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Analysis of a System of Hemivariational Inequalities Arising in Non-Stationary Stokes Equation with Thermal Effects

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Abstract. A non-stationary Stokes equation coupled with an evolution equation of temperature field is studied. Boundary conditions for velocity and temperature fields contain the generalized Clarke gradient. The corresponding variational formulation is governed by a system of hemivariational inequalities. The existence and uniqueness of a weak solution is proved by employing Banach fixed point theorem and hemivariational inequalities. Besides, a fully-discrete problem for this system of hemivariational inequalities is given and error estimates are derived.

AMS subject classifications: 65M15

Key words: Non-stationary Stokes equation, hemivariational inequality, thermal effects, Banach fixed point theorem, numerical analysis.

1. Introduction

Hemivariational inequalities, as a generation of variational inequalities, constitute important tools in studying various nonlinear problems arising in chemistry, physics, biology, engineering, and many other fields. Research on variational inequalities stems from the monotonicity theory and convexity theory, while the study of hemivariational inequalities employs the Clarke subdifferential property of locally Lipschitz functions as the main component and allows the inclusion of non-convex functions. On the one hand, hemivariational inequalities have more advantages than variational inequalities in the characterization of some practical problems, and on the other hand, benefit from the development of non-smooth analysis and multivalued analysis, the theoretical and numerical analysis of hemivariational inequality develop rapidly in past few decades. In particular, variational or

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hemivariational inequalities arising in contact mechanics attracted widespread interests — cf. Refs. [4, 6, 13, 22, 24]. On the other hand, many researchers are interested in applying hemivariational inequalities to fluid mechanic problems [8, 9, 17, 28]. The paper [8] studies a hemivariational inequality arising from a stationary Stokes equation equipped with a nonlinear slip boundary condition, the finite element method is employed to solve the hemivariational inequality and error estimates are provided. Paper [9] is devoted to studying a type of hemivariational inequalities that arises in a non-stationary Navier-Stokes problem. Existence of solution for the abstract hemivariational inequality is sought by a kind of time discretization method, termed as Rothe method. Paper [17] studies Stokes problem for a generalized Newtonian fluid along with unilateral, slip and leak boundary conditions. Existence of a unique weak solution is proven through a surjectivity theorem. A Stokes problem for an incompressible fluid, whose boundary conditions are in the type of subdifferential was stadied in [28]. The associated variational formulation forms a variational-hemivariational inequalities system. The corresponding solution existence as well as the weak compactness of the solution set is established by Schauder fixed point theorem.

The above papers do not consider the interaction of velocity and temperature fields of the fluid, so that the only a pure fluid dynamics problem is studied. However, in fact many parameters of actual fluids are affected by the temperature. In contrast, the flow of fluids also causes changes in temperature. Therefore, numerous studies focus on the fluid problems with thermal effects — cf. [2, 20, 21, 27]. In these papers, either Dirichlet or Neumann boundary conditions are considered, so that all the models lead to a system of equations. However, the physical phenomena can be multitudinous and various boundary conditions can be required. Thus assuming the boundary conditions to include subdifferential nonconvex functions, we can arrive at a system of hemivariational inequalities. Hitherto, there is no works considering hemivariational inequalities arising from Stokes flow with thermal effects, and our aim is to cover this gap.

More exactly, this paper focuses on variational and numerical analysis of a system of hemivariational inequalities arising in a non-stationary incompressible Stokes equation coupled with an evolution equation of temperature field. Inspired by the ideas of [15, 16, 21], we consider the following conservation laws:

$$
\begin{aligned}
&\boldsymbol{u}'(t) - \nu^* \Delta \boldsymbol{u}(t) + \nabla p(t) - c_e \theta(t) = \boldsymbol{q}(t) && \text{in } \Theta \times (0,T),\\
&\operatorname{div} \boldsymbol{u}(t) = \boldsymbol{0} && \text{in } \Theta \times (0,T),\\
&\theta'(t) - \Delta\theta(t) = -c_{ij} \frac{\partial u_i}{\partial x_j}(t) + g(t) && \text{in } \Theta \times (0,T),
\end{aligned}
$$

where $\Theta \subset \mathbb{R}^d, d = 2,3$ is a bounded connected domain, whose boundary Γ is Lipschitz continuous and $0 < T < \infty$. Besides, $\boldsymbol{u}(\boldsymbol{x},t)$ is the flow velocity, $\boldsymbol{q}(\boldsymbol{x},t)$ an external force, ν^* a positive viscosity constant, $p(\boldsymbol{x},t)$ the pressure, $\theta(\boldsymbol{x},t)$ the temperature, $g(\boldsymbol{x},t)$ the density of volume heat sources, and $c_e = (c_{ij})$ the thermal influence operator. Subsequently, the boundary conditions are made up as follows:

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{1.1}$$

$$\theta(t) = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{1.2}$$

$$u_\nu(t) = 0 \qquad \text{on } \Gamma_2 \times (0,T), \tag{1.3}$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j\big(\boldsymbol{u}_\tau(t)\big) \qquad \text{on } \Gamma_2 \times (0,T), \tag{1.4}$$

$$-\frac{\partial \theta(t)}{\partial \nu} \in \partial j_{temp}\big(\theta(t)\big) \qquad \text{on } \Gamma_2 \times (0,T), \tag{1.5}$$

where Γ consists of smooth parts Γ_1, Γ_2 such that meas(Γ_1) is positive. The boundary Γ_1 is endowed with a clamping boundary condition. The condition (1.3) implies that there is no normal velocity on the boundary Γ_2, so that the fluid cannot penetrate outside the domain via Γ_2. Besides, (1.4) is a multivalued friction law modelled by the Clarke subdifferential of a locally Lipschitz function which is non-convex. Assertion (1.5) represents a boundary condition associated to temperature and that it is modelled by a kind of subdifferential type of a non-convex potential j_{temp}. The variational form of this model leads to a hemivariational inequality for the velocity field and a hemivariational inequality for the temperature field. The model is novel and realistic, which makes it interesting and meaningful to study this problem. Meanwhile, we will give a fully discrete scheme of the system of hemivariational inequalities and derive error estimate for the numerical solution. There are a list of papers concerning error estimates for hemivariational inequality arising from viscoelastic contact problem [5, 10, 25, 26]. However, only few papers consider error estimates for hemivariational inequality arising in Stokes flow [11, 14, 23]. In contrast to the above mentioned papers containing only one hemivariational inequality, our work deals with error estimates for a more complicated and challenging problem — viz. for coupled hemivariational inequalities.

Aiming to investigate the above model, we note some elementary material. First, we may not show the dependence of different functions on the variable $\boldsymbol{x}$ explicitly. The deformation-rate and stress tensors are defined as

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\big(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\big), \quad \boldsymbol{\sigma}(\boldsymbol{u}, p) := -p\mathbb{I} + 2\nu^* \boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where $\mathbb{I}$ is the $d \times d$ identity matrix. Denote by $\boldsymbol{\nu}$, we have the unit outward normal vector on boundary. Moreover, if a vector-valued function $\boldsymbol{v}$ on Γ is given, we employ v_ν, $\boldsymbol{v}_\tau$ for its normal and tangential components, and they are respectively defined as $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. In what follows, we sum up the above equations to achieve the following problem.

Problem 1.1. Find a flow velocity $\boldsymbol{u} : \Theta \times (0,T) \to \mathbb{R}^d$, a pressure $p : \Theta \times (0,T) \to \mathbb{R}$ and a temperature $\theta : \Theta \times (0,T) \to \mathbb{R}$ such that for all $t \in (0,T)$

$$\boldsymbol{u}'(t) - \nu^* \Delta \boldsymbol{u}(t) + \nabla p(t) - c_e \theta(t) = \boldsymbol{q}(t) \qquad \text{in } \Theta, \tag{1.6}$$

$$\text{div } \boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{in } \Theta, \tag{1.7}$$

$$\theta'(t) - \Delta \theta(t) = -c_{ij} \frac{\partial u_i}{\partial x_j}(t) + g(t) \qquad \text{in } \Theta, \tag{1.8}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{1.9}$$

$$\theta(t) = 0 \qquad \text{on } \Gamma_1, \tag{1.10}$$

$$u_\nu(t) = 0 \qquad \text{on} \quad \Gamma_2, \tag{1.11}$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j\big(\boldsymbol{u}_\tau(t)\big) \qquad \text{on} \quad \Gamma_2, \tag{1.12}$$

$$-\frac{\partial \theta(t)}{\partial \nu} \in \partial j_{temp}\big(\theta(t)\big) \qquad \text{on} \quad \Gamma_2, \tag{1.13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in} \ \Theta, \tag{1.14}$$

$$\theta(0) = \theta_0 \qquad \text{in} \ \Theta. \tag{1.15}$$

This paper is arranged as follows. In Section 2, we introduce a preliminary material and itemize necessary hypothesis on the data. Afterwards, a system of hemivariational inequalities in accordance with the model is put forward. In Section 3, the existence and uniqueness for the variational problem are proved by using the Banach fixed point theorem and hemivariational inequalities. In Section 4, a fully discrete problem is proposed and error estimates of a finite element method are derived.

2. Notation and Assumptions

In the study of the corresponding mathematical theory, we first recall basic notation, definitions and materials. We first recall Clarke directional derivative. Take Z as a Banach space equipped with a norm $\|\cdot\|_Z$, Z^* the dual of it. The duality pairing between Z^* and Z is denoted by $\langle\cdot,\cdot\rangle_{Z^*\times Z}$.

Definition 2.1. *Take $\vartheta\colon Z \to \mathbb{R}$ as a locally Lipschitz function. It is Clarke directional derivative at $z \in Z$ in the direction $v \in Z$ which is denoted as $\vartheta^0(z;v)$, is given as*

$$\vartheta^0(z;v) = \limsup_{x\to z,\,\mu\downarrow 0} \frac{\vartheta(x+\mu v)-\vartheta(x)}{\mu}$$

and its Clarke subdifferential at z is a subset of the space Z^* defined as

$$\partial\vartheta(z) = \left\{ \eta \in Z^* \mid \vartheta^0(z;v) \ge \langle \eta, v\rangle_{Z^*\times Z} \quad \text{for all} \quad v \in Z \right\}.$$

Definition 2.2. *An operator $B : Z \to Z^*$ is pseudomonotone if for any sequence $\{v_n\}_{n=1}^\infty \subset Z$, $v_n \to v$ weakly in Z and*

$$\limsup_{n\to\infty} \langle Bv_n, v_n - v\rangle_{Z^*\times Z} \le 0$$

indicate that

$$\langle Bv, v - z\rangle_{Z^*\times Z} \le \liminf_{n\to\infty} \langle Bv_n, v_n - z\rangle_{Z^*\times Z}$$

for every $z \in Z$.

The next, we recall two lemmata stated in [12, Lemma 7.24] and [19], respectively.

Lemma 2.1. *Assume that $z, q \in C([a,b])$ satisfy*

$$z(t) \le q(t) + c\int_a^t z(\tau)\,d\tau, \quad t \in [a,b]$$

with a constant $c > 0$. Then

$$z(t) \leq q(t) + c\int_a^t q(\tau)e^{c(t-\tau)}\,d\tau, \quad t \in [a,b].$$

Furthermore, if q is nondecreasing, then

$$z(t) \leq q(t)e^{c(t-a)}.$$

Lemma 2.2. *Take $\Phi : L^2(0,T;Z) \to L^2(0,T;Z)$ as the operator which satisfies*

$$\|(\Phi\omega_1)(t) - (\Phi\omega_2)(t)\|_Z^2 \leq c\int_0^t \|\omega_1(\tau) - \omega_2(\tau)\|_Z^2\,d\tau \tag{2.1}$$

for every $\omega_1, \omega_2 \in L^2(0,T;Z)$, a.e. $t \in (0,T)$. Thus, Φ admits one and only one fixed point in $L^2(0,T;Z)$, thus, there exists a unique $\omega^* \in L^2(0,T;Z)$ which satisfies $\Phi\omega^* = \omega^*$.

We adopt standard notation for Lebesgue and Sobolev spaces. Take $\boldsymbol{v} \in H^1(\Theta;\mathbb{R}^d)$, use identical notation $\boldsymbol{v}$ as the trace of $\boldsymbol{v}$ on $\partial\Theta$, at the same time the symbol v_ν and $\boldsymbol{v}_\tau$ are represented as the normal and tangential traces of it. Moreover, V and $\mathscr{H}$ are introduced as follows:

$$\begin{aligned} &V := \left\{\boldsymbol{v} \in H^1(\Theta;\mathbb{R}^d) \mid \operatorname{div}\boldsymbol{v} = 0 \ \text{ in } \ \Theta, \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1, v_\nu = 0 \ \text{ on } \ \Gamma_2\right\}, \\ &\mathscr{H} := L^2\left(\Theta;\mathbb{S}^d\right), \quad H := L^2\left(\Theta;\mathbb{R}^d\right). \end{aligned}$$

The above sets are real Hilbert spaces, and

$$(\boldsymbol{u},\boldsymbol{v})_V := \left(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathscr{H}}$$

in V. The corresponding norms are denoted by $\|\cdot\|_V$, $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_H$. The Sobolev trace theorem indicates that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_2;\mathbb{R}^d)} \leq \|\gamma\|\|\boldsymbol{v}\|_V \quad \text{for all} \quad \boldsymbol{v} \in V,$$

where $\|\gamma\|$ denotes the norm of the trace operator $\gamma : V \to L^2(\Gamma_2;\mathbb{R}^d)$. As for the pressure field, we will use

$$Q := \left\{q \in L^2(\Theta) \,\middle|\, \int_\Theta q\,dx = 0\right\}.$$

For the temperature field, we let

$$\begin{aligned} &E := \left\{\chi \in H^1(\Theta), \ \chi = 0 \ \text{ on } \ \Gamma_1\right\}, \\ &F := L^2(\Theta). \end{aligned}$$

Note that $V \subset H \subset V^*$ constitutes an evolution triple of function spaces. We introduce the spaces $\mathscr{V} = L^2(0,T;V)$, $\mathscr{Z} = \{\boldsymbol{z} \in \mathscr{V} \mid \boldsymbol{z}' \in \mathscr{V}^*\}$, and $\mathscr{V}^* = L^2(0,T;V^*)$ is taken as the dual of $\mathscr{V}$. The embeddings $\mathscr{Z} \subset C([0,T];H)$ and $\{\boldsymbol{z} \in \mathscr{V} \mid \boldsymbol{z}' \in \mathscr{Z}\} \subset C([0,T];V)$ are continuous, $C([0,T];H)$ denotes the continuous functions space on $[0,T]$ and its values in H.

Similarly, we obtain the evolution triple of spaces $E \subset F \subset E^*$. Let $\mathscr{E} = L^2(0,T;E)$ and $\mathbb{E} = \{\eta \in \mathscr{E} \mid \eta' \in \mathscr{E}^*\}$. The dual of $\mathscr{E}$ is $\mathscr{E}^* = L^2(0,T;E^*)$. We also know that $\mathbb{E} \subset C(0,T;F)$ and $\{\eta \in \mathscr{E} \mid \eta' \in \mathbb{E}\} \subset C([0,T];E)$ are continuous. Besides, we use $\gamma_1 : E \to L^2(\Gamma_2)$ to denote the trace operator for the temperature functions.

In order to study Problem 1.1, we need the following data assumptions:

$\boldsymbol{H(j)}$. The frictional potential $j\colon \Gamma_2 \times \mathbb{R}^d \to \mathbb{R}$ satisfies the conditions:

(a) $j(\cdot,\boldsymbol{r})$ is measurable on Γ_2 for all $\boldsymbol{r} \in \mathbb{R}^d$ and there exists $\boldsymbol{e} \in L^2(\Gamma_2;\mathbb{R}^d)$ such that $j(\cdot,\boldsymbol{e}(\cdot)) \in L^1(\Gamma_2)$.

(b) $j(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_2$.

(c) $\|\partial j(\boldsymbol{x},\boldsymbol{r})\|_{\mathbb{R}^d} \le c_0 + c_1\|\boldsymbol{r}\|_{\mathbb{R}^d}$ for all $\boldsymbol{r} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_2$ with $c_0, c_1 \ge 0$.

(d) $j^0(\boldsymbol{x},\boldsymbol{r}_1;\boldsymbol{r}_2-\boldsymbol{r}_1) + j^0(\boldsymbol{x},\boldsymbol{r}_2;\boldsymbol{r}_1-\boldsymbol{r}_2) \le m_\tau\|\boldsymbol{r}_1-\boldsymbol{r}_2\|^2_{\mathbb{R}^d}$ for all $\boldsymbol{r}_1,\boldsymbol{r}_2 \in \mathbb{R}^d$ with $m_\tau \ge 0$, a.e. $\boldsymbol{x} \in \Gamma_2$.

(e) $j(\boldsymbol{x},\cdot)$ or $-j(\boldsymbol{x},\cdot)$ is regular, a.e. $\boldsymbol{x} \in \Gamma_2$, $\boldsymbol{r} \in \mathbb{R}^d$.

$\boldsymbol{H(j_{temp})}$. The operator $j_{temp}\colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions:

(a) $j_{temp}(\cdot,s)$ is measurable on Γ_2 for all $s \in \mathbb{R}$ and there exists $\rho \in L^2(\Gamma_2)$ such that $j_{temp}(\cdot,\rho(\cdot)) \in L^1(\Gamma_2)$.

(b) $j_{temp}(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_2$.

(c) $|\partial j_{temp}(\boldsymbol{x},s)| \le \overline{c_0} + \overline{c_1}|s|$ for all $s \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_2$ with $\overline{c_0}, \overline{c_1} \ge 0$.

(d) $j^0_{temp}(\boldsymbol{x},s_1;s_2-s_1) + j^0_{temp}(\boldsymbol{x},s_2;s_1-s_2) \le m_1|s_1-s_2|^2$ for all $s_1,s_2 \in \mathbb{R}$ with $m_1 \ge 0$, a.e. $\boldsymbol{x} \in \Gamma_2$.

(e) $j_{temp}(\boldsymbol{x},\cdot)$ or $-j_{temp}(\boldsymbol{x},\cdot)$ is regular, a.e. $\boldsymbol{x} \in \Gamma_2$, $s \in \mathbb{R}$.

$\boldsymbol{H(c_e)}$. The operator $c_e : \Theta \times \mathbb{R} \to \mathbb{R}^d$ satisfies the conditions:

(a) $c_e(\cdot,s)$ is measurable on Θ for all $s \in \mathbb{R}$.

(b) $\|c_e(\boldsymbol{x},s)\|_{\mathbb{R}^d} \le c_{0e} + c_{1e}|s|$ for all $s \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Theta$ with $c_{0e} \in L^2(\Theta), c_{0e}, c_{1e} \ge 0$.

(c) $\|c_e(\boldsymbol{x},s_1) - c_e(\boldsymbol{x},s_2)\|_{\mathbb{R}^d} \le L_e|s_1-s_2|$ for all $s_1,s_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Theta$ with $L_e > 0$.

As for the heat sources density, we assume that

$$g \in H^1(0,T;F). \tag{2.2}$$

In the end, the initial values satisfy $\boldsymbol{u}_0 \in V, \theta_0 \in E$.

We now arrive at the variational formulation of Problem 1.1: For $\boldsymbol{v} \in V$, let the Eq. (1.6) be multiplied by $\boldsymbol{v}$. After integrating the resulting equality, we obtain

$$\begin{aligned}&\int_\Theta \boldsymbol{u}'(t)\cdot\boldsymbol{v}\,dx+\int_\Theta\big(-\nu^*\Delta\boldsymbol{u}(t)\big)\cdot\boldsymbol{v}\,dx\\&\quad-\int_\Theta c_e\theta(t)\cdot\boldsymbol{v}\,dx+\int_\Theta\nabla p(t)\cdot\boldsymbol{v}\,dx\\&=\int_\Theta \boldsymbol{q}(t)\cdot\boldsymbol{v}\,dx.\end{aligned}$$

Considering $\Delta\boldsymbol{u}=2\operatorname{div}\boldsymbol{\varepsilon}(\boldsymbol{u})$ implied by Eq. (1.7), applying Green-type formula, it is easy to obtain

$$\begin{aligned}&\int_\Theta -\operatorname{div}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{v}\,dx=\int_\Theta\boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx-\int_{\partial\Theta}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma,\\&\int_\Theta\nabla p\cdot\boldsymbol{v}\,dx=-\int_\Theta\operatorname{div}\boldsymbol{v}\cdot p\,dx+\int_{\partial\Theta}p\,\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma.\end{aligned}$$

Thus, we deduce

$$\begin{aligned}&\int_\Theta \boldsymbol{u}'(t)\cdot\boldsymbol{v}\,dx+2\nu^*\int_\Theta\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big):\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx\\&\quad-\int_\Theta c_e\theta(t)\cdot\boldsymbol{v}\,dx-\int_\Theta\operatorname{div}\boldsymbol{v}\cdot p(t)\,dx\\&\quad+\int_{\partial\Theta}p(t)\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma-2\nu^*\int_{\partial\Theta}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma\\&=\int_\Theta\boldsymbol{q}(t)\cdot\boldsymbol{v}\,dx.\end{aligned}$$

Since functions are divergence free in V, through the boundary condition and $\boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}=\boldsymbol{\sigma}_\tau\cdot\boldsymbol{v}_\tau+\sigma_\nu v_\nu$, we arrive at the following hemivariational inequality:

$$\begin{aligned}&\int_\Theta \boldsymbol{u}'(t)\cdot\boldsymbol{v}\,dx+2\nu^*\int_\Theta\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big):\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx\\&\quad-\int_\Theta c_e\theta(t)\cdot\boldsymbol{v}\,dx-\int_{\Gamma_2}\boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau\,d\Gamma\\&=\int_\Theta\boldsymbol{q}(t)\cdot\boldsymbol{v}\,dx.\end{aligned}$$

Owing to the Clarke subdifferential, we can easily obtain

$$\big\langle\boldsymbol{u}'(t),\boldsymbol{v}\big\rangle_{V^*\times V}+a\big(\boldsymbol{u}(t),\boldsymbol{v}\big)-\big\langle c_e\theta(t),\boldsymbol{v}\big\rangle_{V^*\times V}+\int_{\Gamma_2}j^0\big(\boldsymbol{u}_\tau(t);\boldsymbol{v}_\tau\big)\,d\Gamma\geq\big\langle\boldsymbol{q}(t),\boldsymbol{v}\big\rangle_{V^*\times V}$$

with the bilinear form

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu^* \int_{\Theta} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx,$$

and linear forms

$$\langle \boldsymbol{u}', \boldsymbol{v} \rangle_{V^* \times V} = \int_{\Theta} \boldsymbol{u}' \cdot \boldsymbol{v}\, dx,$$

$$\langle \boldsymbol{q}, \boldsymbol{v} \rangle_{V^* \times V} = \int_{\Theta} \boldsymbol{q} \cdot \boldsymbol{v}\, dx,$$

$$\langle c_e\theta, \boldsymbol{v} \rangle_{V^* \times V} = \int_{\Theta} c_e\theta \cdot \boldsymbol{v}\, dx.$$

We similarly obtain the inequality as follows:

$$\begin{aligned}&\big\langle \theta'(t), \chi \big\rangle_{E^* \times E} + a_0\big(\theta(t), \chi\big) + \int_{\Gamma_2} j^0_{temp}\big(\theta(t); \chi\big)\, d\Gamma \\ &\geq \big\langle R\boldsymbol{u}(t), \chi \big\rangle_{E^* \times E} + \big\langle g(t), \chi \big\rangle_{E^* \times E},\end{aligned}$$

where

$$a_0(\theta, \chi) = \int_{\Theta} \nabla\theta \cdot \nabla\chi\, dx,$$

and $R : V \to E^*$ is

$$\langle R\boldsymbol{u}, \chi \rangle_{E^* \times E} = -\int_{\Theta} c_{ij} \frac{\partial u_i}{\partial x_j} \chi\, dx.$$

Now we can consider the following problem.

Problem 2.1. Find a flow velocity $\boldsymbol{u} : \Theta \times (0,T) \to \mathbb{R}^d$ and a temperature $\theta : \Theta \times (0,T) \to \mathbb{R}$ such that for a.e. $t \in (0,T)$,

$$\begin{aligned}&\big\langle \boldsymbol{u}'(t), \boldsymbol{v} \big\rangle_{V^* \times V} + a\big(\boldsymbol{u}(t), \boldsymbol{v}\big) - \big\langle c_e\theta(t), \boldsymbol{v} \big\rangle_{V^* \times V} \\ &\quad + \int_{\Gamma_2} j^0\big(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau\big)\, d\Gamma \geq \langle \boldsymbol{q}(t), \boldsymbol{v} \rangle_{V^* \times V} \qquad \text{for all} \quad \boldsymbol{v} \in V, \qquad (2.3)\end{aligned}$$

$$\begin{aligned}&\big\langle \theta'(t), \chi \big\rangle_{E^* \times E} + a_0\big(\theta(t), \chi\big) + \int_{\Gamma_2} j^0_{temp}\big(\theta(t); \chi\big)\, d\Gamma \\ &\geq \big\langle R\boldsymbol{u}(t), \chi \rangle_{E^* \times E} + \langle g(t), \chi \big\rangle_{E^* \times E} \qquad \text{for all} \quad \chi \in E, \qquad (2.4)\end{aligned}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0. \qquad (2.5)$$

3. Existence and Uniqueness of Hemivariational Inequality System

This section contains the proof of the unique solvability of Problem 2.1. For this, we first introduce the operators $B : V \to V^*$, $B_0 : E \to E^*$, $C_1 : F \to V^*$, $C_2 : V \to E^*$,

$J: L^2(\Gamma_2;\mathbb{R}^d)\to\mathbb{R}$, and $J_{temp}: L^2(\Gamma_2)\to\mathbb{R}$ as follows:

$$\langle B\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V}=a(\boldsymbol{u},\boldsymbol{v}),\qquad \boldsymbol{u},\boldsymbol{v}\in V, \tag{3.1}$$

$$\langle B_0\theta,\chi\rangle_{E^*\times E}=a_0(\theta,\chi),\qquad \theta,\chi\in E, \tag{3.2}$$

$$\langle C_1\theta,\boldsymbol{v}\rangle_{V^*\times V}=-\langle c_e\theta,\boldsymbol{v}\rangle_{V^*\times V},\qquad \boldsymbol{v}\in V, \tag{3.3}$$

$$\langle C_2\boldsymbol{u},\chi\rangle_{E^*\times E}=\langle R\boldsymbol{u},\chi\rangle_{E^*\times E},\qquad \chi\in E, \tag{3.4}$$

$$J(\boldsymbol{u})=\int_{\Gamma_2} j(\boldsymbol{x},\boldsymbol{u}_\tau)\,d\Gamma,\qquad \boldsymbol{u}\in L^2\big(\Gamma_2;\mathbb{R}^d\big), \tag{3.5}$$

$$J_{temp}(\theta)=\int_{\Gamma_2} j_{temp}(\boldsymbol{x},\theta)\,d\Gamma,\qquad \theta\in L^2(\Gamma_2). \tag{3.6}$$

Following the assumptions $(Hj)(e)$ and $(Hj_{temp})(e)$ and using [19, Corollary 4.15(vii)], we obtain that $J(\cdot)$ or $-J(\cdot)$ is regular on $L^2(\Gamma_2;\mathbb{R}^d)$, $J_{temp}(\cdot)$ or $-J_{temp}(\cdot)$ is regular on $L^2(\Gamma_2)$. Afterwards, [19, Corollary 4.15(vi) and Lemma 3.39(3)] show that

$$J^0(\boldsymbol{u})=\int_{\Gamma_2} j^0(\boldsymbol{x},\boldsymbol{u}_\tau)\,d\Gamma\qquad\text{for all}\quad \boldsymbol{u}\in W,$$

$$J^0_{temp}(\theta)=\int_{\Gamma_2} j^0_{temp}(\boldsymbol{x},\theta)\,d\Gamma\quad\text{for all}\quad \theta\in Y,$$

where $W=L^2(\Gamma_2;\mathbb{R}^d)$, $Y=L^2(\Gamma_2)$. According to the above equations, the following problem can be obtained.

Problem 3.1. Find a flow velocity $\boldsymbol{u}:\Theta\times(0,T)\to\mathbb{R}^d$ and a temperature $\theta:\Theta\times(0,T)\to\mathbb{R}$ such that for a.e. $t\in(0,T)$,

$$\begin{aligned}&\big\langle \boldsymbol{u}'(t),\boldsymbol{v}\big\rangle_{V^*\times V}+\big\langle B\boldsymbol{u}(t),\boldsymbol{v}\big\rangle_{V^*\times V}+\big\langle C_1\theta(t),\boldsymbol{v}\big\rangle_{V^*\times V}\\ &\quad+J^0\big(\gamma\boldsymbol{u}(t);\gamma\boldsymbol{v}\big)\ge\big\langle \boldsymbol{q}(t),\boldsymbol{v}\big\rangle_{V^*\times V}\qquad\text{for all}\quad \boldsymbol{v}\in V,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\big\langle \theta'(t),\chi\big\rangle_{E^*\times E}+\big\langle B_0\theta(t),\chi\big\rangle_{E^*\times E}+J^0_{temp}\big(\gamma_1\theta(t);\gamma_1\chi\big)\\ &\ge\big\langle C_2\boldsymbol{u}(t),\chi\big\rangle_{E^*\times E}+\big\langle g(t),\chi\big\rangle_{E^*\times E}\qquad\text{for all}\quad \chi\in E.\end{aligned} \tag{3.8}$$

Let us note a few properties of the above defined operators.

1. For $B: V\to V^*$, we have

$$B\text{ is pseudomonotone}, \tag{3.9a}$$

$$\|B\boldsymbol{v}\|_{V^*}\le b_0+b_1\|\boldsymbol{v}\|_V\qquad\text{for all}\quad \boldsymbol{v}\in V,\qquad b_0,b_1\ge 0, \tag{3.9b}$$

$$\begin{aligned}&\langle B\boldsymbol{v}_1-B\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2\rangle_{V^*\times V}\\ &\ge m_B\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V^2\qquad\text{for all}\quad \boldsymbol{v}_1,\boldsymbol{v}_2\in V,\quad m_B>0.\end{aligned} \tag{3.9c}$$

2. For $B_0\colon E \to E^*$, we have

$$
\begin{aligned}
&B_0 \text{ is pseudomonotone}, && && (3.10a)\\
&\|B_0\chi\|_{E^*} \le \bar{b_0} + \bar{b_1}\|\chi\|_E && \text{for all} \quad \chi \in E, \qquad \bar{b_0}, \bar{b_1} \ge 0, && (3.10b)\\
&\langle B_0\chi_1 - B_0\chi_2, \chi_1 - \chi_2\rangle_{E^*\times E} && && \\
&\ge m_{B_0}\|\chi_1 - \chi_2\|_E^2 && \text{for all} \quad \chi_1, \chi_2 \in E, \quad m_{B_0} > 0. && (3.10c)
\end{aligned}
$$

3. For the functional $J : W \to \mathbb{R}$, we have

$$
\begin{aligned}
&J \text{ is locally Lipschitz on } W, && && (3.11a)\\
&\|\partial J(\boldsymbol{u})\|_{W^*} \le c_{0J} + c_{1J}\|\boldsymbol{u}\|_W && \text{for all} \quad \boldsymbol{u} \in W, \quad c_{0J}, c_{1J} \ge 0, && (3.11b)\\
&\langle \xi_1 - \xi_2, \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle_{W^*\times W} && && \\
&\ge -m_J\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_W^2 && \text{for all} \quad \xi_i \in \partial J(\boldsymbol{u}_i), \quad \xi_i \in W^*, \quad \boldsymbol{u}_i \in W, && \\
& && i = 1, 2, \quad m_J \ge 0. && (3.11c)
\end{aligned}
$$

4. For the functional $J_{temp} : Y \to \mathbb{R}$, we have

$$
\begin{aligned}
&J_{temp} \text{ is locally Lipschitz on } Y, && && (3.12a)\\
&\|\partial J_{temp}(\theta)\|_{Y^*} \le c_{0\theta} + c_{1\theta}\|\theta\|_Y && \text{for all} \quad \theta \in Y, \quad c_{0\theta}, c_{1\theta} \ge 0, && (3.12b)\\
&\langle z_1 - z_2, \theta_1 - \theta_2\rangle_{Y^*\times Y} && && \\
&\ge -m_K\|\theta_1 - \theta_2\|_Y^2 && \text{for all} \quad z_i \in \partial J_{temp}(\theta_i), \quad z_i \in Y^*, \quad \theta_i \in Y, && \\
& && i = 1, 2, \quad m_K \ge 0. && (3.12c)
\end{aligned}
$$

5. For the operator $C_2 : V \to E^*$, we have

$$
\begin{aligned}
&C_2\boldsymbol{v} \in F && \text{for all} \quad \boldsymbol{v} \in V, && (3.13a)\\
&\|C_2\boldsymbol{v}_1 - C_2\boldsymbol{v}_2\|_{E^*} \le L_R\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V && \text{for all} \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in V, \quad L_R > 0. && (3.13b)
\end{aligned}
$$

6. For the operator $C_1 : F \to V^*$, we have

$$
\begin{aligned}
&\|C_1\chi\|_{V^*} \le C_{0e} + C_{1e}\|\chi\|_F && \text{for all} \quad \chi \in F, \qquad C_{0e}, C_{1e} \ge 0, && (3.14a)\\
&\|C_1\chi_1 - C_1\chi_2\|_{V^*} \le L_1\|\chi_1 - \chi_2\|_F && \text{for all} \quad \chi_1, \chi_2 \in F, \quad L_1 > 0. && (3.14b)
\end{aligned}
$$

We verify that for B defined by (3.1), (3.9b)-(3.9c) hold with $b_1 = m_B = 2\nu$. Since the operator B is bounded, continuous and monotone, we obtain that B is pseudomonotone. For B_0 defined by (3.2), we have the similar results. Assumptions (3.11) and (3.12) are easy consequences of $H(j)$ and $H(j_{temp})$, respectively. $H(c_e)$ implies (3.14) and (3.13) comes from the definition of R.

Remark 3.1. The assumptions (3.11c) and (3.12c) are equivalent to the conditions

$$
\begin{aligned}
&J^0(\boldsymbol{u}_1; \boldsymbol{u}_2 - \boldsymbol{u}_1) + J^0(\boldsymbol{u}_2; \boldsymbol{u}_1 - \boldsymbol{u}_2) \le m_J\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_W^2,\\
&J_{temp}^0(\theta_1; \theta_2 - \theta_1) + J_{temp}^0(\theta_2; \theta_1 - \theta_2) \le m_K\|\theta_1 - \theta_2\|_Y^2.
\end{aligned}
$$

Now we can present the existence and uniqueness theorem.

Theorem 3.1. *Assume that* (2.2), (3.9)-(3.14) *hold,* $\boldsymbol{q} \in \mathscr{V}^*$ *and*

$$m_B > m_J \|\gamma\|^2, \quad m_{B_0} > m_K \|\gamma_1\|^2, \tag{3.15}$$

then Problem 3.1 *has a unique solution* $(\boldsymbol{u}, \theta)$ *with regularity* $\theta \in H^1(0,T;E) \cap W^{1,\infty}(0,T;F)$*,* $\boldsymbol{u} \in H^1(0,T;V)$*.*

The following steps are carried out to prove Theorem 3.1.

Step 1. Given $\omega \in \mathscr{V}^*$, it follows the following hemivariational inequality.

Problem 3.2. Find $\boldsymbol{u}_\omega \in \mathscr{X}$ such that

$$\begin{aligned} &\left\langle \boldsymbol{u}'_\omega(t) + B\boldsymbol{u}_\omega(t), \boldsymbol{v} \right\rangle_{V^*\times V} + J^0\big(\gamma \boldsymbol{u}_\omega(t); \gamma \boldsymbol{v}\big) \\ \geq &\left\langle \boldsymbol{q}(t) - \omega(t), \boldsymbol{v} \right\rangle_{V^*\times V} \quad \text{for all} \quad \boldsymbol{v} \in V, \quad \text{a.e.} \quad t \in (0,T), \\ &\boldsymbol{u}_\omega(0) = \boldsymbol{u}_0, \end{aligned}$$

where $\omega(t) = C_1\theta(t)$.

Lemma 3.1. *For* $\omega \in \mathscr{V}^*$*, there exists a unique solution* $\boldsymbol{u}_\omega \in \mathscr{X}$ *to Problem* 3.2*. Moreover, given* $\omega_i \in \mathscr{V}^*$ *and the corresponding unique solutions* $\boldsymbol{u}_{\omega_i}$*,* $i = 1, 2$*, there holds the inequality*

$$\int_0^t \|\boldsymbol{u}_{\omega_1}(\tau) - \boldsymbol{u}_{\omega_2}(\tau)\|_V^2 \, d\tau \leq c \int_0^t \|\omega_1(\tau) - \omega_2(\tau)\|_{V^*}^2 \, d\tau \tag{3.16}$$

for all $t \in [0,T]$ *with* $c > 0$*.*

For the first part of this lemma, we notice that Problem 3.2 can be rewritten as follows.

Problem 3.3. Find $\boldsymbol{u}_\omega \in \mathscr{X}$ and $\zeta \in \mathscr{W}^*$ that satisfy

$$\boldsymbol{u}'_\omega(t) + B\boldsymbol{u}_\omega(t) + \zeta(t) = \boldsymbol{q}_\omega(t), \quad \text{a.e.} \quad t \in (0,T),$$

$$\zeta(t) \in \gamma^* \partial J\big(\gamma \boldsymbol{u}_\omega(t)\big), \quad \text{a.e.} \quad t \in (0,T), \tag{3.17}$$

$$\boldsymbol{u}_\omega(0) = \boldsymbol{u}_0, \tag{3.18}$$

where

$$\boldsymbol{q}_\omega(t) = \boldsymbol{q}(t) - \omega(t). \tag{3.19}$$

Following [18, Theorem 5.15], we obtain that Problem 3.3 admits one and only one solution $\boldsymbol{u}_\omega \in \mathscr{X}$. For the second part of this problem, let $\boldsymbol{u}_{\omega_1}, \boldsymbol{u}_{\omega_2} \in \mathscr{X}$ be two solutions to Problem 3.2, then we have

$$\begin{aligned} &\left\langle \boldsymbol{u}'_{\omega_1}(t), \boldsymbol{v} \right\rangle_{V^*\times V} + \left\langle B\boldsymbol{u}_{\omega_1}(t), \boldsymbol{v} \right\rangle_{V^*\times V} + J^0\big(\gamma \boldsymbol{u}_{\omega_1}(t); \gamma \boldsymbol{v}\big) \\ &\quad + \left\langle \omega_1(t), \boldsymbol{v} \right\rangle_{V^*\times V} \geq \left\langle \boldsymbol{q}(t), \boldsymbol{v} \right\rangle_{V^*\times V}, \quad \forall \boldsymbol{v} \in V, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\left\langle \boldsymbol{u}'_{\omega_2}(t), \boldsymbol{v} \right\rangle_{V^*\times V} + \left\langle B\boldsymbol{u}_{\omega_2}(t), \boldsymbol{v} \right\rangle_{V^*\times V} + J^0\big(\gamma \boldsymbol{u}_{\omega_2}(t); \gamma \boldsymbol{v}\big) \\ &\quad + \left\langle \omega_2(t), \boldsymbol{v} \right\rangle_{V^*\times V} \geq \left\langle \boldsymbol{q}(t), \boldsymbol{v} \right\rangle_{V^*\times V}, \quad \forall \boldsymbol{v} \in V. \end{aligned} \tag{3.21}$$

Taking $\boldsymbol{v}=\boldsymbol{u}_{\omega_2}(t)-\boldsymbol{u}_{\omega_1}(t)$ in (3.20) and $\boldsymbol{v}=\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)$ in (3.21) gives

$$\begin{aligned}&\big\langle \boldsymbol{u}'_{\omega_1}(t),\boldsymbol{u}_{\omega_2}(t)-\boldsymbol{u}_{\omega_1}(t)\big\rangle_{V^*\times V}+\big\langle B\boldsymbol{u}_{\omega_1}(t),\boldsymbol{u}_{\omega_2}(t)-\boldsymbol{u}_{\omega_1}(t)\big\rangle_{V^*\times V}\\&\quad+J^0\big(\gamma\boldsymbol{u}_{\omega_1}(t);\gamma\boldsymbol{u}_{\omega_2}(t)-\gamma\boldsymbol{u}_{\omega_1}(t)\big)+\big\langle \omega_1(t),\boldsymbol{u}_{\omega_2}(t)-\boldsymbol{u}_{\omega_1}(t)\big\rangle_{V^*\times V}\\&\geq\big\langle \boldsymbol{q}(t),\boldsymbol{u}_{\omega_2}(t)-\boldsymbol{u}_{\omega_1}(t)\big\rangle_{V^*\times V},\end{aligned}\tag{3.22}$$

$$\begin{aligned}&\big\langle \boldsymbol{u}'_{\omega_2}(t),\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)\big\rangle_{V^*\times V}+\big\langle B\boldsymbol{u}_{\omega_2}(t),\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)\big\rangle_{V^*\times V}\\&\quad+J^0\big(\gamma\boldsymbol{u}_{\omega_2}(t);\gamma\boldsymbol{u}_{\omega_1}(t)-\gamma\boldsymbol{u}_{\omega_2}(t)\big)+\big\langle \omega_2(t),\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)\big\rangle_{V^*\times V}\\&\geq\big\langle \boldsymbol{q}(t),\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)\big\rangle_{V^*\times V}.\end{aligned}\tag{3.23}$$

Adding the above inequalities, we obtain

$$\begin{aligned}&\big\langle \boldsymbol{u}'_{\omega_1}(t)-\boldsymbol{u}'_{\omega_2}(t)+B\boldsymbol{u}_{\omega_1}(t)-B\boldsymbol{u}_{\omega_2}(t),\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)\big\rangle_{V^*\times V}\\&\leq J^0\big(\gamma\boldsymbol{u}_{\omega_1}(t);\gamma\boldsymbol{u}_{\omega_2}(t)-\gamma\boldsymbol{u}_{\omega_1}(t)\big)+J^0\big(\gamma\boldsymbol{u}_{\omega_2}(t);\gamma\boldsymbol{u}_{\omega_1}(t)-\gamma\boldsymbol{u}_{\omega_2}(t)\big)\\&\quad+\big\langle \omega_1(t)-\omega_2(t),\boldsymbol{u}_{\omega_2}(t)-\boldsymbol{u}_{\omega_1}(t)\big\rangle_{V^*\times V}.\end{aligned}\tag{3.24}$$

Integrating (3.24) gives

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)\|_H^2+m_B\int_0^t\|\boldsymbol{u}_{\omega_1}(\tau)-\boldsymbol{u}_{\omega_2}(\tau)\|_V^2\,d\tau\\&\quad-\frac{1}{2}\|\boldsymbol{u}_{\omega_1}(0)-\boldsymbol{u}_{\omega_2}(0)\|_H^2\\&\leq\int_0^t\big(m_J\|\gamma\|^2\|\boldsymbol{u}_{\omega_1}(\tau)-\boldsymbol{u}_{\omega_2}(\tau)\|_V^2\big)\,d\tau\\&\quad+\int_0^t\|\omega_1(\tau)-\omega_2(\tau)\|_{V^*}\|\boldsymbol{u}_{\omega_1}(\tau)-\boldsymbol{u}_{\omega_2}(\tau)\|_V\,d\tau.\end{aligned}\tag{3.25}$$

Since $\boldsymbol{u}_{\omega_1}(0)=\boldsymbol{u}_{\omega_2}(0)$, the Cauchy Schwarz inequality yields

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{u}_{\omega_1}(t)-\boldsymbol{u}_{\omega_2}(t)\|_H^2+\big(m_B-m_J\|\gamma\|^2-\epsilon\big)\int_0^t\|\boldsymbol{u}_{\omega_1}(\tau)-\boldsymbol{u}_{\omega_2}(\tau)\|_V^2\,d\tau\\&\leq c\int_0^t\|\omega_1(\tau)-\omega_2(\tau)\|_{V^*}^2\,d\tau.\end{aligned}$$

Finally, using the inequality $m_B>m_J\|\gamma\|^2$, we deduce that if $\epsilon>0$ is a sufficiently small, then

$$\int_0^t\|\boldsymbol{u}_{\omega_1}(\tau)-\boldsymbol{u}_{\omega_2}(\tau)\|_V^2\,d\tau\leq c\int_0^t\|\omega_1(\tau)-\omega_2(\tau)\|_{V^*}^2\,d\tau.\tag{3.26}$$

Step 2. Using the velocity field $\boldsymbol{u}_\omega$ obtained in Lemma 3.1 and focus on the following hemivariational inequality.

Problem 3.4. Find $\theta_\omega \in \mathbb{E}$ such that

$$\begin{aligned}
&\left\langle \theta'_\omega(t) + B_0\theta_\omega(t), \chi \right\rangle_{E^*\times E} + J^0_{temp}\big(\gamma_1\theta_\omega(t); \chi\big) \\
\geq &\left\langle g(t) + C_2\boldsymbol{u}_\omega(t), \chi \right\rangle_{E^*\times E}, \\
&\theta(0) = \theta_0.
\end{aligned}$$

The following lemma ensures that Problem 3.4 has a unique solution.

Lemma 3.2. *For $\omega \in \mathcal{V}^*$, Problem* 3.4 *has a unique solution $\theta_\omega \in \mathcal{E}$. In addition, if θ_{ω_i}, $i = 1,2$ are two solutions to Problem* 3.4 *corresponding to $\omega = \omega_i$, then there exists a constant k such that*

$$\|\theta_{\omega_1}(t) - \theta_{\omega_2}(t)\|_F^2 \leq k\int_0^t \|\boldsymbol{u}_{\omega_1}(\tau) - \boldsymbol{u}_{\omega_2}(\tau)\|_V^2\, d\tau. \tag{3.27}$$

For the first part of this lemma, we point out that $\theta_\omega \in \mathbb{E}$ is a solution to Problem 3.4 if and only if θ satisfies the evolution inclusion as follows.

Problem 3.5. Find $\theta_\omega \in \mathbb{E}$ and $\pi \in \mathcal{Y}^*$ such that

$$\begin{aligned}
&\theta'_\omega(t) + B_0\theta_\omega(t) + \pi(t) = C_2\boldsymbol{u}_\omega(t) + g(t), && \text{a.e.} \quad t \in (0,T), \\
&\pi(t) \in \gamma_1^*\partial J_{temp}\big(\gamma_1\theta_\omega(t)\big), && \text{a.e.} \quad t \in (0,T), \\
&\theta_\omega(0) = \theta_0.
\end{aligned} \tag{3.28}$$

According to [19], Problem 3.5 has at least one solution. We prove the uniqueness. If $\theta_{\omega,1}, \theta_{\omega,2} \in \mathcal{E}$ are solutions to Problem 3.5, then for a.e. $t \in (0,T)$ we have

$$\theta'_{\omega,1}(t) + B_0\theta_{\omega,1}(t) + \pi_1(t) = C_2\boldsymbol{u}_\omega(t) + g(t), \tag{3.29}$$

$$\theta'_{\omega,2}(t) + B_0\theta_{\omega,2}(t) + \pi_2(t) = C_2\boldsymbol{u}_\omega(t) + g(t), \tag{3.30}$$

$$\pi_1(t) \in \gamma_1^*\partial J_{temp}\big(\gamma_1\theta_{\omega,1}(t)\big), \quad \pi_2(t) \in \gamma_1^*\partial J_{temp}\big(\gamma_1\theta_{\omega,2}(t)\big), \tag{3.31}$$

$$\theta_{\omega,1}(0) = \theta_{\omega,2}(0) = \theta_0. \tag{3.32}$$

Subtracting (3.30) from (3.29), multiplying the resulting equation by $\theta_1(t) - \theta_2(t)$, and integrating the result by parts yield

$$\begin{aligned}
&\frac{1}{2}\|\theta_{\omega,1}(t) - \theta_{\omega,2}(t)\|_F^2 + m_{B_0}\int_0^t \|\theta_{\omega,1}(\tau) - \theta_{\omega,2}(\tau)\|_E^2\, d\tau \\
&\quad - \frac{1}{2}\|\theta_{\omega_1}(0) - \theta_{\omega_2}(0)\|_F^2 \\
\leq &\int_0^t \big(m_K\|\gamma_1\theta_{\omega,1}(\tau) - \gamma_1\theta_{\omega,2}(\tau)\|_E^2\big)\, d\tau.
\end{aligned} \tag{3.33}$$

Since $\theta_{\omega,1}(0) = \theta_{\omega,2}(0) = \theta_0$, we obtain

$$\frac{1}{2}\|\theta_{\omega,1}(t) - \theta_{\omega,2}(t)\|_F^2 + \big(m_{B_0} - m_K\|\gamma_1\|^2\big)\int_0^t \|\theta_{\omega,1}(\tau) - \theta_{\omega,2}(\tau)\|_E^2\, d\tau \leq 0.$$

Thus, $\theta_{\omega,1} = \theta_{\omega,2}$, so that Problem 3.5 admits a unique solution θ_ω, i.e. the existence and uniqueness result of the solution to Problem 3.4 is proven. It remains to show the following statement. If θ_{ω_i} is a unique solutions to Problem 3.4 corresponding to ω_i, $i = 1,2$, then

$$\begin{aligned}&\left\langle\theta'_{\omega_1}(t) + B_0\theta_{\omega_1}(t), \chi\right\rangle_{E^*\times E} + J^0_{temp}\big(\gamma_1\theta_{\omega_1}(t); \gamma_1\chi\big)\\ \geq &\left\langle C_2\boldsymbol{u}_{\omega_1}(t), \chi\right\rangle_{E^*\times E} + \left\langle g(t), \chi\right\rangle_{E^*\times E},\\ &\left\langle\theta'_{\omega_2}(t) + B_0\theta_{\omega_2}(t), \chi\right\rangle_{E^*\times E} + J^0_{temp}\big(\gamma_1\theta_{\omega_2}(t); \gamma_1\chi\big)\\ \geq &\left\langle C_2\boldsymbol{u}_{\omega_2}(t), \chi\right\rangle_{E^*\times E} + \left\langle g(t), \chi\right\rangle_{E^*\times E}.\end{aligned}$$

Take $\chi = \theta_{\omega_2}(t) - \theta_{\omega_1}(t)$ in the first inequality and $\chi = \theta_{\omega_1}(t) - \theta_{\omega_2}(t)$ in the second one to achieve

$$\begin{aligned}&\left\langle\theta'_{\omega_1}(t) - \theta'_{\omega_2}(t) + B_0\theta_{\omega_1}(t) - B_0\theta_{\omega_2}(t), \theta_{\omega_1}(t) - \theta_{\omega_2}(t)\right\rangle_{E^*\times E}\\ \leq\, &J^0_{temp}\big(\gamma_1\theta_{\omega_1}(t); \gamma_1\theta_{\omega_2}(t) - \gamma_1\theta_{\omega_1}(t)\big) + J^0_{temp}\big(\gamma_1\theta_{\omega_2}(t); \gamma_1\theta_{\omega_1}(t) - \gamma_1\theta_{\omega_2}(t)\big)\\ &+ \left\langle C_2\boldsymbol{u}_{\omega_1}(t) - C_2\boldsymbol{u}_{\omega_2}(t), \theta_{\omega_1}(t) - \theta_{\omega_2}(t)\right\rangle_{E^*\times E}.\end{aligned}$$

Integrating this inequality gives

$$\begin{aligned}&\frac{1}{2}\|\theta_{\omega_1}(t) - \theta_{\omega_2}(t)\|_F^2 + m_{B_0}\int_0^t \|\theta_{\omega_1}(\tau) - \theta_{\omega_2}(\tau)\|_E^2\, d\tau\\ &-\frac{1}{2}\|\theta_{\omega_1}(0) - \theta_{\omega_2}(0)\|_F^2\\ \leq\, &m_K\|\gamma_1\|^2\int_0^t \|\theta_{\omega_1}(\tau) - \theta_{\omega_2}(\tau)\|_E^2\, d\tau\\ &+\int_0^t \|C_2\boldsymbol{u}_{\omega_1}(\tau) - C_2\boldsymbol{u}_{\omega_2}(\tau)\|_{E^*}\|\theta_{\omega_1}(\tau) - \theta_{\omega_2}(\tau)\|_E\, d\tau.\end{aligned}$$

Applying the inequality $ab \leq \epsilon a^2 + c\,b^2$, under (3.13b) and the condition $\theta_{\omega_1}(0) = \theta_{\omega_2}(0)$, we obtain

$$\begin{aligned}&\frac{1}{2}\|\theta_{\omega_1}(t) - \theta_{\omega_2}(t)\|_F^2 + \big(m_{B_0} - m_K\|\gamma_1\|^2 - \epsilon L_R\big)\int_0^t \|\theta_{\omega_1}(\tau) - \theta_{\omega_2}(\tau)\|_E^2\, d\tau\\ \leq\, &cL_R\int_0^t \|\boldsymbol{u}_{\omega_1}(\tau) - \boldsymbol{u}_{\omega_2}(\tau)\|_V^2\, d\tau.\end{aligned}$$

Since $m_{B_0} - m_K\|\gamma_1\|^2 > 0$, for a sufficiently small constant $\epsilon > 0$ we obtain

$$\|\theta_{\omega_1}(t) - \theta_{\omega_2}(t)\|_F^2 \leq k\int_0^t \|\boldsymbol{u}_{\omega_1}(\tau) - \boldsymbol{u}_{\omega_2}(\tau)\|_V^2\, d\tau,$$

where $k = 2cL_R$.

Step 3. Towards the end of the proof, we write $\boldsymbol{u}_\omega$ and θ_ω to represent the solutions of Problems 3.2 and 3.4 and consider the operator Φ defined by

$$\Phi\omega = C_1\theta_\omega. \tag{3.34}$$

Lemma 3.3. *The operator Φ has a unique fixed point $\omega^* \in \mathscr{V}^*$.*

Proof. Let $\omega_1, \omega_2 \in \mathscr{V}^*$. Combining (3.16), (3.27), and (3.34) leads to the following inequality:

$$\begin{aligned}
&\|\Phi\omega_1(t) - \Phi\omega_2(t)\|_{V^*}^2 \\
&= \|C_1\theta_{\omega_1}(t) - C_1\theta_{\omega_2}(t)\|_{V^*}^2 \\
&\le L_1^2\|\theta_{\omega_1}(t) - \theta_{\omega_2}(t)\|_F^2 \\
&\le kL_1^2 \int_0^t \|\boldsymbol{u}_{\omega_1}(\tau) - \boldsymbol{u}_{\omega_2}(\tau)\|_V^2\, d\tau \\
&\le ckL_1^2 \int_0^t \|\omega_1(\tau) - \omega_2(\tau)\|_{V^*}^2\, d\tau
\end{aligned}$$

for all $t \in [0, T]$ with $ckL_1^2 > 0$. Using Lemma 2.2, we deduce that there exists a unique $\omega^* \in \mathscr{V}^*$ such that $\Phi\omega^* = \omega^*$, which fulfills the proof of the lemma. □

We are ready to prove the main result of this section.

Proof of Theorem 3.1. Take $\omega^* \in \mathscr{V}^*$ as the unique fixed point of the operator Φ, i.e.

$$\omega^*(t) = C_1\theta_{\omega^*}(t),$$

a.e. $t \in (0, T)$. Let $\boldsymbol{u}^* = \boldsymbol{u}_{\omega^*}$ be the unique solution of Problem 3.2 corresponding to ω^* established in Lemma 3.1. In addition, let $\theta^* = \theta_{\omega^*}$ be the unique solution of Problem 3.4 proved in Lemma 3.2. Therefore, $(\boldsymbol{u}^*, \theta^*)$ can be the unique solution to Problem 3.1. Moreover, we have $\boldsymbol{u}^* \in \mathscr{X}$ and $\theta^* \in \mathbb{E}$. The uniqueness of this theorem is a direct result of the uniqueness of the fixed point of Φ, Lemmata 3.1 and 3.2. The proof is finished. □

4. A Fully Discrete Approximation Problem

The main goal of this section is to construct a full-discrete scheme for the coupled system of hemivariational inequalities formed in Problem 3.1 and derive an error estimate result.

Let $h > 0$ be a spatial discretization parameter and V^h and E^h be finite dimensional subspaces of V and E, respectively. Here we adopt an equidistant time grid $t_n = nk, n = 0, 1, \ldots, N$, $N \in \mathbb{N}$, $k = T/N$. If $f = f(t)$ is a time continuous function, we write f_n for $f(t_n)$.

Let $\boldsymbol{u}_0^h, \theta_0^h$ be appropriate approximation of initial data $\boldsymbol{u}_0, \theta_0$, respectively. In what follows, we employ c for different values from time to time.

Problem 4.1. Find a discrete flow velocity $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_0^{hk}, \ldots, \boldsymbol{u}_N^{hk}\}$ and a discrete temperature $\{\theta^{hk} = \theta_0^{hk}, \ldots, \theta_N^{hk}\}$ such that for $1 \le n \le N$ the following equations hold:

$$\begin{aligned}
&\left(\frac{\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}}{k}, \boldsymbol{v}^h\right)_H + \left\langle B\boldsymbol{u}_n^{hk}, \boldsymbol{v}^h\right\rangle_{V^*\times V} + J^0\left(\gamma \boldsymbol{u}_n^{hk}; \gamma \boldsymbol{v}^h\right) \\
&\quad + \left\langle C_1\theta_n^{hk}, \boldsymbol{v}^h\right\rangle_{V^*\times V} \ge \left\langle \boldsymbol{q}_n, \boldsymbol{v}^h\right\rangle_{V^*\times V} \quad \text{for all} \quad \boldsymbol{v}^h \in V^h, \qquad (4.1)\\
&\left(\frac{\theta_n^{hk} - \theta_{n-1}^{hk}}{k}, \chi^h\right)_F + \left\langle B_0\theta_n^{hk}, \chi^h\right\rangle_{E^*\times E} + J^0_{temp}\left(\gamma_1\theta_n^{hk}; \gamma_1\chi^h\right) \\
&\ge \left\langle C_2\boldsymbol{u}_n^{hk}, \chi^h\right\rangle_{E^*\times E} + \left\langle g_n, \chi^h\right\rangle_{E^*\times E}, \qquad \forall \chi^h \in E^h, \qquad (4.2)\\
&\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \quad \theta_0^{hk} = \theta_0^h.
\end{aligned}$$

We now focus on estimating the residues $\delta_n = \boldsymbol{u}_n - \boldsymbol{u}_n^{hk}$ and $\boldsymbol{\varepsilon}_n = \theta_n - \theta_n^{hk}$ starting with δ_n. Take $\boldsymbol{v} = \boldsymbol{u}_n^{hk} - \boldsymbol{v}_n^h$ at $t = t_n$ in (3.7) and replace $\boldsymbol{v}^h$ by $\boldsymbol{v}_n^h - \boldsymbol{u}_n^{hk}$ in (4.1), so that

$$\begin{aligned}
&\left\langle \boldsymbol{u}_n', \boldsymbol{u}_n^{hk} - \boldsymbol{v}_n^h\right\rangle_{V^*\times V} + \left\langle B\boldsymbol{u}_n, \boldsymbol{u}_n^{hk} - \boldsymbol{v}_n^h\right\rangle_{V^*\times V} \\
&\quad + \left\langle C_1\theta_n, \boldsymbol{u}_n^{hk} - \boldsymbol{v}_n^h\right\rangle_{V^*\times V} + J^0\left(\gamma\boldsymbol{u}_n; \gamma\boldsymbol{u}_n^{hk} - \gamma\boldsymbol{v}_n^h\right) \\
&\ge \left\langle \boldsymbol{q}_n, \boldsymbol{u}_n^{hk} - \boldsymbol{v}_n^h\right\rangle_{V^*\times V}, \\
&\left(\frac{\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk}}{k}, \boldsymbol{v}_n^h - \boldsymbol{u}_n^{hk}\right)_H + \left\langle B\boldsymbol{u}_n^{hk}, \boldsymbol{v}_n^h - \boldsymbol{u}_n^{hk}\right\rangle_{V^*\times V} \\
&\quad + \left\langle C_1\theta_n^{hk}, \boldsymbol{v}_n^h - \boldsymbol{u}_n^{hk}\right\rangle_{V^*\times V} + J^0\left(\gamma\boldsymbol{u}_n^{hk}; \gamma\boldsymbol{v}_n^h - \gamma\boldsymbol{u}_n^{hk}\right) \\
&\ge \left\langle \boldsymbol{q}_n, \boldsymbol{v}_n^h - \boldsymbol{u}_n^{hk}\right\rangle_{V^*\times V}.
\end{aligned}$$

Adding these inequalities to each other and using (3.9c) yields

$$\begin{aligned}
m_B\|\delta_n\|_V^2 &\le \left\langle B\boldsymbol{u}_n - B\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right\rangle_{V^*\times V} \\
&\le \left\langle B\boldsymbol{u}_n - B\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n - \boldsymbol{v}_n^h\right\rangle_{V^*\times V} + \left(\boldsymbol{u}_n' - \frac{\boldsymbol{u}_n - \boldsymbol{u}_{n-1}}{k}, \boldsymbol{u}_n^{hk} - \boldsymbol{v}_n^h\right)_H \\
&\quad - \frac{1}{k}(\delta_n - \delta_{n-1}, \delta_n)_H + \frac{1}{k}\left(\delta_n - \delta_{n-1}, \boldsymbol{u}_n - \boldsymbol{v}_n^h\right)_H + I_1 + I_2, \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= J^0\left(\gamma\boldsymbol{u}_n; \gamma\boldsymbol{u}_n^{hk} - \gamma\boldsymbol{v}_n^h\right) + J^0\left(\gamma\boldsymbol{u}_n^{hk}; \gamma\boldsymbol{v}_n^h - \gamma\boldsymbol{u}_n^{hk}\right), \\
I_2 &= \left\langle C_1\theta_n - C_1\theta_n^{hk}, \boldsymbol{u}_n^{hk} - \boldsymbol{v}_n^h\right\rangle_{V^*\times V}.
\end{aligned}$$

We first evaluate the term I_1. The subadditivity of generalized directional derivative (3.11) gives

$$J^0\left(\gamma\boldsymbol{u}_n; \gamma\boldsymbol{u}_n^{hk} - \gamma\boldsymbol{v}_n^h\right) + J^0\left(\gamma\boldsymbol{u}_n^{hk}; \gamma\boldsymbol{v}_n^h - \gamma\boldsymbol{u}_n^{hk}\right)$$

$$\begin{aligned}
&\le J^0\left(\gamma \boldsymbol{u}_n;\gamma \boldsymbol{u}_n^{hk}-\gamma \boldsymbol{u}_n\right)+J^0\left(\gamma \boldsymbol{u}_n;\gamma \boldsymbol{u}_n-\gamma \boldsymbol{v}_n^h\right)\\
&\quad +J^0\left(\gamma \boldsymbol{u}_n^{hk};\gamma \boldsymbol{v}_n^h-\gamma \boldsymbol{u}_n\right)+J^0\left(\gamma \boldsymbol{u}_n^{hk};\gamma \boldsymbol{u}_n-\gamma \boldsymbol{u}_n^{hk}\right)\\
&\le m_J\|\gamma\|^2\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+c\,\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\|_W. \qquad (4.4)
\end{aligned}$$

Now we consider the remaining terms on the right side of (4.3). It is easily seen that

$$\begin{aligned}
(\delta_n-\delta_{n-1},\delta_n)_H &= \frac{1}{2}\left(\|\delta_n\|_H^2-\|\delta_{n-1}\|_H^2+\|\delta_n-\delta_{n-1}\|_H^2\right)\\
&\ge \frac{1}{2}\left(\|\delta_n\|_H^2-\|\delta_{n-1}\|_H^2\right),
\end{aligned}$$

so that

$$-\frac{1}{k}(\delta_n-\delta_{n-1},\delta_n)_H \le -\frac{1}{2k}\left(\|\delta_n\|_H^2-\|\delta_{n-1}\|_H^2\right). \qquad (4.5)$$

Set

$$\boldsymbol{E}_n := \boldsymbol{u}_n'-\frac{\boldsymbol{u}_n-\boldsymbol{u}_{n-1}}{k}$$

and note that

$$\left\langle \boldsymbol{E}_n,\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\right\rangle_{V^*\times V}\le \|\boldsymbol{E}_n\|_{V^*}\left\|\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\right\|_V.$$

Applying the elementary inequality

$$ab\le \epsilon a^2+c\,b^2 \qquad (4.6)$$

with an $\epsilon>0$ leads to the estimate

$$\left\langle \boldsymbol{E}_n,\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\right\rangle_{V^*\times V}\le \epsilon\|\delta_n\|_V^2+c\,\|\boldsymbol{E}_n\|_{V^*}^2+c\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_V^2. \qquad (4.7)$$

It follows from (3.9b) and (4.6) that

$$\left\langle B\boldsymbol{u}_n-B\boldsymbol{u}_n^{hk},\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\rangle_{V^*\times V}\le \epsilon\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V^2+c\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_V^2. \qquad (4.8)$$

Finally, we have

$$\begin{aligned}
&\left\langle C_1\theta_n-C_1\theta_n^{hk},\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\right\rangle_{V^*\times V}\\
&\le c\left\|\theta_n-\theta_n^{hk}\right\|_E\left\|\boldsymbol{u}_n^{hk}-\boldsymbol{v}_n^h\right\|_V\\
&\le \epsilon\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V^2+c\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_V^2+c\left\|\theta_n-\theta_n^{hk}\right\|_E^2. \qquad (4.9)
\end{aligned}$$

Using (4.4)-(4.9) in (4.3) and taking into account the assumption $m_B>m_J\|\gamma\|^2$ yields

$$\begin{aligned}
&k\|\delta_n\|_V^2+\|\delta_n\|_H^2-\|\delta_{n-1}\|_H^2\\
&\le c\,k\left(\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_V^2+\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_W+\|E_n\|_{V^*}^2\right)\\
&\qquad +c\,k\left\|\theta_n-\theta_n^{hk}\right\|_E^2+\left(\delta_n-\delta_{n-1},\boldsymbol{u}_n-\boldsymbol{v}_n^h\right)_H. \qquad (4.10)
\end{aligned}$$

Replacing n by l in (4.10) and summing the corresponding estimates in l gives

$$
\begin{aligned}
& k\sum_{l=1}^{n}\|\delta_l\|_V^2+\|\delta_n\|_H^2 \\
&\le \|\delta_0\|_H^2+c\,k\sum_{l=1}^{n}\Big(\big\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\big\|_V^2+\big\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\big\|_W+\|E_l\|_{V^*}^2\Big) \\
&\quad + c\,k\sum_{l=1}^{n}\big\|\theta_l-\theta_l^{hk}\big\|_E^2+\sum_{l=1}^{n}\big(\delta_l-\delta_{l-1},\boldsymbol{u}_l-\boldsymbol{v}_l^h\big)_H .
\end{aligned}
$$

We now estimate $\boldsymbol{\varepsilon}_n$. Taking $\chi=\theta_n^{hk}-\chi_n^h$ at $t=t_n$ in (3.8) and replacing χ^h by $\chi_n^h-\theta_n^{hk}$ in (4.2), we obtain

$$
\begin{aligned}
& \big\langle\theta_n',\theta_n^{hk}-\chi_n^h\big\rangle_{E^*\times E}+\big\langle B_0\theta_n,\theta_n^{hk}-\chi_n^h\big\rangle_{E^*\times E} \\
&\quad + J_{temp}^0\big(\gamma_1\theta_n;\gamma_1\theta_n^{hk}-\gamma_1\chi_n^h\big) \\
&\ge \big\langle C_2\boldsymbol{u}_n,\theta_n^{hk}-\chi_n^h\big\rangle_{E^*\times E}+\big\langle g_n,\theta_n^{hk}-\chi_n^h\big\rangle_{E^*\times E}, \\
& \left(\frac{\theta_n^{hk}-\theta_{n-1}^{hk}}{k},\chi_n^h-\theta_n^{hk}\right)_F+\big\langle B_0\theta_n^{hk},\chi_n^h-\theta_n^{hk}\big\rangle_{E^*\times E} \\
&\quad + J_{temp}^0\big(\gamma_1\theta_n^{hk};\gamma_1\chi_n^h-\gamma_1\theta_n^{hk}\big) \\
&\ge \big\langle C_2\boldsymbol{u}_n^{hk},\chi_n^h-\theta_n^{hk}\big\rangle_{E^*\times E}+\big\langle g_n,\chi_n^h-\theta_n^{hk}\big\rangle_{E^*\times E}.
\end{aligned}
$$

Adding these inequalities to each other and using (3.10c) implies

$$
\begin{aligned}
m_{B_0}\|\boldsymbol{\varepsilon}_n\|_E^2 &\le \big\langle B_0\theta_n-B_0\theta_n^{hk},\theta_n-\theta_n^{hk}\big\rangle_{E^*\times E} \\
&\le \big\langle B_0\theta_n-B_0\theta_n^{hk},\theta_n-\chi_n^h\big\rangle_{E^*\times E}+\left(\theta_n'-\frac{\theta_n-\theta_{n-1}}{k},\theta_n^{hk}-\chi_n^h\right)_F \\
&\quad -\frac{1}{k}(\boldsymbol{\varepsilon}_n-\boldsymbol{\varepsilon}_{n-1},\boldsymbol{\varepsilon}_n)_F+\frac{1}{k}\big(\boldsymbol{\varepsilon}_n-\boldsymbol{\varepsilon}_{n-1},\theta_n-\chi_n^h\big)_F+I_3+I_4, \qquad (4.11)
\end{aligned}
$$

where

$$
\begin{aligned}
I_3 &= \big\langle C_2\boldsymbol{u}_n-C_2\boldsymbol{u}_n^{hk},\theta_n-\theta_n^{hk}\big\rangle_{E^*\times E}, \\
I_4 &= J_{temp}^0\big(\gamma_1\theta_n;\gamma_1\theta_n^{hk}-\gamma_1\chi_n^h\big)+J_{temp}^0\big(\gamma_1\theta_n^{hk};\gamma_1\chi_n^h-\gamma_1\theta_n^{hk}\big).
\end{aligned}
$$

Using again the subadditivity of the generalized directional derivative (3.12) shows that

$$
\begin{aligned}
& J_{temp}^0\big(\gamma_1\theta_n;\gamma_1\theta_n^{hk}-\gamma_1\chi_n^h\big)+J_{temp}^0\big(\gamma_1\theta_n^{hk};\gamma_1\chi_n^h-\gamma_1\theta_n^{hk}\big) \\
&\le J_{temp}^0\big(\gamma_1\theta_n;\gamma_1\theta_n^{hk}-\gamma_1\theta_n\big)+J_{temp}^0\big(\gamma_1\theta_n;\gamma_1\theta_n-\gamma_1\chi_n^h\big) \\
&\quad + J_{temp}^0\big(\gamma_1\theta_n^{hk};\gamma_1\chi_n^h-\gamma_1\theta_n\big)+J_{temp}^0\big(\gamma_1\theta_n^{hk};\gamma_1\theta_n-\gamma_1\theta_n^{hk}\big) \\
&\le m_K\|\gamma_1\|^2\big\|\theta_n-\theta_n^{hk}\big\|_E^2+c\,\big\|\theta_n-\chi_n^h\big\|_Y.
\end{aligned}
$$

We estimate the remaining terms similar to the evaluation of δ_n, thus obtaining

$$\begin{aligned}
&k\sum_{l=1}^{n}\|\boldsymbol{\varepsilon}_l\|_E^2+\|\boldsymbol{\varepsilon}_n\|_F^2\\
&\le\|\boldsymbol{\varepsilon}_0\|_F^2+c\,k\sum_{l=1}^{n}\Big(\big\|\theta_l-\chi_l^h\big\|_E^2+\big\|\theta_l-\chi_l^h\big\|_Y+\|G_l\|_{E^*}^2\Big)\\
&\quad+c\,k\sum_{l=1}^{n}\big\|\boldsymbol{u}_l-\boldsymbol{u}_l^{hk}\big\|_V^2+c\sum_{l=1}^{n}\big(\boldsymbol{\varepsilon}_l-\boldsymbol{\varepsilon}_{l-1},\theta_l-\chi_l^h\big)_F\,,
\end{aligned}$$

where $G_l=\theta_l'-(\theta_l-\theta_{l-1})/k$.

Next we consider the term $\sum\limits_{l=1}^{n}(\delta_l-\delta_{l-1},\boldsymbol{u}_l-\boldsymbol{v}_l^h)_H$. Following the approach of [12], we obtain

$$\begin{aligned}
&\sum_{l=1}^{n}\big(\delta_l-\delta_{l-1},\boldsymbol{u}_l-\boldsymbol{v}_l^h\big)_H\\
&\le\frac{1}{2}\left(\|\delta_n\|_H^2+\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\|_H^2\right)+\frac{k}{2}\sum_{l=1}^{n-1}\Big(\|\delta_l\|_H^2+k^{-2}\left\|\big(\boldsymbol{u}_l-\boldsymbol{v}_l^h\big)-\big(\boldsymbol{u}_{l+1}-\boldsymbol{v}_{l+1}^h\big)\right\|_H^2\Big)\\
&\quad+\frac{1}{2}\left(\|\delta_0\|_H^2+\left\|\boldsymbol{u}_1-\boldsymbol{v}_1^h\right\|_H^2\right).
\end{aligned}$$

On the other hand, since

$$\boldsymbol{E}_l=\frac{1}{k}\int_{t_{l-1}}^{t_l}(t-t_{l-1})\boldsymbol{u}''(t)\,dt,$$

we have

$$\begin{aligned}
\|\boldsymbol{E}_l\|_{V^*}^2&\le\frac{1}{k^2}\int_{t_{l-1}}^{t_l}(t-t_{l-1})^2\,dt\int_{t_{l-1}}^{t_l}\|\boldsymbol{u}''(t)\|_{V^*}^2\,dt\\
&=\frac{k}{3}\int_{t_{l-1}}^{t_l}\|\boldsymbol{u}''(t)\|_{V^*}^2\,dt.
\end{aligned}$$

Consequently,

$$k\sum_{l=1}^{n}\|\boldsymbol{E}_l\|_{V^*}^2\le\frac{k^2}{3}\|\boldsymbol{u}''\|_{L^2(0,T;V^*)}^2.$$

Similarly, for $\sum_{l=1}^{n}(\boldsymbol{\varepsilon}_l-\boldsymbol{\varepsilon}_{l-1},\theta_l-\chi_l^h)_F$, we get

$$\begin{aligned}
&\sum_{l=1}^{n}\big(\boldsymbol{\varepsilon}_l-\boldsymbol{\varepsilon}_{l-1},\theta_l-\chi_l^h\big)_F\\
&\le\frac{1}{2}\left(\|\boldsymbol{\varepsilon}_n\|_F^2+\left\|\theta_n-\chi_n^h\right\|_F^2\right)+\frac{k}{2}\sum_{l=1}^{n-1}\Big(\|\boldsymbol{\varepsilon}_l\|_F^2+k^{-2}\left\|\big(\theta_l-\chi_l^h\big)-\big(\theta_{l+1}-\chi_{l+1}^h\big)\right\|_F^2\Big)\\
&\quad+\frac{1}{2}\left(\|\boldsymbol{\varepsilon}_0\|_F^2+\left\|\theta_1-\chi_1^h\right\|_F^2\right),
\end{aligned}$$

which yields

$$k\sum_{l=1}^{n}\|G_l\|_{E^*}^2 \le \frac{k^2}{3}\|\theta''\|_{L^2(0,T;E^*)}^2.$$

Finally, we have the following inequality:

$$\begin{aligned}
&\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_H^2+k\sum_{l=1}^{n}\left\|\boldsymbol{u}_l-\boldsymbol{u}_l^{hk}\right\|_V^2+\left\|\theta_n-\theta_n^{hk}\right\|_F^2+k\sum_{l=1}^{n}\left\|\theta_l-\theta_l^{hk}\right\|_E^2\\
&\le c\left\{k\sum_{l=1}^{n}\left(\left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_V^2+\left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_W\right)+k^2\|\theta\|_{H^2(0,T;E^*)}^2+k^2\|\boldsymbol{u}\|_{H^2(0,T;V^*)}^2\right.\\
&\quad+\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_H^2+k\sum_{l=1}^{n}\left(\left\|\theta_l-\chi_l^h\right\|_E^2+\left\|\theta_l-\chi_l^h\right\|_Y\right)\\
&\quad+\|\boldsymbol{u}_1-\boldsymbol{v}_1\|_H^2+\|\varepsilon_0\|_F^2+\|\theta_1-\chi_1\|_F^2+\|\theta_n-\chi_n^h\|_F^2+\|\delta_0\|_H^2\\
&\quad+k^{-1}\sum_{l=1}^{n-1}\left\|\left(\boldsymbol{u}_l-\boldsymbol{v}_l^h\right)-\left(\boldsymbol{u}_{l+1}-\boldsymbol{v}_{l+1}^h\right)\right\|_H^2\\
&\quad+k^{-1}\sum_{l=1}^{n-1}\left\|\left(\theta_l-\chi_l^h\right)-\left(\theta_{l+1}-\chi_{l+1}^h\right)\right\|_F^2\\
&\quad\left.+k\sum_{l=1}^{n}\left(\|\delta_l\|_H^2+k\sum_{j=1}^{l}\left\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\right\|_V^2\right)+k\sum_{l=1}^{n}\left(\|\boldsymbol{\varepsilon}_l\|_F^2+k\sum_{j=1}^{l}\left\|\theta_j-\theta_j^{hk}\right\|_E^2\right)\right\}.
\end{aligned}$$

Using the Gronwall inequality gives

$$\begin{aligned}
&\max_{1\le n\le N}\|\delta_n\|_H^2+k\sum_{n=1}^{N}\|\delta_n\|_V^2+\max_{1\le n\le N}\|\boldsymbol{\varepsilon}_n\|_F^2+k\sum_{n=1}^{N}\|\boldsymbol{\varepsilon}_n\|_E^2\\
&\le c\,k^2\left(\|\boldsymbol{u}\|_{H^2(0,T;V^*)}^2+\|\theta\|_{H^2(0,T;E^*)}^2\right)+c\left(\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_H^2+\left\|\theta_0-\theta_0^h\right\|_F^2\right)+c\max_{1\le n\le N}\overline{R}_n,
\end{aligned}$$

where

$$\begin{aligned}
\overline{R}_n=\inf_{\boldsymbol{v}_l^h\in V^h,\,\chi_l^h\in E^h}&\left\{k\sum_{l=1}^{n}\left(\left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_V^2+\left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_W+\left\|\theta_l-\chi_l^h\right\|_E^2+\left\|\theta_l-\chi_l^h\right\|_Y\right)\right.\\
&\quad+\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_H^2+k^{-1}\sum_{l=1}^{n-1}\left\|\left(\boldsymbol{u}_l-\boldsymbol{v}_l^h\right)-\left(\boldsymbol{u}_{l+1}-\boldsymbol{v}_{l+1}^h\right)\right\|_H^2\\
&\quad\left.+\left\|\theta_n-\chi_n^h\right\|_F^2+k^{-1}\sum_{l=1}^{n-1}\left\|\left(\theta_l-\chi_l^h\right)-\left(\theta_{l+1}-\chi_{l+1}^h\right)\right\|_F^2\right\}.
\end{aligned}$$

Taking into account the above results, we arrive at the following theorem.

Theorem 4.1. *Let* $(\boldsymbol{u},\theta)$ *and* $(\boldsymbol{u}^{hk},\theta^{hk})$ *be the solutions to Problem* 3.1 *and Problem* 4.1*, respectively. If* $\boldsymbol{q}\in\mathcal{V}^*$ *and* (2.2)*,* (3.9)*-*(3.14)*,* $m_B > m_J\|\gamma\|^2$*,* $m_{B_0} > m_K\|\gamma_1\|^2$ *hold, then*

$$\begin{aligned}&\max_{1\le n\le N}\|\delta_n\|_H^2 + k\sum_{n=1}^N\|\delta_n\|_V^2 + \max_{1\le n\le N}\|\boldsymbol{\varepsilon}_n\|_F^2 + k\sum_{n=1}^N\|\boldsymbol{\varepsilon}_n\|_E^2 \qquad (4.12)\\ &\le c\,k^2\left(\|\boldsymbol{u}\|^2_{H^2(0,T;V^*)} + \|\theta\|^2_{H^2(0,T;E^*)}\right) + c\left(\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_H^2 + \left\|\theta_0-\theta_0^h\right\|_F^2\right) + c\max_{1\le n\le N}\overline{R}_n,\end{aligned}$$

where

$$\begin{aligned}\overline{R}_n = \inf_{\boldsymbol{v}_l^h\in V^h,\,\chi_l^h\in E^h}&\left\{k\sum_{l=1}^n\left(\left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_V^2 + \left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_W + \left\|\theta_l-\chi_l^h\right\|_E^2 + \left\|\theta_l-\chi_l^h\right\|_Y\right)\right.\\ &+\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_H^2 + k^{-1}\sum_{l=1}^{n-1}\left\|\left(\boldsymbol{u}_l-\boldsymbol{v}_l^h\right)-\left(\boldsymbol{u}_{l+1}-\boldsymbol{v}_{l+1}^h\right)\right\|_H^2\\ &\left.+\left\|\theta_n-\chi_n^h\right\|_F^2 + k^{-1}\sum_{l=1}^{n-1}\left\|\left(\theta_l-\chi_l^h\right)-\left(\theta_{l+1}-\chi_{l+1}^h\right)\right\|_F^2\right\}.\end{aligned}$$

Under Theorem 4.1, we analyze the error estimate. Let Θ be a polygonal or polyhedral domain and ϑ^h a regular triangulation of Θ consistent with the partition of the boundary $\Gamma=\partial\Theta$ into Γ_1 and Γ_2. For $T\in\vartheta^h$, denote by $P_1(T;\mathbb{R}^d)$ a polynomials space with a total degree at most one in T. Now we can use the linear element space of piecewise continuous affine functions

$$V^h := \left\{\boldsymbol{v}^h\in V\cap C(\overline{\Theta};\mathbb{R}^d):\ \boldsymbol{v}^h|_T\in P_1(T;\mathbb{R}^d)\quad\text{for all}\quad T\in\vartheta^h\right\}, \qquad (4.13)$$

$$E^h := \left\{\chi^h\in E\cap C(\overline{\Theta}):\ \chi^h|_T\in P_1(T)\quad\text{for all}\quad T\in\vartheta^h\right\}. \qquad (4.14)$$

Corollary 4.1. *Under the assumptions of Theorem* 4.1*, let* $\{V^h\}$*,* $\{E^h\}$ *be respectively the family of linear element spaces* (4.13) *and* (4.14) *of continuous and piecewise affine functions. If* $\boldsymbol{u}\in C([0,T];H^2(\Theta;\mathbb{R}^d))$*,* $\boldsymbol{u}_{|_{\Gamma_2}}\in C([0,T];H^2(\Gamma_2;\mathbb{R}^d))$*,* $\theta\in C([0,T];H^2(\Theta))$*, then the following error estimate holds:*

$$\max_{1\le n\le N}\|\delta_n\|_H^2 + k\sum_{n=1}^N\|\delta_n\|_V^2 + \max_{1\le n\le N}\|\varepsilon_n\|_F^2 + k\sum_{n=1}^N\|\varepsilon_n\|_E^2 \le c\left(k^2+h^2\right). \qquad (4.15)$$

Proof. The standard finite element interpolation error estimates [1,3,7] will be applied. Using $\boldsymbol{v}_l^h\in V^h$ as the finite element interpolation of $\boldsymbol{u}_l$ yields

$$\left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_V \le ch\|\boldsymbol{u}_l\|_{H^2(\Theta;\mathbb{R}^d)}. \qquad (4.16)$$

Take $\chi_l^h\in E^h$ as the finite element interpolation of θ_l. It can be deduced that

$$\left\|\theta_l-\chi_l^h\right\|_E \le ch\|\theta_l\|_{H^2(\Theta)}. \qquad (4.17)$$

Now that $\boldsymbol{v}_l^h$ interpolates $\boldsymbol{v}_l$ on Γ_2. According to [25], we have

$$\left\|\boldsymbol{u}_l-\boldsymbol{v}_l^h\right\|_{L^2(\Gamma_2;\mathbb{R}^d)}\le ch^2\|\boldsymbol{u}_l\|_{H^2(\Gamma_2;\mathbb{R}^d)}. \tag{4.18}$$

Since $(\boldsymbol{v}_l^h-\boldsymbol{v}_{l+1}^h)$ is the finite element interpolation of $(\boldsymbol{u}_l-\boldsymbol{u}_{l+1})$, we have

$$\left\|\left(\boldsymbol{u}_l-\boldsymbol{v}_l^h\right)-\left(\boldsymbol{u}_{l+1}-\boldsymbol{v}_{l+1}^h\right)\right\|_H^2\le ch^2\|\boldsymbol{u}_l-\boldsymbol{u}_{l+1}\|_V^2\le ch^2k\int_{t_l}^{t_{l+1}}\|\boldsymbol{u}'(t)\|_V^2\,dt.$$

Consequently,

$$k^{-1}\sum_{l=1}^{n-1}\left\|\left(\boldsymbol{u}_l-\boldsymbol{v}_l^h\right)-\left(\boldsymbol{u}_{l+1}-\boldsymbol{v}_{l+1}^h\right)\right\|_H^2\le ch^2\|\boldsymbol{u}\|^2_{H^1(0,T;H^2(\Theta;\mathbb{R}^d))},\quad 1\le n\le N. \tag{4.19}$$

Similar considerations show that

$$k^{-1}\sum_{l=1}^{n-1}\left\|\left(\theta_l-\chi_l^h\right)-\left(\theta_{l+1}-\chi_{l+1}^h\right)\right\|_F^2\le ch^2\|\theta\|^2_{H^1(0,T;H^2(\Theta))},\quad 1\le n\le N. \tag{4.20}$$

Since $\boldsymbol{u}_0^h$ and θ_0^h are the finite element interpolations of $\boldsymbol{u}_0$ and θ_0, we note that

$$\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V+\left\|\theta_0-\theta_0^h\right\|_F\le ch. \tag{4.21}$$

Finally, we have

$$\max_{1\le n\le N}\left\|\boldsymbol{u}_n-\boldsymbol{v}_n^h\right\|_H\le ch^2\|\boldsymbol{u}\|_{C([0,T];H^2(\Theta;\mathbb{R}^d))}, \tag{4.22}$$

$$\max_{1\le n\le N}\left\|\theta_n-\chi_n^h\right\|_F\le ch^2\|\theta\|_{C([0,T];H^2(\Theta))}. \tag{4.23}$$

Combining (4.16)-(4.23) and (4.12), we arrive at the estimate (4.15). □

References

[1] K.E. Atkinson and W. Han, *Therotical Numerical Analysis: A Functional Analysis Framework*, Springer-Verlag (2008).

[2] M. Beneš and P. Kučera, *On the Navier-Stokes flows for heat-conducting fluids with mixed boundary conditions*, J. Math. Anal. Appl. **389**, 769–780 (2012).

[3] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Elements Methods*, Springer-Verlag (2008).

[4] A. Capatina, *Variational Inequalities Frictional Contact Problems, Advances in Mechanics and Mathematics*, Springer (2014).

[5] X. Cheng, Q. Xiao, S. Migórski and A. Ochal, *Error estimate for quasistatic history-dependent contact model*, Comput. Math. Appl. **77**, 2943–2952 (2019).

[6] X. Cheng, H. Xuan and Q. Xiao, *Analysis of Rothe method for a variational-hemivariational inequality in adhesive contact problem for locking materials*, Int. J. Numer. Anal. Model. **18**, 287–310 (2021).

[7] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland (1978).

[8] C. Fang, K. Czuprynski, W. Han, X. Cheng and X. Dai, *Finite element method for a stationary*

Stokes hemivariational inequality with slip boundary condition, IMAJ. Numer. Anal. **40**, 2696–2716 (2020).

[9] C. Fang, W. Han, S. Migórski and M. Sofonea, *A class of hemivariational inequalities for nonstationary Navier-Stokes equations*, Nonlinear Anal. Real World Appl. **31**, 257–276 (2016).

[10] D. Han and W. Han, *Numerical analysis of an evolutionary variational-hemivariational inequality with application to a dynamic contact problem*, J. Comput. Appl. Math. **358**, 163–178 (2019).

[11] W. Han, K. Czuprynski and F. Jing, *Mixed finite element method for a hemivariational inequality of stationary Navier-Stokes equations*, J. Sci. Comput. **89**, 1–22 (2021).

[12] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, AMS/IP (2002).

[13] A. Klarbring, A. Mikelic and M. Shillor, *Frictional contact problems with normal compliance*, Internat. J. Engrg. Sci. **26**, 811–832 (1988).

[14] M. Ling, F. Wang and W. Han, *The nonconforming virtual element method for a stationary Stokes hemivariational inequality with slip boundary condition*, J. Sci. Comput. **85**, 56–85 (2020).

[15] G. Lukaszewicz, *On the stationary flows of viscous, incompressible and heat-conducting fluids*, Math. Methods Appl. Sci. **10**, 329–337 (1988).

[16] T. Masahisa and T. Daisuke, *Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients*, Numer. Math. **100**, 351–372 (2005).

[17] S. Migórski and S. Dudek, *Steady flow with unilateral and leak/slip boundary conditions by the Stokes variational-hemivariational inequality*, Appl. Anal. 1–17 (2020).

[18] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities, Models and Analysis of Contact Problems*, Springer (2013).

[19] S. Migórski and P. Szafraniec, *A class of dynamic frictional contact problems governed by a system of hemivariational inequalities in thermoviscoelasticity*, Nonlinear Anal. Real World Appl. **15**, 158–171 (2014).

[20] J. Naumann, *An Existence Theorem for Weak Solutions to The Equations of Non-Stationary Motion of Heat-Conducting Incompressible Viscous Fluids*, Birkhuser (2005).

[21] C. Pérez, J. Thomas, S. Blancher and R. Creff, *The steady Navier-Stokes/energy system with temperature-dependent viscosity-Part 1: Analysis of the continuous problem*, Internat. J. Numer. Methods Fluids **56**, 63–89 (2008).

[22] M. Sofonea and S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Chapman & Hall (Baca Raton), CRC Press (2018).

[23] F. Wang, M. Ling, W. Han and F. Jing, *Adaptive discontinuous Galerkin methods for solving an incompressible Stokes flow problem with slip boundary condition of frictional type*, J. Comput. Appl. Math. **371**, 112270 (2020).

[24] H. Xuan and X. Cheng, *Numerical analysis and simulation of an adhesive contact problem with damage and long memory*, Discrete Contin. Dyn. Syst. Ser. B. **26**, 2781–2804 (2021).

[25] H. Xuan, X. Cheng, W. Han and Q. Xiao, *Numerical analysis of a dynamic contact problem with history-dependent operators*, Numer. Math. Theor. Meth. Appl. **13**, 569–594 (2020).

[26] W. Xu, Z. Huang, W. Han, W. Chen and C. Wang, *Numerical analysis of history-dependent hemivariational inequalities and applications to viscoelastic contact problems with normal penetration*, Int. J. Numer. Anal. Model. **77**, 2596–2607 (2019).

[27] W. Yan and Z. Gao, *Shape optimization in the Navier-Stokes flow with thermal effects*, Numer. Methods Partial Differential Equations. **30**, 1700–1715 (2014).

[28] J. Zhao, S. Migórski and S. Dudek, *Analysis of Stokes system with solution-dependent subdifferential boundary conditions*, Fixed Point Theory Algorithms Sci. Eng. **2021**, 1–18 (2021).