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Convergence Rates of Split-Step Theta Methods for SDEs with Non-Globally Lipschitz Diffusion Coefficients

Xiaojuan Wu and Siqing Gan*

School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, Hunan, China.

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Abstract. The present work analyzes the mean-square approximation error of split-step theta methods in a non-globally Lipschitz regime. We show that under a coupled monotonicity condition and polynomial growth conditions, the considered methods with the parameters $\theta \in [1/2, 1]$ have convergence rate of order 1/2. This covers a class of stochastic differential equations with super-linearly growing diffusion coefficients such as the popular 3/2-model in finance. Numerical examples support the theoretical results.

AMS subject classifications: 60H35, 60H15, 65C30

Key words: Stochastic differential equation, non-globally Lipschitz coefficient, split-step theta method, strong convergence rate.

1. Introduction

Stochastic differential equations (SDEs) play an important role in various fields of natural and social sciences. However, most of SDEs can not be solved analytically, so that numerical simulations become a vital tool for understanding SDE models. Various numerical schemes are developed, with strong and weak approximation errors well studied under the classical conditions that the coefficients of SDEs are globally Lipschitz continuous [21, 30]. However, since the majority of nonlinear SDEs arising in applications have super-linearly growing coefficients, the study of their numerical approximations is a non-trivial task. As is shown in [16], for a large class of SDEs with super-linearly growing coefficients the popularly used Euler-Maruyama (EM) method can produce numerical solutions with divergent moment bounds as the time step-size tends to zero. This results in strong and weak divergence of the numerical approximations. Such observations can be also found in the early reference [11, Section 3], where a motivating example was given. Note that a large number of works devoted to the numerical analysis of SDEs under non-globally Lipschitz conditions makes an emphasis on implicit schemes — cf. [1–3, 9, 10, 12, 13, 19, 23, 26, 27, 34, 36], and on

*Corresponding author. *Email addresses:* `wuxiaojuan@csu.edu.cn` (X. Wu), `sqgan@csu.edu.cn` (S. Gan)

developing approximation methods based on modifications of traditional explicit schemes — cf. [4–7, 14, 15, 17, 18, 20, 22, 24, 25, 29, 31–33, 35, 39], to just mention a few.

This work is concerned with a kind of split-step implicit schemes for SDEs with non-globally Lipschitz coefficients, where the drift and diffusion coefficients are assumed to obey the coupled monotonicity condition (2.3). This setting allows super-linearly growing diffusion coefficients and covers the popular 3/2-model in finance. The schemes under consideration are called the split-step theta (SST) methods — cf. the Eqs. (2.2) below. They have been introduced by Huang [13], where the exponential mean square stability of SST methods and the usual stochastic theta methods (STMs), was examined under the monotonicity condition (2.3). In particular, it was shown that the SST methods with $\theta > 1/2$ have better nonlinear stability properties than the STMs do. The SST methods extend the split-step backward Euler (SSBE) method proposed by Mattingly *et al.* [28], where the ergodicity of SDEs with locally Lipschitz coefficients and their approximations have been studied. They showed that the explicit EM method does not inherit the geometric ergodicity of such SDEs while the SSBE scheme was able to reproduce the ergodicity. The strong convergence rate of the SSBE scheme was first established in [9] for SDEs with non-globally Lipschitz drift but globally Lipschitz diffusion coefficients. Similar strong convergence results are derived in [38] for the SST methods with $\theta \in [1/2, 1]$ and in [19] for semi-implicit split-step numerical methods and globally Lipschitz continuous diffusion coefficients. If the diffusion coefficients can grow super-linearly, some efforts have been made to prove the strong convergence rate of split-step type methods. Thus Liu *et al.* [23] proposed a family of split-step balanced θ-methods for SDEs with non-globally Lipschitz continuous coefficients. Using the fundamental strong convergence theorem [33], they obtained the desired strong convergence rate. Besides, using the notions of stochastic C-stability and stochastic B-consistency, Andersson and Kruse [2] obtained the mean-square convergence rate of the SSBE scheme under the coupled monotonicity condition (2.3) for non-globally Lipschitz diffusion coefficients. However, to the best of the authors knowledge, in the case of non-globally Lipschitz diffusion coefficients, the convergence rates of general SST methods (2.2) with $\theta \in [1/2, 1]$ has not been studied. As pointed out in [9, p. 1060], the split-step implicit method with $\theta = 1/2$, may be of practical interest for Hamiltonian problems perturbed by damping and/or noise.

Motivated by the above results, we study the mean-square error of the general SST methods for SDEs with possibly super-linearly growing diffusion coefficients. In particular, we show that SST methods with $\theta \in [1/2, 1]$ converge with the rate 1/2 under a coupled monotonicity condition and polynomial growth conditions. This setting covers a class of SDEs with super-linearly growing diffusion coefficients including the popular 3/2-model in finance. Wang *et al.* [36] proposed a new approach to the mean-square error analysis for STMs. It does not require a priori high-order moment estimates of numerical approximations and allows to recover mean-square convergence rates of STMs with $\theta \in [1/2, 1]$ under the coupled monotonicity condition (2.3).

The present article extends the ideas of [36] to general SST methods with $\theta \in [1/2, 1]$. Unlike [36], we have to introduce an auxiliary process $\widetilde{X}(t_n)$ and develop a new techniques in the error analysis — cf. the proof of Theorem 3.1 and comments at the end of Section 3.

Finally, we note that the fundamental strong convergence theorem in [33] for one-step approximations in a non-globally Lipschitz setting requires a priori high-order moment estimates of numerical approximations. However, we have not used this result because we were not able to derive high-order moment estimates for general SST methods (2.2) with $\theta \in [1/2, 1]$ in the non-globally Lipschitz setting.

The remainder of this paper is organized as follows. Section 2 presents a numerical scheme and formulates some assumptions. In Section 3, the expected rate of mean-square convergence is established. Numerical simulations in Section 4 verify the theoretical results.

2. SST Methods

Let $\mathbb{N}$ denote the set of all positive integers and $d, m \in \mathbb{N}$. If $x, y \in \mathbb{R}^d$, then $|x|$ refers to the Euclidean norm of x and $\langle x, y\rangle$ the inner product of x and y. Besides, if A is an $d \times m$ matrix, then $|A| := \sqrt{\text{trace}(A^T A)}$ denotes the trace norm of A. In addition, for a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$, let $\mathbb{E}$ and $L^r(\Omega;\mathbb{R}^d), r \geq 1$ respectively denote the expectation and the Banach space consisting of $\mathbb{R}^d$-valued random variables ξ such that

$$\|\xi\|_{L^r(\Omega;\mathbb{R}^d)} := \big(\mathbb{E}\big[|\xi|^r\big]\big)^{\frac{1}{r}} < \infty.$$

By C we denote generic h-independent deterministic positive constants that may take different values at different occasions.

The present work is concerned with the mean-square approximations of the following Itô SDEs:

$$\mathrm{d}X(t) = f\big(t, X(t)\big)\,\mathrm{d}t + g\big(t, X(t)\big)\,\mathrm{d}W(t), \quad t \in (0, T], \quad X(0) = X_0, \tag{2.1}$$

where $W : [0, T] \times \Omega \to \mathbb{R}^m$ is an $\mathbb{R}^m$-valued standard Brownian motion with respect to $\{\mathscr{F}_t\}_{t\in[0,T]}$. Besides, we assume that the initial data $X_0 : \Omega \to \mathbb{R}^d$ are $\mathscr{F}_0$-measurable and the drift coefficient $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ satisfy the coupled monotonicity condition — cf. (2.3).

Note that in general, analytical solutions of SDEs (2.1) are not known, so that numerical methods have to be used. In order to approximate (2.1), we construct uniform meshes $t_i = ih$, $h = T/N$, $i = 1, 2, \ldots, N$ and consider split-step theta methods (SSTMs), given by $Y_0 = X_0$, i.e.

$$Z_n = Y_n + \theta h f(t_n + \theta h, Z_n), \tag{2.2a}$$

$$Y_{n+1} = Y_n + h f(t_n + \theta h, Z_n) + g(t_n + \theta h, Z_n)\Delta W_n, \tag{2.2b}$$

where

$$\Delta W_n := W(t_{n+1}) - W(t_n), \quad n = 0, 1, \ldots, N - 1.$$

If $\theta = 0$, this scheme becomes the popular EM method and if or $\theta = 1$, this is the split-step backward Euler method [28].

Assumption 2.1 (Monotonicity Condition)**.** There are non-negative constants $q \in (2, \infty)$ and $L \in [0, \infty)$ such that

$$\begin{aligned}&\left\langle x-y, f(t,x)-f(t,y)\right\rangle + \frac{q-1}{2}|g(t,x)-g(t,y)|^2 \\ &\le L|x-y|^2, \quad t \in [0,T], \quad x,y \in \mathbb{R}^d,\end{aligned} \tag{2.3}$$

where f and g are the drift and diffusion coefficients of the SDEs (2.1), respectively.

The monotonicity condition (2.3) guarantees that if $h \in (0, 1/(\theta L))$, then the implicit methods ($\theta > 0$) (2.2) admit unique $\mathscr{F}_{t_n}$-adapted solutions Z_n, Y_n in $\mathbb{R}^d$, cf. [2, Corollary 4.2].

We also need another condition, related to polynomial growth and coercivity of the drift and diffusion coefficients.

Assumption 2.2. There are positive constants $\gamma \in [1, \infty)$, $\nu \in (0, \infty)$, and $p_0 \in [4\gamma - 2, \infty)$ such that

$$\begin{aligned}&|f(t,x)-f(s,x)| + |g(t,x)-g(s,x)| \\ &\le C\big(1+|x|^\gamma\big)|t-s|^{\frac{1}{2}}, && t,s \in [0,T], \quad x \in \mathbb{R}^d, && (2.4)\\ &|f(t,x)-f(t,y)| \le C\big(1+|x|^{\gamma-1}+|y|^{\gamma-1}\big)|x-y|, && t \in [0,T], \quad x,y \in \mathbb{R}^d, && (2.5)\\ &\left\langle x, f(t,x)\right\rangle + \frac{p_0-1}{2}|g(t,x)|^2 \le \nu\big(1+|x|^2\big), && t \in [0,T], \quad x \in \mathbb{R}^d. && (2.6)\end{aligned}$$

Moreover, we also assume that X_0 is $\mathscr{F}_0$-measurable and $\|X_0\|_{L^{p_0}(\Omega;\mathbb{R}^d)} < \infty$.

Using (2.3) and (2.5), one can show that for any $t \in [0,T]$ and $x, y \in \mathbb{R}^d$ the following estimates holds:

$$|f(t,x)| \le C\big(1+|x|^\gamma\big), \quad |g(t,x)| \le C\big(1+|x|^{\frac{\gamma+1}{2}}\big), \tag{2.7}$$

and

$$\begin{aligned}|g(t,x)-g(t,y)|^2 &\le \frac{2L}{q-1}|x-y|^2 + \frac{2}{q-1}|x-y||f(t,x)-f(t,y)| \\ &\le C\big(1+|x|^{\gamma-1}+|y|^{\gamma-1}\big)|x-y|^2.\end{aligned} \tag{2.8}$$

Assumptions 2.1 and 2.2 ensure that the SDE (2.1) possesses a unique adapted solution $X : [0,T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths such that for any $p \in [2, p_0]$, one has

$$\sup_{t\in[0,T]} \|X(t)\|_{L^p(\Omega;\mathbb{R}^d)} < \infty. \tag{2.9}$$

It follows from (2.7) and (2.9) that for any $t, s \in [0,T]$ the following estimate holds:

$$\|X(t)-X(s)\|_{L^\delta(\Omega;\mathbb{R}^d)} \le C|t-s|^{\frac{1}{2}}, \quad \delta \in [1, p_0/\gamma]. \tag{2.10}$$

One can consult [2, Section 2, Proposition 5.4] for the above assertions.

3. Convergence Rates

This section is devoted to the mean-square convergence of SST schemes for $\theta \in [1/2, 1]$. First, we consider an auxiliary process $\widetilde{X}(t_n)$, $n = 0, 1, \dots, N$ defined by

$$\widetilde{X}(t_n) = X(t_n) + \theta h f\big(t_n + \theta h, \widetilde{X}(t_n)\big). \tag{3.1}$$

It is easily checked that since f satisfies the monotonicity condition (2.3), for $h\theta L < 1$ the process $\widetilde{X}(t_n) \in \mathscr{F}_{t_n}$ uniquely determined by (3.1) is well-posed — cf. [2, Theorem 4.1]. Besides, $\widetilde{X}(0) = Z_0$.

Further, we have the following moment estimates for $\widetilde{X}(t_n)$.

Lemma 3.1. *Suppose that $\theta \in [0, 1]$, $2\theta\nu h \le \varrho$ for a $\varrho < 1$, Assumptions 2.1, 2.2 hold, and $\widetilde{X}(t_n)$, $n = 0, 1, \dots, N$ is the process defined by (3.1). Then*

$$\big|\widetilde{X}(t_n)\big|^2 \le \frac{1}{1-\varrho}|X(t_n)|^2 + \frac{2\theta\nu h}{1-\varrho}. \tag{3.2}$$

Proof. Using (2.6), we write

$$\begin{aligned}
|X(t_n)|^2 &= \big|\widetilde{X}(t_n) - \theta h f\big(t_n + \theta h, \widetilde{X}(t_n)\big)\big|^2 \\
&\ge \big|\widetilde{X}(t_n)\big|^2 - 2\theta h \big\langle \widetilde{X}(t_n), f\big(t_n + \theta h, \widetilde{X}(t_n)\big)\big\rangle \\
&\ge (1 - 2\theta\nu h)\big|\widetilde{X}(t_n)\big|^2 - 2\theta\nu h,
\end{aligned}$$

and the proof is complete. □

For $j = 1, 2, \dots, N$ we set

$$\begin{aligned}
\mathscr{R}_j := &\int_{t_{j-1}}^{t_j} f\big(s, X(s)\big) - f\big(t_{j-1} + \theta h, \widetilde{X}(t_{j-1})\big)\,\mathrm{d}s \\
&+ \int_{t_{j-1}}^{t_j} g\big(s, X(s)\big) - g\big(t_{j-1} + \theta h, \widetilde{X}(t_{j-1})\big)\,\mathrm{d}W(s),
\end{aligned} \tag{3.3}$$

and evaluate the approximation error as follows.

Lemma 3.2. *Let $\theta \in [1/2, 1]$. If Assumptions 2.1 and 2.2 hold, then*

$$\sup_{1\le j\le N} \mathbb{E}\big[|\mathscr{R}_j|^2\big] \le Ch^2, \quad \sup_{1\le j\le N} \mathbb{E}\big[|\mathbb{E}(\mathscr{R}_j|\mathscr{F}_{t_{j-1}})|^2\big] \le Ch^3. \tag{3.4}$$

Proof. The Hölder inequality and the Itô isometry give

$$\begin{aligned}
\mathbb{E}\big[|\mathscr{R}_j|^2\big] \le\, &2h \int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|f\big(s, X(s)\big) - f\big(t_{j-1} + \theta h, \widetilde{X}(t_{j-1})\big)\big|^2\Big]\,\mathrm{d}s \\
&+ 2\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|g\big(s, X(s)\big) - g\big(t_{j-1} + \theta h, \widetilde{X}(t_{j-1})\big)\big|^2\Big]\,\mathrm{d}s \\
=:\, &\mathrm{Err}_1 + \mathrm{Err}_2.
\end{aligned}$$

The term Err_1 can be estimated as

$$\begin{aligned}\mathrm{Err}_1 \le\, & 6h\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|f\big(s,X(s)\big)-f\big(t_{j-1}+\theta h,X(s)\big)\big|^2\Big]\,\mathrm{d}s \\ & + 6h\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|f\big(t_{j-1}+\theta h,X(s)\big)-f\big(t_{j-1}+\theta h,X(t_{j-1})\big)\big|^2\Big]\,\mathrm{d}s \\ & + 6h\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|f\big(t_{j-1}+\theta h,X(t_{j-1})\big)-f\big(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\big)\big|^2\Big]\,\mathrm{d}s \\ =:\, & \mathrm{Err}_{11}+\mathrm{Err}_{12}+\mathrm{Err}_{13}. \end{aligned} \tag{3.5}$$

By the inequality (2.4), we have

$$\mathrm{Err}_{11} \le Ch\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big(1+|X(s)|^{\gamma}\big)^2\Big]\big|t_{j-1}+\theta h-s\big|\,\mathrm{d}s \le Ch^3.$$

Applying the Hölder inequality and the relations (2.5), (2.9), (2.10) yields

$$\begin{aligned} & \mathbb{E}\Big[\big|f\big(t_{j-1}+\theta h,X(s)\big)-f\big(t_{j-1}+\theta h,X(t_{j-1})\big)\big|^2\Big] \\ & \le C\mathbb{E}\Big[\big(1+|X(s)|^{2\gamma-2}+|X(t_{j-1})|^{2\gamma-2}\big)\big|X(s)-X(t_{j-1})\big|^2\Big] \\ & \le C\left(1+\Big(\mathbb{E}\big[|X(s)|^{4\gamma-2}\big]\Big)^{\frac{2\gamma-2}{4\gamma-2}}+\Big(\mathbb{E}\big[|X(t_{j-1})|^{4\gamma-2}\big]\Big)^{\frac{2\gamma-2}{4\gamma-2}}\right) \\ & \quad \times\left(\mathbb{E}\Big[|X(s)-X(t_{j-1})|^{\frac{4\gamma-2}{\gamma}}\Big]\right)^{\frac{2\gamma}{4\gamma-2}} \le Ch, \end{aligned}$$

Therefore, the term Err_{12} can be now estimated as

$$\mathrm{Err}_{12} \le Ch^3.$$

Considering the last term and using (2.5), (2.7), (3.1) and (3.2), we write

$$\begin{aligned} & \mathbb{E}\Big[\big|f\big(t_{j-1}+\theta h,X(t_{j-1})\big)-f\big(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\big)\big|^2\Big] \\ & \le C\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{2\gamma-2}+|\widetilde{X}(t_{j-1})|^{2\gamma-2}\big)\big|X(t_{j-1})-\widetilde{X}(t_{j-1})\big|^2\Big] \\ & \le Ch^2\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{2\gamma-2}+|\widetilde{X}(t_{j-1})|^{2\gamma-2}\big)\big|f\big(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\big)\big|^2\Big] \\ & \le Ch^2\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{2\gamma-2}+|\widetilde{X}(t_{j-1})|^{2\gamma-2}\big)\big(1+|\widetilde{X}(t_{j-1})|^{2\gamma}\big)\Big] \\ & \le Ch^2\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{4\gamma-2}+|\widetilde{X}(t_{j-1})|^{4\gamma-2}\big)\Big] \\ & \le Ch^2\Big(1+\mathbb{E}\big[|X(t_{j-1})|^{4\gamma-2}\big]\Big). \end{aligned}$$

Consequently,

$$\mathrm{Err}_{13} \le Ch^4,$$

and recalling the above estimates gives

$$\mathrm{Err}_1 \leq Ch^3.$$

Similar to the previous considerations we write

$$\begin{aligned}\mathrm{Err}_2 \leq 6 &\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|g\big(s,X(s)\big)-g\big(t_{j-1}+\theta h,X(s)\big)\big|^2\Big]\,\mathrm{d}s\\ &+6\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|g\big(t_{j-1}+\theta h,X(s)\big)-g\big(t_{j-1}+\theta h,X(t_{j-1})\big)\big|^2\Big]\,\mathrm{d}s\\ &+6\int_{t_{j-1}}^{t_j} \mathbb{E}\Big[\big|g\big(t_{j-1}+\theta h,X(t_{j-1})\big)-g\big(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\big)\big|^2\Big]\,\mathrm{d}s\\ =:&\ \mathrm{Err}_{21}+\mathrm{Err}_{22}+\mathrm{Err}_{23}.\end{aligned}$$

The term Err_{21} is estimated in the same way as Err_{11}. In order to bound Err_{22}, we use (2.8)-(2.10), the inequality

$$\frac{8\gamma-4}{3\gamma-1} \leq \frac{4\gamma-2}{\gamma}$$

valid for $\gamma \geq 1$, and the Hölder inequality, thus obtaining

$$\begin{aligned}&\mathbb{E}\Big[\big|g\big(t_{j-1}+\theta h,X(s)\big)-g\big(t_{j-1}+\theta h,X(t_{j-1})\big)\big|^2\Big]\\ &\leq C\mathbb{E}\Big[\big(1+|X(s)|^{\gamma-1}+|X(t_{j-1})|^{\gamma-1}\big)|X(s)-X(t_{j-1})|^2\Big]\\ &\leq C\left(1+\Big(\mathbb{E}\big[|X(s)|^{4\gamma-2}\big]\Big)^{\frac{\gamma-1}{4\gamma-2}}+\Big(\mathbb{E}\big[|X(t_{j-1})|^{4\gamma-2}\big]\Big)^{\frac{\gamma-1}{4\gamma-2}}\right)\\ &\qquad\times\left(\mathbb{E}\Big[|X(s)-X(t_{j-1})|^{\frac{8\gamma-4}{3\gamma-1}}\Big]\right)^{\frac{6\gamma-2}{8\gamma-4}} \leq Ch.\end{aligned}$$

Analogously, we show that

$$\begin{aligned}&\mathbb{E}\Big[\big|g\left(t_{j-1}+\theta h,X(t_{j-1})\right)-g\left(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\right)\big|^2\Big]\\ &\leq C\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{\gamma-1}+|\widetilde{X}(t_{j-1})|^{\gamma-1}\big)\big|X(t_{j-1})-\widetilde{X}(t_{j-1})\big|^2\Big]\\ &\leq Ch^2\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{\gamma-1}+|\widetilde{X}(t_{j-1})|^{\gamma-1}\big)\big|f\left(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\right)\big|^2\Big]\\ &\leq Ch^2\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{\gamma-1}+|\widetilde{X}(t_{j-1})|^{\gamma-1}\big)\big(1+|\widetilde{X}(t_{j-1})|^{2\gamma}\big)\Big]\\ &\leq Ch^2\mathbb{E}\Big[\big(1+|X(t_{j-1})|^{3\gamma-1}+|\widetilde{X}(t_{j-1})|^{3\gamma-1}\big)\Big]\\ &\leq Ch^2\Big(1+\mathbb{E}\big[|X(t_{j-1})|^{3\gamma-1}\big]\Big).\end{aligned}$$

Combining the corresponding estimates gives

$$\mathrm{Err}_2 \leq Ch^2.$$

Noting the stochastic integral vanishes under the conditional expectation and recalling estimates for Err_1 as shown in (3.5), we get

$$\begin{aligned}\mathbb{E}\Big[\big|\mathbb{E}\big(\mathscr{R}_j|\mathscr{F}_{t_{j-1}}\big)\big|^2\Big]&=\mathbb{E}\left[\left|\mathbb{E}\left(\int_{t_{j-1}}^{t_j} f\big(s,X(s)\big)-f\big(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\big)\,\mathrm{d}s\Big|\mathscr{F}_{t_{j-1}}\right)\right|^2\right]\\&\le\mathbb{E}\left[\left|\int_{t_{j-1}}^{t_j} f\big(s,X(s)\big)-f\big(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\big)\,\mathrm{d}s\right|^2\right]\\&\le h\int_{t_{j-1}}^{t_j}\mathbb{E}\Big[\big|f\big(s,X(s)\big)-f\big(t_{j-1}+\theta h,\widetilde{X}(t_{j-1})\big)\big|^2\Big]\,\mathrm{d}s\le Ch^3.\end{aligned}$$

The proof is complete. □

Theorem 3.1. *Let* $\theta\in[1/2,1]$, $\varrho<1$, *and Assumptions* 2.1-2.2 *hold. If*

$$h\in\big(0,\varrho/\big(2\theta\max(L,\nu)\big)\big],$$

then there is a constant C *independent of* n*, such that*

$$\mathbb{E}\big[|X(t_n)-Y_n|^2\big]\le Ch,\quad n\in\{1,2,\dots,N\},\quad N\in\mathbb{N}. \tag{3.6}$$

Proof. It follows from (3.3) that

$$X(t_{n+1})=X(t_n)+hf\big(t_n+\theta h,\widetilde{X}(t_n)\big)+g\big(t_n+\theta h,\widetilde{X}(t_n)\big)\Delta W_n+\mathscr{R}_{n+1}. \tag{3.7}$$

Setting

$$\begin{aligned}&e_n:=X(t_n)-Y_n,\\&\delta f_{\widetilde{X}(t_n),Z_n}:=f\big(t_n+\theta h,\widetilde{X}(t_n)\big)-f(t_n+\theta h,Z_n),\\&\delta g_{\widetilde{X}(t_n),Z_n}:=g\big(t_n+\theta h,\widetilde{X}(t_n)\big)-g(t_n+\theta h,Z_n),\end{aligned}$$

and subtracting the Eq. (2.2a) from the Eq. (3.1) gives

$$\begin{aligned}\widetilde{X}(t_n)-Z_n&=e_n+\theta h\big[f\big(t_n+\theta h,\widetilde{X}(t_n)\big)-f(t_n+\theta h,Z_n)\big]\\&=e_n+\theta h\delta f_{\widetilde{X}(t_n),Z_n}.\end{aligned} \tag{3.8}$$

The monotonicity condition (2.3) yields

$$\begin{aligned}|e_n|^2&\ge\big|\widetilde{X}(t_n)-Z_n\big|^2-2\theta h\big\langle\widetilde{X}(t_n)-Z_n,\delta f_{\widetilde{X}(t_n),Z_n}\big\rangle\\&\ge\big(1-2\theta Lh\big)\big|\widetilde{X}(t_n)-Z_n\big|^2,\end{aligned}$$

so that

$$\big|\widetilde{X}(t_n)-Z_n\big|^2\le\frac{1}{1-2\theta Lh}|e_n|^2. \tag{3.9}$$

Further, subtracting the Eq. (2.2b) from (3.7) and using the above notation we get

$$e_{n+1} = e_n + h\delta f_{\widetilde{X}(t_n),Z_n} + \delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n + \mathscr{R}_{n+1}. \tag{3.10}$$

Consequently,

$$\begin{aligned}|e_{n+1}|^2 = & |e_n|^2 + h^2|\delta f_{\widetilde{X}(t_n),Z_n}|^2 + |\delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n|^2 + |\mathscr{R}_{n+1}|^2 \\ & + 2h\big\langle e_n, \delta f_{\widetilde{X}(t_n),Z_n}\big\rangle + 2\big\langle e_n, \delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n\big\rangle + 2\big\langle e_n, \mathscr{R}_{n+1}\big\rangle \\ & + 2h\big\langle \delta f_{\widetilde{X}(t_n),Z_n}, \delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n\big\rangle + 2h\big\langle \delta f_{\widetilde{X}(t_n),Z_n}, \mathscr{R}_{n+1}\big\rangle \\ & + 2\big\langle \delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n, \mathscr{R}_{n+1}\big\rangle. \end{aligned} \tag{3.11}$$

Taking into account the relation (3.8), we write

$$\begin{aligned} 2h\langle e_n, \delta f_{\widetilde{X}(t_n),Z_n}\rangle &= 2h\big\langle \widetilde{X}(t_n) - Z_n - \theta h \delta f_{\widetilde{X}(t_n),Z_n}, \delta f_{\widetilde{X}(t_n),Z_n}\big\rangle \\ &= 2h\big\langle \widetilde{X}(t_n) - Z_n, \delta f_{\widetilde{X}(t_n),Z_n}\big\rangle - 2\theta h^2\big|\delta f_{\widetilde{X}(t_n),Z_n}\big|^2, \\ 2h\langle \delta f_{\widetilde{X}(t_n),Z_n}, \mathscr{R}_{n+1}\rangle &= \frac{2}{\theta}\big\langle \widetilde{X}(t_n) - Z_n - e_n, \mathscr{R}_{n+1}\big\rangle. \end{aligned}$$

Substituting the above equation into (3.11) gives

$$\begin{aligned}|e_{n+1}|^2 = & |e_n|^2 + (1-2\theta)h^2|\delta f_{\widetilde{X}(t_n),Z_n}|^2 + |\delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n|^2 + |\mathscr{R}_{n+1}|^2 \\ & + 2h\big\langle \widetilde{X}(t_n) - Z_n, \delta f_{\widetilde{X}(t_n),Z_n}\big\rangle + 2\big\langle e_n, \delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n\big\rangle + 2\langle e_n, \mathscr{R}_{n+1}\rangle \\ & + 2h\big\langle \delta f_{\widetilde{X}(t_n),Z_n}, \delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n\big\rangle + \frac{2}{\theta}\big\langle \widetilde{X}(t_n) - Z_n, \mathscr{R}_{n+1}\big\rangle - \frac{2}{\theta}\langle e_n, \mathscr{R}_{n+1}\rangle \\ & + 2\big\langle \delta g_{\widetilde{X}(t_n),Z_n}\Delta W_n, \mathscr{R}_{n+1}\big\rangle. \end{aligned}$$

In order to proceed, we need the following auxiliary result:

$$\mathbb{E}\big[|e_n|^2\big] < \infty, \quad n = 0, 1, \ldots, N. \tag{3.12}$$

It is proved by the mathematical induction. Noting that

$$\mathbb{E}\big[|e_0|^2\big] = \mathbb{E}\big[|X_0 - Y_0|^2\big] = 0,$$

we assume that $\mathbb{E}[|e_k|^2] < \infty$ for $k = 0, 1, \ldots, N-1$. The relations (3.9) and (3.8) yield

$$\mathbb{E}\Big[\big|\widetilde{X}(t_k) - Z_k\big|^2\Big] < \infty \tag{3.13}$$

and

$$\mathbb{E}\Big[\big|\delta f_{\widetilde{X}(t_k),Z_k}\big|^2\Big] = \frac{1}{\theta^2 h^2}\mathbb{E}\Big[\big|\widetilde{X}(t_k) - Z_k - e_k\big|^2\Big] < \infty,$$

respectively. Using these relations and the monotonicity condition (2.3), we write

$$\begin{aligned}\mathbb{E}\Big[\big|\delta g_{\widetilde{X}(t_k),Z_k}\big|^2\Big] &\leq \frac{2L}{q-1}\mathbb{E}\Big[\big|\widetilde{X}(t_k) - Z_k\big|^2\Big] - \frac{2}{q-1}\mathbb{E}\Big[\big\langle \widetilde{X}(t_k) - Z_k, \delta f_{\widetilde{X}(t_k),Z_k}\big\rangle\Big] \\ &\leq \frac{2L+1}{q-1}\mathbb{E}\Big[\big|\widetilde{X}(t_k) - Z_k\big|^2\Big] + \frac{1}{q-1}\mathbb{E}\Big[\big|\delta f_{\widetilde{X}(t_k),Z_k}\big|^2\Big] < \infty. \end{aligned} \tag{3.14}$$

Therefore,

$$\mathbb{E}\Big[\big\langle e_k, \delta g_{\widetilde{X}(t_k),Z_k}\Delta W_k\big\rangle\Big] = 0,$$
$$\mathbb{E}\Big[\big\langle \delta f_{\widetilde{X}(t_k),Z_k}, \delta g_{\widetilde{X}(t_k),Z_k}\Delta W_k\big\rangle\Big] = 0$$

and

$$\begin{aligned}\mathbb{E}\big[|e_{k+1}|^2\big] = {} & \mathbb{E}\big[|e_k|^2\big] + (1-2\theta)h^2\mathbb{E}\Big[|\delta f_{\widetilde{X}(t_k),Z_k}|^2\Big] + h\mathbb{E}\Big[|\delta g_{\widetilde{X}(t_k),Z_k}|^2\Big] \\ & + \mathbb{E}\big[|\mathscr{R}_{k+1}|^2\big] + 2h\mathbb{E}\Big[\big\langle \widetilde{X}(t_k) - Z_k, \delta f_{\widetilde{X}(t_k),Z_k}\big\rangle\Big] \\ & + \left(2 - \frac{2}{\theta}\right)\mathbb{E}\Big[\big\langle e_k, \mathbb{E}\big[\mathscr{R}_{k+1}|\mathscr{F}_{t_k}\big]\big\rangle\Big] \\ & + \frac{2}{\theta}\mathbb{E}\Big[\big\langle \widetilde{X}(t_k) - Z_k, \mathbb{E}\big[\mathscr{R}_{k+1}|\mathscr{F}_{t_k}\big]\big\rangle\Big] \\ & + 2\mathbb{E}\Big[\big\langle \delta g_{\widetilde{X}(t_k),Z_k}\Delta W_k, \mathscr{R}_{k+1}\big\rangle\Big] < \infty, \end{aligned} \tag{3.15}$$

where we also used the estimate (3.4) in order to ensure that

$$\mathbb{E}\big[|\mathscr{R}_{k+1}|^2\big] < \infty, \quad \mathbb{E}\big[|\mathbb{E}(\mathscr{R}_{k+1}|\mathscr{F}_{t_k})|^2\big] < \infty.$$

By induction arguments, the estimates (3.12) hold for all $n = 0, 1, \ldots, N$. Therefore, (3.13)-(3.14) are valid for any $k = 0, 1, \ldots, N$ and (3.15) is valid for any $k = 0, 1, \ldots, N-1$. Since $\theta \geq 1/2$, we can use the Cauchy-Schwarz inequality and the monotonicity condition (2.3), thus obtaining

$$\begin{aligned}\mathbb{E}[|e_{n+1}|^2] \leq {} & (1+h)\mathbb{E}\big[|e_n|^2\big] + h(q-1)\mathbb{E}\Big[|\delta g_{\widetilde{X}(t_n),Z_n}|^2\Big] \\ & + \frac{q-1}{q-2}\mathbb{E}\big[|\mathscr{R}_{n+1}|^2\big] + 2h\mathbb{E}\Big[\big\langle \widetilde{X}(t_n) - Z_n, \delta f_{\widetilde{X}(t_n),Z_n}\big\rangle\Big] \\ & + \frac{3}{h}\mathbb{E}\Big[\big|\mathbb{E}\big[\mathscr{R}_{n+1}|\mathscr{F}_{t_n}\big]\big|^2\Big] + 2h\mathbb{E}\Big[\big|\widetilde{X}(t_n) - Z_n\big|^2\Big] \\ \leq {} & (1+h)\mathbb{E}\big[|e_n|^2\big] + \frac{q-1}{q-2}\mathbb{E}\big[|\mathscr{R}_{n+1}|^2\big] + \frac{3}{h}\mathbb{E}\Big[\big|\mathbb{E}\big[\mathscr{R}_{n+1}|\mathscr{F}_{t_n}\big]\big|^2\Big] \\ & + 2h(L+1)\mathbb{E}\Big[\big|\widetilde{X}(t_n) - Z_n\big|^2\Big], \quad \forall n = 0, 1, \ldots, N-1.\end{aligned}$$

Taking into account (3.9), we show that

$$\begin{aligned}\mathbb{E}\big[|e_{n+1}|^2\big] \leq {} & \left(1 + h + \frac{2(L+1)}{1-2\theta Lh}h\right)\mathbb{E}\big[|e_n|^2\big] + \frac{q-1}{q-2}\mathbb{E}\big[|\mathscr{R}_{n+1}|^2\big] \\ & + \frac{3}{h}\mathbb{E}\Big[\big|\mathbb{E}\big[\mathscr{R}_{n+1}|\mathscr{F}_{t_n}\big]\big|^2\Big].\end{aligned}$$

Since

$$e_0 = 0, \quad \frac{1}{1-2\theta Lh} \leq \frac{1}{1-\varrho},$$

the iterations lead to the estimate

$$\mathbb{E}\left[|e_n|^2\right] \le C\left(\sum_{j=1}^{n}\mathbb{E}\left[|\mathscr{R}_j|^2\right] + h^{-1}\sum_{j=1}^{n}\mathbb{E}\left[\left|\mathbb{E}(\mathscr{R}_j|\mathscr{F}_{t_{j-1}})\right|^2\right]\right). \tag{3.16}$$

Combining (3.16) and (3.4), we arrive at (3.6). □

Let us point out that our approach to the error analysis is inspired by the work [36], where the convergence rates of stochastic theta methods are established. However, here we consider an auxiliary process $\widetilde{X}(t_n)$ and employ a new technique, since the error remainders $\mathscr{R}_j$ involve $X(s)$, $\widetilde{X}(t_{j-1})$ and are more complicated than the respective remainders in [36]. Moreover, the error equation $e_n := X(t_n) - Y_n$ in (3.10) depends on $\widetilde{X}(t_n)$, Z_n, instead of depending on $X(t_n), Y_n$.

In addition, our approach can be extended to split-step implicit Milstein-type schemes [37]. To briefly illustrate the results in [37], we take $d = m = 1$. In this setting, the work considers the following split-step theta Milstein methods with $Y_0 = X_0$ and a parameter $\theta \in [0,1]$:

$$\begin{aligned} Z_n &= Y_n + \theta h f(t_n + \theta h, Z_n),\\ Y_{n+1} &= Y_n + h f(t_n + \theta h, Z_n) + g(t_n + \theta h, Z_n)\Delta W_n\\ &\quad + \frac{1}{2} g' g\big(t_n + \theta h, Z_n\big)\big(\Delta W_n^2 - h\big), \end{aligned}$$

where

$$\Delta W_n := W(t_{n+1}) - W(t_n), \quad n = 0, 1, \ldots, N-1.$$

We choose $\theta \in [0,1]$ and assume that there are constants $q \in (2,\infty)$, $\zeta \in (1,\infty)$, $L_0, L_1 \in (0,\infty)$ and $h_0 \in (0,T]$ such that for all $x, y \in \mathbb{R}^d$, $h \in (0,h_0)$ the following inequalities hold:

$$\begin{aligned} &\big\langle x - y, f(t,x) - f(t,y)\big\rangle \le L_0|x-y|^2,\\ &2\big\langle x - y, f(t,x) - f(t,y)\big\rangle + (q-1)|g(t,x) - g(t,y)|^2\\ &\quad + (1-2\theta)h|f(t,x) - f(t,y)|^2\\ &\quad + \frac{\zeta}{2} h|g'g(t,x) - g'g(t,y)|^2 \le L_1|x-y|^2. \end{aligned}$$

These monotonicity conditions and polynomial growth conditions similar to Assumption 2.2 allows to show that for $\theta \in [1/2, 1]$ one has

$$\mathbb{E}\left[|X(t_n) - Y_n|^2\right] \le C h^2,$$

cf. [37] for more detail.

4. Numerical Experiments

This section presents the results of numerical simulations aimed to support the theoretical analysis. We start with the 3/2-model from finance

$$\mathrm{d}X(t) = \lambda X(t)\big(\mu - |X(t)|\big)\,\mathrm{d}t + \sigma |X(t)|^{\frac{3}{2}}\,\mathrm{d}W(t), \quad X(0) = X_0 > 0,$$

cf. [8,10]. It is shown in [32, Appendix] that in this case, Assumptions 2.1-2.2 are satisfied with $q = (\lambda + \sigma^2)/\sigma^2$, $L = \nu = 2\lambda\mu$, $\gamma = 2$, $p_0 = (2\lambda + \sigma^2)/\sigma^2$. We assume that $\lambda/\sigma^2 \geq 5/2$, which implies $q \geq 7/2 > 2$, $p_0 \geq 4\gamma - 2 = 6$, so that Theorem 3.1 is applicable. Therefore, in numerical simulations we choose $\lambda = 5/2, \mu = 2/5, \sigma = 1, X_0 = 1$. As the benchmark solutions we use the ones obtained by the SSBE approximations with the fine step size $h_{\text{exact}} = 2^{-13}$. Besides, the mean-square errors $\mathbb{E}[|X(T) - Y_N|^2]$ are computed by the Monte Carlo (MC) approximations over $M = 10000$ samples — i.e.

$$\mathbb{E}\big[|X(T) - Y_N|^2\big] \approx \frac{1}{M}\sum_{i=1}^{M}\big|X_T^{(i)} - Y_N^{(i)}\big|^2.$$

The resulting root mean square error (RMSE)

$$\text{RMSE} = \sqrt{\frac{\text{Var}(|X_T - Y_N|^2)}{M}}$$

of the Monte Carlo approximation is computed approximately by using the sample variance instead of the variance. For SST methods with $\theta = 1/2$, $\theta = 1$, Tables 1 and 2 show the mean-square approximation errors $1/M\sum_{i=1}^{M}|X_T^{(i)} - Y_N^{(i)}|^2$ and the RMSE for the MC approximation, respectively. Note that the RMSE for the MC approximation is negligible. Fig. 1 demonstrates the approximation errors $\sqrt{1/M\sum_{i=1}^{M}|X_T^{(i)} - Y_N^{(i)}|^2}$ of SST methods with $\theta = 1/2$ and $\theta = 1$ for six step sizes $h = 2^{-i}$, $i = 5, 6, \ldots, 10$. The resulting errors decrease linearly, at a slope of about 1/2, consistent with the theoretical convergence rate.

Our second example concerns double well dynamics with multiplicative noise — viz.

$$\mathrm{d}X(t) = X(t)\big(1 - X(t)^2\big)\,\mathrm{d}t + \sigma\big(1 - X(t)^2\big)\,\mathrm{d}W(t), \quad t \in [0,1], \quad X(0) = X_0 = 2.$$

Table 1: Mean-square approximation errors $1/M\sum_{i=1}^{M}|X_T^{(i)} - Y_N^{(i)}|^2$.

h	$\theta = 0.5$	$\theta = 1$
$h = 2^{-5}$	0.0016	0.0018
$h = 2^{-6}$	6.5860e-04	7.0704e-04
$h = 2^{-7}$	3.1292e-04	3.2641e-04
$h = 2^{-8}$	1.5234e-04	1.5732e-04
$h = 2^{-9}$	7.3609e-05	7.4433e-05
$h = 2^{-10}$	3.2224e-05	3.2313e-05

Table 2: The RMSE for the MC approximation.

h	$\theta = 0.5$	$\theta = 1$
$h = 2^{-5}$	7.5128e-05	8.6959e-05
$h = 2^{-6}$	2.9832e-05	3.6426e-05
$h = 2^{-7}$	1.6934e-05	1.9320e-05
$h = 2^{-8}$	1.0868-05	1.2161e-05
$h = 2^{-9}$	3.4798e-06	3.5724e-06
$h = 2^{-10}$	1.4789e-06	1.4818e-06

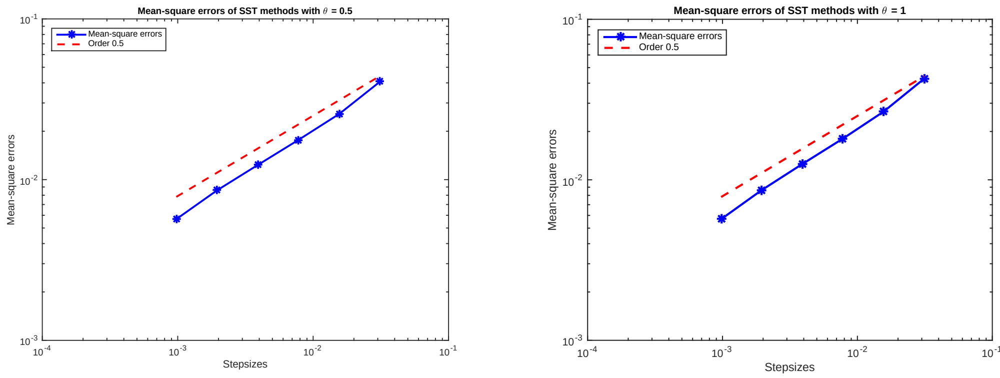

Figure 1: Convergence rates of SST methods with $\theta = 1/2$ (Left) and $\theta = 1$ (Right).

Such a model has been considered in [3, Section 8.1] and [22, Section 1]. It is easily seen that the conditions (2.4)-(2.5) are satisfied with $\gamma = 3$ and it is thus required that $p_0 \geq 10$. We choose $p_0 = 10$ and compute that

$$\begin{aligned}&\big\langle x, f(x)\big\rangle + \frac{p_0-1}{2}|g(x)|^2 = x^2 - x^4 + \frac{9\sigma^2}{2}\big(1-x^2\big)^2\\ =&-\left(1-\frac{9\sigma^2}{2}\right)x^4 + (1-9\sigma^2)x^2 + \frac{9\sigma^2}{2} \leq \nu\end{aligned}$$

for $\nu > 0$. Note that (2.6) holds for $\sigma^2 < 2/9$, and (2.3) is obviously satisfied with $L = 1$. Furthermore, if $\sigma^2 < 2/9$ and $2 < q < 11/2$, then

$$\begin{aligned}&2\langle x-y, f(x)-f(y)\rangle + (q-1)|g(x)-g(y)|^2\\ =&\big[2-2\big(x^2+xy+y^2\big)+(q-1)\sigma^2(x+y)^2\big](x-y)^2\\ =&\big[2-x^2-y^2+\big((q-1)\sigma^2-1\big)(x+y)^2\big]|x-y|^2\\ \leq&\, 2|x-y|^2,\end{aligned}$$

where we used the inequality $(q-1)\sigma^2 - 1 < 0$. Thus Assumptions 2.1-2.2 hold true provided that $\sigma^2 < 2/9$. Now we take $\sigma = 0.45$, $\theta = 1/2, 1$ so that $\sigma^2 < 2/9$ and Theorem 3.1 is applicable. Fig. 2 shows the approximation errors $\sqrt{1/M\sum_{i=1}^{M}|X_T^{(i)} - Y_N^{(i)}|^2}$ of the SST methods with $\theta = 1/2, \theta = 1$ for six step sizes $h = 2^{-i}, i = 5, 6, \ldots, 10$. Similar to the first test, the expectations are approximated by computing averages over $M = 10000$ samples and the benchmark solutions are obtained by using the fine step size $h_{\text{exact}} = 2^{-13}$. Numerical results indicate that the RMSE for the MC approximation here are negligible. Thus the resulting errors decrease almost linearly at a slope of about 1/2, consistent with the theoretical convergence rate.

As the third example, we consider the SDE system

$$\mathrm{d}X(t) = \big[AX(t) + F(X(t))\big]\,\mathrm{d}t + G\big(X(t)\big)\,\mathrm{d}W(t), \quad t \in [0,1], \quad X(0) = X_0, \tag{4.1}$$

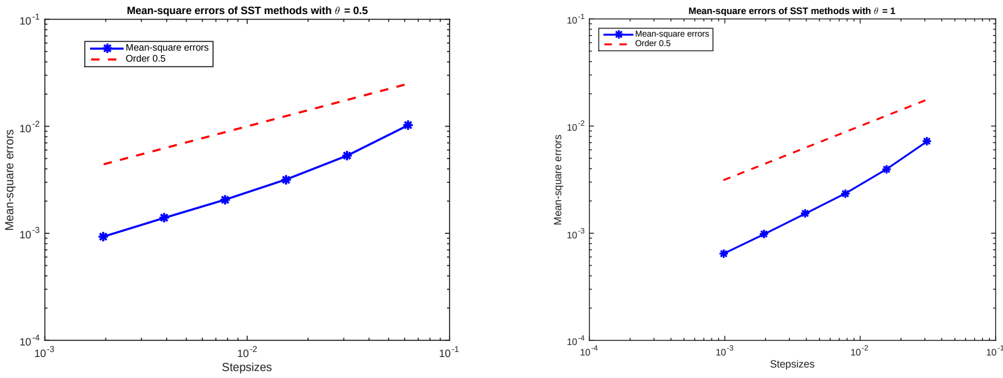

Figure 2: Convergence rates of SST methods with $\theta = 1/2$ (Left) and $\theta = 1$ (Right).

where $X_0 = (1,1,\dots,1)^T \in R^{m\times 1}, A \in R^{m\times m}$ and

$$A = (m+1)^2 \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix},$$

$$F(x) = \begin{bmatrix} 1 + x_1 - x_1^3 \\ 1 + x_2 - x_2^3 \\ \vdots \\ 1 + x_m - x_m^3 \end{bmatrix}, \quad G(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_m \end{bmatrix}.$$

Such SDE system arise in the finite difference spatial discretization with the spatial stepsize $\Delta x := 1/(m+1)$, of the following stochastic partial differential equation

$$\begin{aligned} \mathrm{d}u(t,x) &= \left[\frac{\partial^2}{\partial x^2} u(t,x) + f\big(u(t,x)\big)\right] \mathrm{d}t \\ &\quad + g\big(u(t,x)\big)\,\mathrm{d}W(t), \quad t \in (0,1], \quad x \in (0,1), \\ u(t,0) &= u(t,1) = 0, \\ u(0,x) &= u_0(x), \end{aligned}$$

where $u_0(x) \equiv 1$, $f(u) = 1 + u - u^3$ and $g(u) = 2u$. We use SST methods with $\theta = 0.5, 1$ and the tamed Euler method proposed by [17] to discretize the SDE system (4.1) with five step sizes $h = 2^{-i}, i = 2,5,\dots,6$. As before, the expectations are again approximated by the averages over $M = 10000$ samples and the benchmark solutions are obtained by approximations using the fine step size $h_{\text{exact}} = 2^{-12}$. Numerical results show that the RMSE for the MC approximation here are negligible.

Tables 3 and 4 show the approximation errors $\sqrt{1/M \sum_{i=1}^{M} |X_T^{(i)} - Y_N^{(i)}|^2}$ of these three methods for two dimensions $m = 4$ and $m = 9$. It is easily seen that the tamed Euler

Table 3: Mean-square approximation errors for three schemes ($m=4$).

h	$\theta=0.5$	$\theta=1$	Tamed Euler
$h=2^{-2}$	0.3202	0.2040	5.0269
$h=2^{-3}$	0.1156	0.1229	3.6509
$h=2^{-4}$	0.0653	0.0723	1.2767
$h=2^{-5}$	0.0424	0.0453	0.0528
$h=2^{-6}$	0.0276	0.0288	0.0292

Table 4: Mean-square approximation errors for three schemes ($m=9$).

h	$\theta=0.5$	$\theta=1$	Tamed Euler
$h=2^{-2}$	0.8619	0.2852	9.1844
$h=2^{-3}$	0.2645	0.1721	8.9198
$h=2^{-4}$	0.0923	0.1013	5.6077
$h=2^{-5}$	0.0591	0.0632	1.8393
$h=2^{-6}$	0.0384	0.0401	0.4821

method can give satisfactory results in the low dimension case $m = 4$ if the time step size is small — viz. $h = 2^{-5}, 2^{-6}$. When dimension increases to 9, the tamed Euler method produces rather large errors and the approximations obtained become unreliable for all step sizes. However, the SST methods with step sizes $2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}$ always produce reliable results with small errors, even in the high dimensional case $m = 9$. This happens because the problem (4.1) turns out to be a very stiff system [30] as m increases. The tamed Euler method, as a kind of explicit methods, faces severe time step-size reduction due to stability issues. On the other hand, the SST methods, as implicit methods, have excellent stability property and perform very well.

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