_{2\le\ell\le L}\|u-\mathscr{I}u\|^2_{H^1(J_\ell)}\right). \tag{3.27}$$

On the first subinterval I_1, we use Lemma 3.1 with $s_{0,1}=s_1=1$ and (3.25) thus obtaining

$$\|u-\mathscr{I}u\|^2_{H^1(I_1)}\le C\left(\frac{k_1}{2}\right)^2\|u\|^2_{H^2(I_1)}\le Ck_1^{2\theta-1}=C\sigma^{M(2\theta-1)}. \tag{3.28}$$

On the subintervals I_m, $2\le m\le M+1$, we use Lemma 3.1 and get

$$\begin{aligned}\sum_{2\le m\le M+1}\|u-\mathscr{I}u\|^2_{H^1(I_m)}&\le C\sum_{2\le m\le M+1}\left(\frac{k_m}{2}\right)^{2s_m}\frac{\Gamma(r_m+1-s_m)}{\Gamma(r_m+1+s_m)}\|u\|^2_{H^{s_m+1}(I_m)}\\&\le C\sum_{2\le m\le M+1}\left(\frac{k_m}{2}\right)^{2s_m+1}\frac{\Gamma(r_m+1-s_m)}{\Gamma(r_m+1+s_m)}\|u\|^2_{W^{s_m+1,\infty}(I_m)}\end{aligned} \tag{3.29}$$

for $0\le s_m\le\min\{r_m,s_{0,m}\}$. Since away from $t=0$ the solution u is analytic, for $2\le m\le M+1$ the regularity exponents $s_{0,m}$ can be chosen arbitrarily large. For convenience, we set

$$e_m=\left(\frac{k_m}{2}\right)^{2s_m+2}\frac{\Gamma(r_m+1-s_m)}{\Gamma(r_m+1+s_m)}\|u\|^2_{W^{s_m+1,\infty}(I_m)},\quad 2\le m\le M+1.$$

In view of (3.25), for any $s_m\ge0$ we have

$$\|u\|^2_{W^{s_m+1,\infty}(I_m)}\le Cd^{2s_m}\Gamma(2s_m+1)\sigma^{2(M-m+2)(\theta-s_m-1)},\quad 2\le m\le M+1.$$

Since $k_m=\lambda t_{m-1}$ with $t_{m-1}=\sigma^{M-m+2}T_1$, it yields

$$\begin{aligned}e_m&\le C\left(\frac{\lambda\sigma^{M-m+2}}{2}\right)^{2s_m+2}\frac{\Gamma(r_m+1-s_m)}{\Gamma(r_m+1+s_m)}d^{2s_m}\Gamma(2s_m+1)\big(\sigma^{M-m+2}\big)^{2(\theta-s_m-1)}\\&\le C\sigma^{(M-m+2)2\theta}\left((\lambda d)^{2s_m}\frac{\Gamma(r_m+1-s_m)}{\Gamma(r_m+1+s_m)}\Gamma(2s_m+1)\right).\end{aligned}$$

Selecting $s_m = \varepsilon_m r_m$ with an $\varepsilon_m \in (0,1)$ and using Stirling's formula gives

$$e_m \le C\sigma^{(M-m+2)2\theta} r_m^{1/2} \left((\lambda d \varepsilon_m)^{2\varepsilon_m} \frac{(1-\varepsilon_m)^{1-\varepsilon_m}}{(1+\varepsilon_m)^{1+\varepsilon_m}} \right)^{r_m}.$$

Since the function

$$g_{\lambda,d}(\varepsilon) = (\lambda d \varepsilon)^{2\varepsilon} \frac{(1-\varepsilon)^{1-\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}$$

satisfies the inequality

$$0 < \inf_{0<\varepsilon<1} g_{\lambda,d}(\varepsilon) =: g_{\lambda,d}(\varepsilon_{\min}) < 1$$

with $\varepsilon_{\min} = 1/\sqrt{1+\lambda^2 d^2}$, we set

$$g_{\min} := g_{\lambda,d}(\varepsilon_{\min})$$

and choose

$$\varepsilon_m = \varepsilon_{\min}, \quad 2 \le m \le M+1.$$

Then

$$e_m \le C\sigma^{(M-m+2)2\theta} r_m^{1/2} g_{\min}^{r_m} \le C\sigma^{2M\theta} \big(\nu(M+1)\big)^{1/2} \big(\sigma^{(2-m)2\theta} g_{\min}^{\nu m}\big).$$

If

$$\nu_0 := \max\left\{ \frac{2\theta \log\sigma}{\log(g_{\min})}, 1 \right\},$$

then $g_{\min}^{\nu m} \le \sigma^{2m\theta}$ for $\nu \ge \nu_0$. Consequently,

$$e_m \le C\sigma^{2M\theta}\big(\nu(M+1)\big)^{1/2}\sigma^{4\theta} \le C\sigma^{2M\theta}\big(\nu(M+1)\big)^{1/2}, \quad 2 \le m \le M+1. \tag{3.30}$$

Substituting (3.30) into (3.29) yields

$$\begin{aligned} &\sum_{2\le m\le M+1} \|u - \mathscr{I}u\|^2_{H^1(I_m)} \\ &\le C \sum_{2\le m\le M+1} \frac{1}{k_m} \sigma^{2M\theta}\big(\nu(M+1)\big)^{1/2} \le C\big(\nu(M+1)\big)^{1/2} \sum_{2\le m\le M+1} \frac{\sigma^{2M\theta}}{(1-\sigma)\sigma^{M-m+1}} \\ &\le C\big(\nu(M+1)\big)^{1/2}\sigma^{(2\theta-1)M} \sum_{2\le m\le M+1} \sigma^{(m-1)} \le C\big(\nu(M+1)\big)^{1/2}\sigma^{(2\theta-1)M}. \end{aligned} \tag{3.31}$$

Moreover, taking into account standard approximation properties for analytic functions — cf. [26], we have

$$\sum_{2\le \ell \le L} \|u - \mathscr{I}u\|^2_{H^1(J_\ell)} = \|u - \mathscr{I}u\|^2_{H^1(T_1,T)} \le Ce^{-br_\ell} \le Ce^{-b\lfloor \nu(M+1)\rfloor}. \tag{3.32}$$

Finally, combining (3.27), (3.28), (3.31) and (3.32) gives

$$\|u - U\|^2_{H^1(0,T)} \le C\left(\sigma^{M(2\theta-1)} + \big(\nu(M+1)\big)^{1/2}\sigma^{(2\theta-1)M} + e^{-b\lfloor \nu(M+1)\rfloor}\right) \le Ce^{-bM}$$

as $M \to \infty$. Since $D = \dim(S^{\mathbf{r}-1,0}(\mathscr{T}_{M,\sigma})) \le CM^2$ for sufficiently large M, the estimate (3.26) follows. □

4. Numerical Examples

In order to illustrate the performance of the hp-version of C^0-CPG time-stepping method, we apply it to the VIDE (1.1) with

$$p(t)=\frac{t}{1+t},\quad q(t)=1,\quad K(t,s)=e^s$$

and $T=1$. The right-hand side f is chosen so that the corresponding equation (1.1) has the solution $u(t)=t^{3-\alpha}e^{-t}$ with $0\le\alpha<1$. The discrete L^∞-errors are calculated as

$$\|u-U\|_{L^\infty(I)}\approx\max_{1\le n\le N,0\le j\le 20}\left\{\left|u(x_{j,n})-U(x_{j,n})\right|\right\},$$

where $x_{j,n}=t_{n-1}+k_nj/20$ for $0\le j\le 20$.

We start with the situation, when the solution is smooth — i.e. we choose $\alpha=0$ and obtain analytic solution $u=t^3e^{-t}$ on $[0,T]$. Fig. 1 shows the performance of the h-version of C^0-CPG time-stepping method with uniform step-size k and fixed uniform approximation order r. The H^1-errors are plotted against the number of the degrees of freedom in a log-log scale for $r=1,2,3,4$. Note that the convergence of H^1-errors is $\mathscr{O}(k^r)$, consistent with the error estimate (3.24). Besides, Table 1 presents various numerical errors and experimental rates of convergence for different approximation orders r. Note that $\max e(t_n):=\max\limits_{1\le n\le N}|(u-U)(t_n)|$ is the maximum nodal error and $\max e'(t_n^-)=\max\limits_{1\le n\le N}|(u-U)'(t_n^-)|$ is the maximum nodal error of the derivative. Table 1 clearly shows the convergence orders

$$\|u-U\|_{L^2(I)}=\mathscr{O}\big(k^{r+1}\big),\quad \|u-U\|_{H^1(I)}=\mathscr{O}\big(k^{r}\big),\quad \|u-U\|_{L^\infty(I)}=\mathscr{O}\big(k^{r+1}\big),$$

and the superconvergence orders

$$\max_{1\le n\le N}|(u-U)(t_n)|=\mathscr{O}\big(k^{2r-1}\big),$$
$$\max_{1\le n\le N}\left|(u-U)'(t_n^-)\right|=\mathscr{O}\big(k^{2r-1}\big)$$

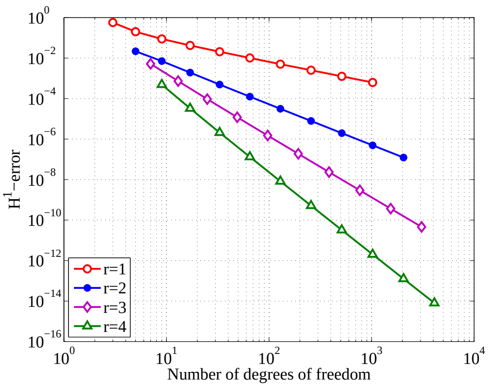

Figure 1: H^1-errors of the h-version, $\alpha=0$.

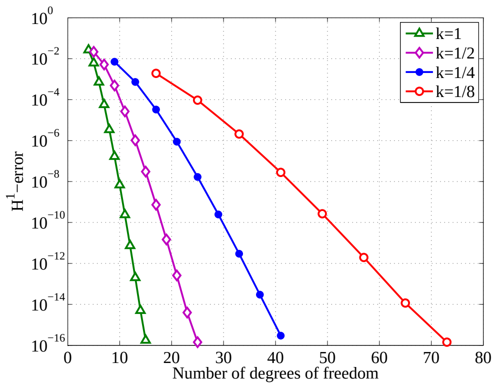

Figure 2: H^1-errors of the p-version, $\alpha=0$.

Table 1: Numerical errors and convergence orders of the h-version, $\alpha = 0$.

r	k	$\|e\|_{L^2(I)}$	order	$\|e\|_{H^1(I)}$	order	$\|e\|_{L^\infty(I)}$	order	$\max e(t_n)$	order	$\max e'(t_n^-)$	order
1	1/64	4.25E-03	1.04	1.01E-02	1.04	8.13E-03	1.04	8.13E-03	1.04	8.04E-03	1.04
	1/128	2.11E-03	1.02	5.04E-03	1.02	4.04E-03	1.02	4.04E-03	1.02	3.99E-03	1.02
	1/256	1.05E-03	1.01	2.51E-03	1.01	2.01E-03	1.01	2.01E-03	1.01	1.99E-03	1.01
2	1/32	1.01E-06	3.05	1.25E-04	2.03	4.19E-06	2.96	1.78E-07	2.96	3.98E-07	2.99
	1/64	1.27E-07	3.02	3.14E-05	2.02	5.44E-07	2.98	2.31E-08	2.98	5.10E-08	3.00
	1/128	1.60E-08	3.01	7.86E-06	2.01	6.92E-08	2.99	2.94E-09	2.99	6.46E-09	3.00
3	1/16	1.01E-07	4.10	1.20E-05	3.07	3.80E-07	3.99	1.04E-09	5.15	1.76E-09	5.15
	1/32	6.38E-09	4.05	1.50E-06	3.04	2.48E-08	4.00	3.21E-11	5.09	5.46E-11	5.09
	1/64	4.00E-10	4.03	1.88E-07	3.02	1.58E-09	4.00	9.97E-13	5.05	1.70E-12	5.05
4	1/8	2.48E-08	5.19	2.09E-06	4.15	7.68E-08	5.02	7.04E-11	7.25	1.19E-10	7.26
	1/16	7.82E-10	5.10	1.31E-07	4.08	2.57E-09	5.01	5.54E-13	7.15	9.36E-13	7.15
	1/32	2.45E-11	5.05	8.24E-09	4.04	8.30E-11	5.01	4.50E-15	7.02	7.77E-15	6.99

for $r > 1$. For $r = 1$, the convergence rates of the errors in all norms are of order 1. A theoretical explanation of these observations (except the errors in H^1-norm) remains an open problem in our context.

We next test the performance of the p-version C^0-CPG time-stepping method on uniform partitions with fixed uniform step-size k. In Fig. 2, we increase the polynomial degree r and plot the H^1-errors against the number of degrees of freedom in a linear-log scale. It can be seen that for any fixed time partition, an exponential rate of convergence is achieved, consistent with Remark 3.3. We also observe that in p-version, the global H^1-errors of 10^{-15} can be reached for at most 15 degrees of freedom. However, this is not the case for h-version, as shown in Fig. 1. Therefore, for smooth solutions, it is more advantageous to use p-refinement rather than h-refinement.

Consider next the situation with $\alpha = 0.5$, when the solution $u = t^{2.5}e^{-t}$ has a singularity at $t = 0$. The third-order derivative of u is unbounded near $t = 0$ and $u \in H^{3-\varepsilon}(I)$ for any $\varepsilon > 0$. Fig. 3 shows the performance of the h-version of the method on uniform partitions. The H^1-errors of the h-version method against the degrees of freedom are plotted in the log-log scale. The optimal order $r + 1$ is not obtained due to the loss of smoothness of u at $t = 0$. Fig. 4 shows that the p-version method also achieves the algebraic rates of convergence, although performing slightly better than the h-version method.

We now consider the performance of the hp-version of C^0-CPG time-stepping method based on geometrically refined time steps and linearly increasing degree of polynomials. In Fig. 5, we plot the H^1-errors against the square root of the number of degrees of freedom for fixed slope $\nu = 2.5$ and various grading factors σ. Each curve exhibits the exponential convergence rate, confirming Theorem 3.3. In Fig. 6, we plot the H^1-errors for $\sigma = 0.1$ and various slopes ν. Note that exponential rates of convergence are achieved for each ν. However, the optimal choice of the parameters σ and ν remains an open problem in our context.

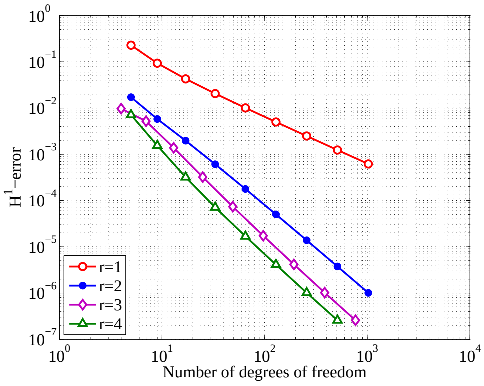

Figure 3: H^1-errors of the h-version, $\alpha = 0.5$.

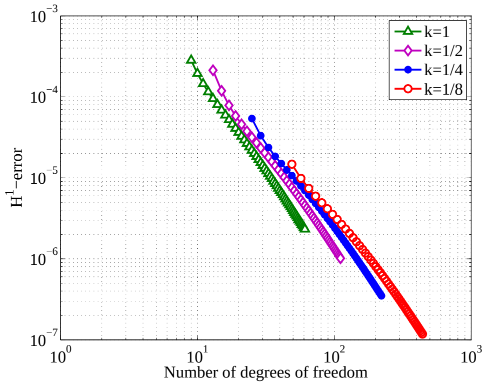

Figure 4: H^1-errors of the p-version, $\alpha = 0.5$.

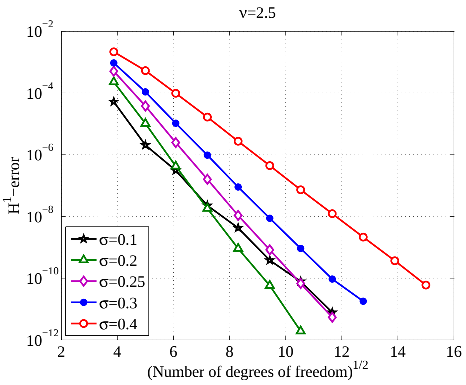

Figure 5: H^1-errors of the hp-version with fixed ν, $\alpha = 0.5$.

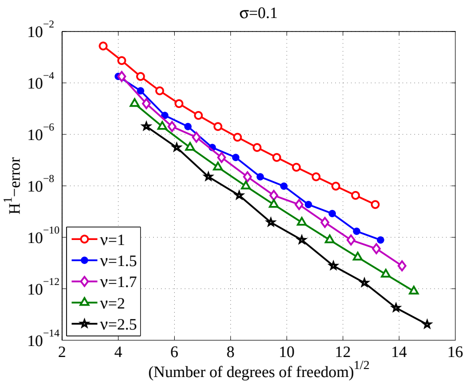

Figure 6: H^1-errors of the hp-version with fixed σ, $\alpha = 0.5$.

5. Concluding Remarks

In this work we study an hp-version of C^0-CPG time-stepping method for the second-order VIDEs with weakly singular kernels. In contrast to the methods transforming second-order problems into the first-order systems, here we combine the CG and DG methodologies thus obtaining a direct discretisation of the second-order derivative. We derive an a priori error estimate in the H^1-norm fully explicit with respect to the local discretisation and regularity parameters. For analytic solutions with start-up singularities, we prove that exponential rates of convergence can be attained by using geometrically refined time steps and linearly increasing approximation orders.

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