East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-093.130723

Vol. **14**, No. 1, pp. 195-222
February 2024

Iterative Method with Inertia for Variational Inequalities on Hadamard Manifolds with Lower Bounded Curvature

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Received 24 March 2023; Accepted (in revised version) 13 July 2023.

Abstract. In this paper, we are concerned with solving variational inequalities on Hadamard manifolds, the curvature of which is bounded from below. The underlying vector field is assumed to be continuous and pseudomonotone. By combining the hyperplane projection method and the inertial extrapolation technique, a Halpern-type method is proposed. Under some mild assumptions, global convergence of the proposed algorithm is established. Numerical experiments are reported to show the efficiency of the proposed algorithm.

AMS subject classifications: 47J20, 65K05, 90C33

Key words: Variational inequality, pseudomonotone vector field, Hadamard manifold, hyperplane projection method, inertial term.

1. Introduction

In the past few decades, various optimization problems on Riemannian manifolds have been studied [1, 3, 7, 11, 12, 14, 16, 18, 21, 22, 26, 30–33, 35–37, 41–43]. Among these research problems, variational inequality problems on Riemannian manifolds have received much attention. Specially, a complete and simply connected Riemannian manifold with nonpositive curvature is called the Hadamard manifold. Variational inequality problems for univalued vector fields on Hadamard manifolds was first introduced and investigated by Németh [22]. The existence and uniqueness results were established. Li *et al.* [19] introduced variational inequality problems for univalued vector fields on general Riemannian manifolds, the existence results of solution for problems defined on locally convex subsets

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with weak pole interior points were established. For set-valued vector fields on general Riemannian manifolds, the existence of the solutions and the convexity of the solution set of related variational inequality problems are investigated by Li *et al.* [17]. In addition, nonsmooth variational inequalities on Hadamard manifolds were also studied by Ansaria *et al*. [4]. Some existence results in terms of a bifunction in the setting of Hadamard manifolds were given.

By the Cartan-Hadamard theorem, any Hadamard manifold is diffeomorphic to a Euclidean space. Thus variational inequality problems on a Hadamard manifold can be reformulated as some equivalent problem in a Euclidean space by using the underlying diffeomorphic map. However, the monotonicity/pseudomonotonicity of the underlying vector field may be destroyed. Conversely, by endowing some appropriate Riemannian metric on a subset of a Euclidean space, this set may become a Hadamard manifold, and some nonmonotone problems on the original set may become monotone problems with respect to the endowed Riemannian metric [10]. Thus how to use the geometric structure of Hadamard manifolds directly to devise efficient algorithms becomes an important research topic.

In the past few years, some iterative methods have been proposed for solving variational inequalities on Riemannian manifolds. Li *et al.* [17] proposed a proximal point algorithm for variational inequality problems for set-valued monotone vector fields on general Riemannian manifolds. Tang *et al.* [25] constructed a variant of Korpelevich's method for variational inequality problems for univalued pseudomonotone vector fields on Hadamard manifolds. Tang *et al*. [27] proposed a modified projection-type method for variational inequality problems for univalued pseudomonotone vector fields on Hadamard manifolds. Batista *et al.* [5, 6] introduced an inexact proximal point algorithm and an extragradient-type algorithm for variational inequality for set-valued monotone vector fields on Hadamard manifolds. Ansaria *et al.* [2] considered a proximal point algorithm for inclusion problems on Hadamard manifolds and its application to univalued monotone variational inequality problem.

In this paper, we are concerned with the construction of an efficient iterative method for solving variational inequality problems for univalued pseudomonotone vector field on Hadamard manifolds. Let $\mathscr{H}$ be a Hadamard manifold. Without abuse of notation, we use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to denote the inner products and norms on different tangent spaces of $\mathscr{H}$ by its Riemannian metric. A tangent vector field $V:\mathscr{H}\to T\mathscr{H}$ is called pseudomonotone if it holds

$$\left\langle \mathrm{Exp}_p^{-1}q, V(p)\right\rangle \geq 0 \implies \left\langle \mathrm{Exp}_q^{-1}p, V(q)\right\rangle \leq 0 \quad \text{for all} \quad p,q\in\mathscr{H}, \tag{1.1}$$

where $\mathrm{Exp}_p^{-1}:\mathscr{H}\to T_p\mathscr{H}$ denotes the inverse of the exponential map $\mathrm{Exp}_p:T_p\mathscr{H}\to\mathscr{H}$. A subset $C\subset\mathscr{H}$ is said to be convex if for any two points $p,q\in C$, the Riemannian geodesic connecting p and q is contained in C; that is, if $\gamma:[a,b]\to C$ is a Riemannian geodesic with $\gamma(a)=p$ and $\gamma(b)=q$, then $\gamma((1-t)a+tb)\in C$ for any $t\in[0,1]$. Let C be a closed and convex subset of $\mathscr{H}$ and $V:\mathscr{H}\to T\mathscr{H}$ be a pseudomonotone vector field. The variational inequality problem for V and C is to find a point $p\in C$ such that

$$\left\langle V(p), \mathrm{Exp}_p^{-1}q\right\rangle \geq 0 \quad \text{for all} \quad q\in C. \tag{1.2}$$

Sometimes, the solution set of problem (1.2) is not a singleton, one may need to compute the point in the solution set which is closest to a prescribed given point. In this case, among all these existing iterative methods, only the modified projection-type method in [27] can generate an iterative sequence converging to the desired point. However, the metric projection of a given point to the intersection of three closed convex subsets needs to be computed at each iteration, this may be computational costly for some problems. This motivates us to devise a new efficient projection-type method for problem (1.2). Different from the technique used in [27], we construct a Halpern-type iterative scheme.

In the past few years, different iterative methods with inertial terms have been proposed for soling variational inequalities in Hilbert spaces [24, 28, 34, 40]. By using inertial extrapolation techniques, the convergence speed of some iterative methods can be effectively accelerated. This motives us to apply this technique to the projection-type methods for variational inequalities on Hadamard manifolds. To devise an efficient iterative method satisfying the above considerations, we combine the hyperplane projection method and the inertial extrapolation technique to construct a Halpern-type algorithm for solving problem (1.2). To analyze the global convergence property of the proposed new algorithm, the sectional curvature of the underlying Hadamard manifold is assumed to be bounded below. Numerical experiments show the efficiency of the proposed new algorithm.

The rest of this paper is organized as follows. Some basic concepts and results on Riemannian manifolds are reviewed in Section 2. In Section 3, we propose a Riemannian inertial Halpern-type algorithm for variational inequalities on Hadamard manifolds. The global convergence of the algorithm is proved under some mild assumptions in Section 4. In Section 5, some numerical results are reported to show the efficiency of this method. Finally, some concluding remarks are given in Section 6.

2. Preliminaries

In this section, we review necessary concepts and basic results, which are used for construction and analysis of iterative methods for solving problem (1.2). In the following, $\mathscr{H}$ is assumed to be a Hadamard manifold. Let C be a closed and convex subset of $\mathscr{H}$. Given a point $p \in \mathscr{H}$, the metric projection of p onto C defined by

$$\Pi_C p := \big\{p_0 \in C \mid \operatorname{dist}(p, p_0) \le \operatorname{dist}(p, q) \ \text{ for all } \ q \in C\big\}.$$

This metric projection has the following properties [29].

Proposition 2.1. *Let C be a closed and convex subset of a Hadamard manifold $\mathscr{H}$. For any point $p \in \mathscr{H}$, $\Pi_C p$ is a singleton, and the following inequality holds:*

$$\left\langle \operatorname{Exp}^{-1}_{\Pi_C p} p, \operatorname{Exp}^{-1}_{\Pi_C p} q \right\rangle \le 0 \quad \textit{for all} \quad q \in C. \tag{2.1}$$

Furthermore, the metric projection $\Pi_C : \mathscr{H} \to C$ is a nonexpansive mapping — i.e.

$$\operatorname{dist}(\Pi_C p, \Pi_C q) \le \operatorname{dist}(p, q) \quad \textit{for all} \quad p, q \in \mathscr{H}, \tag{2.2}$$

where $\operatorname{dist} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ *denotes the Riemannian distance function on* $\mathscr{H}$.

For any two points $p,q \in \mathscr{H}$, there exists a unique normalized Riemannian geodesic connecting p and q, which is a minimal geodesic [23, Theorem 4.1]. A geodesic triangle $\triangle(p,q,r)$ of a Riemannian manifold is a set consisting of three points p, q, and r, and three minimal geodesics joining these three points. Let $\triangle(p,q,r)$ be a geodesic triangle of $\mathscr{H}$. Then, according to [8, p.24], there are three points $p',q',r' \in \mathbb{R}^2$ such that

$$\operatorname{dist}(p,q) = \|p'-q'\|_2, \quad \operatorname{dist}(q,r) = \|q'-r'\|_2, \quad \operatorname{dist}(r,p) = \|r'-p'\|_2. \tag{2.3}$$

The triangle $\triangle(p',q',r') \in \mathbb{R}^2$ is called the comparison triangle of the geodesic triangle $\triangle(p,q,r)$, which is unique with respect to the isometry of $\mathbb{R}^2$. For angles and distances between points, a geodesic triangle and its comparison triangle have the following special relationship.

Lemma 2.1 (cf. Li *et al.* [15, Lemma 3.5]). *Let $\triangle(p,q,r)$ be a geodesic triangle of a Hadamard manifold $\mathscr{H}$ and $\triangle(p',q',r')$ be its comparison triangle.*

1. *Let α,β,γ and α',β',γ' be the angles of $\triangle(p,q,r)$ and $\triangle(p',q',r')$ at the vertices p,q,r and p',q',r', respectively. Then it holds that*
$$\alpha \le \alpha', \quad \beta \le \beta', \quad \gamma \le \gamma'. \tag{2.4}$$
2. *Let w be a point in the Riemannian geodesic connecting p and q. Given a point w' in the interval $[p',q']$. Suppose that*
$$\operatorname{dist}(w,p) = \|w'-p'\|_2, \quad \operatorname{dist}(w,q) = \|w'-q'\|_2.$$
Then it holds that
$$\operatorname{dist}(w,r) \le \|w'-r'\|_2. \tag{2.5}$$

For any two points $p,q \in \mathscr{H}$ and any tangent vector $\xi_p \in T_p\mathscr{H}$, it holds

$$\left\|\operatorname{Exp}_p^{-1} q\right\| = \operatorname{dist}(p,q), \quad \operatorname{dist}\big(\operatorname{Exp}_p(\xi_p), p\big) = \|\xi_p\|. \tag{2.6}$$

Since any two points in a Hadamard manifold can be joined by a unique minimal geodesic, for any two points $p,q \in \mathscr{H}$, we use $P_{p,q} : T_p\mathscr{H} \to T_q\mathscr{H}$ to denote the parallel translation along the minimal geodesic joining p and q. For the parallel translation operator, it holds

$$\langle P_{p,q}\xi_p, \eta_q\rangle = \langle \xi_p, P_{q,p}\eta_q\rangle \quad \text{for all} \quad p,q \in \mathscr{H}, \quad \xi_p \in T_p\mathscr{M}, \quad \eta_q \in T_q\mathscr{H}. \tag{2.7}$$

Denote

$$w_t := \operatorname{Exp}_p\big(t\operatorname{Exp}_p^{-1} q\big), \quad w_t' := (1-t)p' + tq' \quad \text{for} \quad t \in [0,1].$$

By (2.3), (2.5), and (2.6), we have

$$\begin{aligned}
&\operatorname{dist}\Big(\operatorname{Exp}_p\big(t\operatorname{Exp}_p^{-1} q\big), r\Big) = \operatorname{dist}(w_t, r) \\
&\le \|w_t' - r'\|_2 = \|(1-t)p' + tq' - r'\|_2 \\
&= \|(1-t)(p'-r') + t(q'-r')\|_2 \\
&\le (1-t)\|p'-r'\|_2 + t\|q'-r'\|_2 \\
&= (1-t)\cdot\operatorname{dist}(p,r) + t\cdot\operatorname{dist}(q,r) \quad \text{for all} \quad t \in [0,1].
\end{aligned} \tag{2.8}$$

Proposition 2.2 (cf. Sakai [23, Proposition 4.5]). *Let $\triangle(p_1,p_2,p_3)$ be a geodesic triangle of a Hadamard manifold $\mathscr{H}$. Denote, for each $i=1,2,3 \pmod 3$, by $\gamma_i:[0,l_i]\to\mathscr{H}$ the geodesic joining p_i to p_{i+1}, and set $l_i:=\int_0^{l_i}\|\gamma_i'(t)\|dt$ and $\alpha_i:=\angle(\gamma_i'(0),-\gamma_{i-1}'(l_{i-1}))$. Then*

1. $\alpha_1+\alpha_2+\alpha_3\le\pi$,
2. $l_i^2+l_{i+1}^2-2l_il_{i+1}\cos\alpha_{i+1}\le l_{i-1}^2$,
3. $l_{i+1}\cos\alpha_{i+2}+l_i\cos\alpha_i\ge l_{i+2}$.

In terms of distance function and exponential mapping, the second conclusion of Proposition 2.2 can be rewritten as following form:

$$\mathrm{dist}^2(p_i,p_{i+1})+\mathrm{dist}^2(p_{i+1},p_{i+2})-2\left\langle \mathrm{Exp}_{p_{i+1}}^{-1}p_i,\mathrm{Exp}_{p_{i+1}}^{-1}p_{i+2}\right\rangle\le\mathrm{dist}^2(p_i,p_{i+2}). \tag{2.9}$$

For Hadamard manifolds with curvature bounded from below, the following trigonometric distance bound holds.

Lemma 2.2 (cf. Zhang & Sra [38, Lemma 6]). *Let $\triangle(p_1,p_2,p_3)$ be a geodesic triangle of a Hadamard manifold $\mathscr{H}$. Let l_i and α_i, $i=1,2,3 \pmod 3$ be defined as in Proposition 2.2. Suppose that the curvature of $\mathscr{H}$ is bounded below by $-\kappa<0$, then it holds that*

$$l_{i-1}^2\le\frac{\sqrt{\kappa}l_{i+1}}{\tanh(\sqrt{\kappa}l_{i+1})}l_i^2+l_{i+1}^2-2l_il_{i+1}\cos\alpha_{i+1}. \tag{2.10}$$

For the boundedness of the inverse of exponential mapping on Hadamard manifolds with curvature bounded from blow, the following result holds.

Lemma 2.3 (cf. Zhang & Sra [39, Theorem 2 (17)]). *Let $\triangle(p,q,x_*)$ be a geodesic triangle of a Hadamard manifold $\mathscr{H}$. Suppose that the curvature of $\mathscr{H}$ is bounded from below by $-\kappa<0$ and $\mathrm{dist}(p,x_*)\le1/(4\sqrt{\kappa})$, then it holds that*

$$\left\|\mathrm{Exp}_{x_*}^{-1}(p)-\mathrm{Exp}_{x_*}^{-1}(q)\right\|\le\sqrt{1+2\kappa\cdot\mathrm{dist}^2(p,x_*)}\cdot\mathrm{dist}(p,q). \tag{2.11}$$

For exponential mapping, parallel translation, and inner product on Hadamard manifolds, we have the following properties.

Lemma 2.4 (cf. Batista *et al*. [6, Lemmas 1.1-1.2]). *Let $p_*,q_*\in\mathscr{H}$ and $\{p_k\},\{q_k\}\subset\mathscr{H}$ be such that $\lim_{k\to\infty}p_k=p_*$ and $\lim_{k\to\infty}q_k=q_*$. Then the following assertions hold:*

1. *For any $q\in\mathscr{H}$, we have*
$$\lim_{k\to\infty}\mathrm{Exp}_{p_k}^{-1}q=\mathrm{Exp}_{p_*}^{-1}q,\quad \lim_{k\to\infty}\mathrm{Exp}_q^{-1}p_k=\mathrm{Exp}_q^{-1}p_*.$$
2. *If $\xi_{p_k}\in T_{p_k}\mathscr{H}$ and $\lim_{k\to\infty}\xi_{p_k}=\xi_{p_*}$, then $\xi_{p_*}\in T_{p_*}\mathscr{H}$.*

3. *Let* $\xi_{p_k}, \eta_{p_k} \in T_{p_k}\mathscr{H}$ *and* $\xi_{p_*}, \eta_{p_*} \in T_{p_*}\mathscr{H}$. *If*

$$\lim_{k\to\infty} \xi_{p_k} = \xi_{p_*}, \quad \lim_{k\to\infty} \eta_{p_k} = \eta_{p_*},$$

then we have

$$\lim_{k\to\infty} \langle \xi_{p_k}, \eta_{p_k} \rangle = \langle \xi_{p_*}, \eta_{p_*} \rangle = \langle \lim_{k\to\infty} \xi_{p_k}, \lim_{k\to\infty} \eta_{p_k} \rangle.$$

4. $\lim_{k\to\infty} \mathrm{Exp}_{p_k}^{-1}(q_k) = \mathrm{Exp}_{p_*}^{-1} q_*$.

5. *Let* $\xi_{p_*} \in T_{p_*}\mathscr{H}$, $t_* \in [0,1)$, $\xi_{p_k} \in T_{p_k}\mathscr{H}$, *and* $\{t_k\} \subset (0,1)$ *be such that*

$$\lim_{k\to\infty} \xi_{p_k} = \xi_{p_*}, \quad \lim_{k\to\infty} t_k = t_*.$$

Define

$$\gamma_k(t) := \mathrm{Exp}_{p_k} t\xi_{p_k}, \quad x_k := \gamma_k(t_k) = \mathrm{Exp}_{p_k} t_k \xi_{p_k}$$

for each $k \geq 0$. *In addition, denote*

$$\gamma_*(t) := \mathrm{Exp}_{p_*} t\xi_{p_*}, \quad x_* := \gamma_*(t_*) = \mathrm{Exp}_{p_*} t_* \xi_{p_*} = q_*.$$

Then it holds

$$\lim_{k\to\infty} P_{\gamma_k(0),\gamma_k(t_k)} \xi_{p_k} = P_{\gamma_*(0),\gamma_*(t_*)} \xi_{p_*}.$$

Lemma 2.5 (cf. Maingé [20]). *Let* $\{a_n\}$ *be a sequence in* $(0,1)$, $\{b_n\}$ *a real sequence, and* $\{c_n\}$ *a real sequence such that* $\sum_{n=1}^{\infty} c_n < \infty$. *Let* $\{\Psi_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$\Psi_{n+1} \leq (1-a_n)\Psi_n + b_n + c_n, \quad n \geq 1.$$

Then the following results hold:

1. *If* $b_n \leq a_n M$ *for some* $M \geq 0$, *then* $\{\Psi_n\}$ *is a bounded sequence.*
2. *If* $\sum_{n=1}^{\infty} a_n = \infty$ *and* $\limsup_{n\to\infty} b_n/a_n \leq 0$, *then* $\lim_{n\to\infty} \Psi_n = 0$.

Lemma 2.6 (cf. Cholamjiak *et al*. [9, Lemma 2.6]). *Let* $\{\Phi_n\}$ *be a sequence of nonnegative real numbers such that*

$$\Phi_{n+1} \leq (1-a_n)\Phi_n + a_n\zeta_n, \quad \Phi_{n+1} \leq \Phi_n - \vartheta_n + \pi_n$$

for $n \geq 0$, *where* $\{a_n\}$, $\{\vartheta_n\}$, *and* $\{\pi_n\}$ *are real sequences satisfying the following conditions:*

1. $a_n \in (0,1)$, $\lim_{n\to\infty} a_n = 0$, *and* $\sum_{n=1}^{\infty} a_n = \infty$,
2. $\lim_{n\to\infty} \pi_n = 0$,
3. $\lim_{k\to\infty} \vartheta_{n_k} = 0$ *implies that* $\limsup_{k\to\infty} \zeta_{n_k} \leq 0$ *for any subsequence* $\{\vartheta_{n_k}\}$ *of* $\{\vartheta_n\}$.

Then $\lim_{n\to\infty} \Phi_n = 0$.

3. An Inertial Halpern-Type Algorithm

In this section, we first review Korpelevich's method [25] for solving the problem (1.2). After that, we combine the inertial extrapolation technique and the Halpern-type iterative technique together to modify this algorithm. Specially, the exact form of the proposed new algorithm for the case that $\mathscr{H} = \mathbb{R}^n$ is also given. In the following, $\mathscr{H}$ is assumed to be a Hadamard manifold and C is assumed to be a closed and convex subset of $\mathscr{H}$.

Algorithm 3.1 Korpelevich's Method

1: Choose a constant $\delta \in (0,1)$, two parameters $\overline{\beta}$ and $\widetilde{\beta}$ satisfying $0 < \overline{\beta} < \widetilde{\beta}$, a sequence $\{\beta_n\} \subset [\overline{\beta}, \widetilde{\beta}]$, and an initial point $x_0 \in \mathscr{H}$. Set $n := 0$.

2: Compute

$$z_n := \Pi_C \left(\mathrm{Exp}_{x_n}\big(- \beta_n V(x_n) \big) \right).$$

If $\mathrm{dist}(x_n, z_n) = 0$, then stop. Otherwise, compute

$$\eta_n := \max\{2^{-j}\}, \quad j = 0, 1, 2, \ldots$$

such that

$$-\big\langle V\big(\gamma_n(\eta_n)\big), \gamma_n'(\eta_n)\big\rangle \geq \frac{\delta}{\beta_n} \mathrm{dist}^2(x_n, z_n),$$

where

$$\gamma_n(t) := \mathrm{Exp}_{x_n} \left(t \mathrm{Exp}_{x_n}^{-1}(z_n) \right).$$

Compute $y_n := \gamma_n(\eta_n)$.

3: Define

$$H_n := \big\{ x \in \mathscr{H} \mid h_n(x) \leq 0 \big\}, \quad h_n(x) := \big\langle V(y_n), \mathrm{Exp}_{y_n}^{-1} x \big\rangle,$$

and compute

$$w_n := \Pi_{H_n} x_n, \quad x_{n+1} := \Pi_C w_n.$$

4: Replace n by $n+1$ and go to step 2.

The above method is a generalization of the extragradient method for solving variational inequalities in Euclidean spaces, for which differentiability of the underlying vector field is not required. Compared to the proximal point method, this method is an explicit iterative method, no specific subproblems need to be solved at each iteration. At each iteration, only two metric projections onto C and one metric projection onto H_n are required, the computational cost at each iteration is relatively low. Noting that inertial extrapolation helps to accelerate iterative methods, we consider the application of this technique to the above algorithm. To approximate the point in the solution set of problem (1.2) closest to a prescribed given point, Halpern-type iterative scheme can be applied.

To develop a new efficient iterative method satisfying the above requirements, we assume that the following conditions hold:

(C1) The curvature of $\mathscr{H}$ is bounded below by $-\kappa$, $\kappa \geq 0$.

(C2) $\{a_n\}$ and $\{\epsilon_n\}$ are positive sequences such that

$$a_n \in (0,1), \quad \lim_{n\to\infty} a_n = 0, \quad \sum_{n=0}^{\infty} a_n = \infty, \quad \epsilon_n \geq 0, \quad \lim_{n\to\infty} \frac{\epsilon_n}{a_n} = 0. \tag{3.1}$$

(C3) $\{\tau_n\}$ is a sequence such that

$$\begin{aligned} &0 < \tau_n \leq \frac{1}{4\sqrt{\kappa}} \quad \text{and} \quad 0 < \liminf_{n\to\infty} \tau_n, \quad \text{if} \quad \kappa > 0, \\ &0 < \tau_n \leq \infty \quad\quad \text{and} \quad 0 < \liminf_{n\to\infty} \tau_n, \quad \text{if} \quad \kappa = 0. \end{aligned} \tag{3.2}$$

(C4) $\{\mu_n\}$ is a nonnegative sequence such that

$$\lim_{n\to\infty} \mu_n = 0. \tag{3.3}$$

Combining the inertial extrapolation technique and the Halpern-type iterative scheme together, we propose the following new algorithm for solving problem (1.2).

Algorithm 3.2 Inertial Halpern-Type Algorithm

1: Choose four sequences $\{a_n\}$, $\{\epsilon_n\}$, $\{\tau_n\}$, and $\{\mu_n\}$ satisfying conditions (C1)-(C4), and two constants $\eta \in (0,1)$ and $\delta \in (0,1/2)$. Give a prescribed point $u \in \mathscr{H}$. Select arbitrary points $x_{-1}, x_0 \in C$ and $\theta \in [0,1)$. Set $n := 0$.

2: Choose θ_n such that $0 \leq \theta_n \leq \overline{\theta}_n$, where

$$\overline{\theta}_n := \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{\|\mathrm{Exp}_{x_n}^{-1} x_{n-1}\|}\right\}, & \text{if} \quad x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.4}$$

3: Compute

$$w_n := \Pi_C \Big(\mathrm{Exp}_{x_n}\big(-\theta_n \mathrm{Exp}_{x_n}^{-1} x_{n-1}\big)\Big), \tag{3.5}$$

$$z_n := \Pi_C \big(\mathrm{Exp}_{w_n}\big(-V(w_n)\big)\big). \tag{3.6}$$

4: If $\mathrm{dist}(w_n, z_n) = 0$, then stop. Otherwise, compute

$$\eta_n := \max\big\{\widehat{\eta}_n \cdot \eta^j\big\}, \quad j = 0,1,2,\ldots$$

such that

$$-\big\langle V\big(\gamma_n(\eta_n)\big), \gamma_n'(\eta_n)\big\rangle \geq \delta \cdot \mathrm{dist}^2(w_n, z_n) - \mu_n, \tag{3.7}$$

where

$$\widehat{\eta}_n := \min\left\{\eta, \frac{\tau_n}{\mathrm{dist}(w_n, z_n)}\right\}, \quad \gamma_n(t) := \mathrm{Exp}_{w_n}\big(t\mathrm{Exp}_{w_n}^{-1} z_n\big). \tag{3.8}$$

Compute

$$y_n := \gamma_n(\eta_n). \tag{3.9}$$

5: Compute

$$x_{n+1} := \Pi_C\Big(\mathrm{Exp}_u\big((1-a_n)\mathrm{Exp}_u^{-1}(\Pi_{H_n} w_n)\big)\Big), \tag{3.10}$$

where

$$H_n := \big\{x \in \mathscr{H} \mid h_n(x) \le 0\big\}, \quad h_n(x) := \big\langle V(y_n), \mathrm{Exp}_{y_n}^{-1} x\big\rangle. \tag{3.11}$$

6: Replace n by $n+1$ and go to step 2.

For this new algorithm, we have the following remarks.

Remark 3.1. The computational cost of Algorithm 3.2 at each iteration is almost the same as Algorithm 3.1. The backtracking line search method of Algorithm 3.2 is different from that of Algorithm 3.1. Numerical experiments in Section 5 show the efficiency of this modified line search method. If the solution set of problem (1.2) is nonempty, then the iterative sequence $\{x_n\}$ generated by Algorithm 3.2 converges to the point in the solution set, which is closest to the prescribed point $u \in \mathscr{H}$ (see Theorem 4.1).

Remark 3.2. Let $\gamma_n : [0,1] \to \mathscr{H}$ denote the Riemannian geodesic connecting w_n and z_n with $\gamma_n(0) = w_n$ and $\gamma_n(1) = z_n$. Since $w_n, z_n \in C$, by the definition of y_n in (3.9), we get $y_n \in C$. By the definition of $\gamma_n(t)$ in (3.8), its velocity $\gamma_n'(t)$ is a parallel vector field along γ_n. By the definition and property of parallel translation, for all $t, t_1, t_2, t_3 \in \mathbb{R}$ it holds

$$\gamma_n'(t) = P_{w_n,\gamma_n(t)}\mathrm{Exp}_{w_n}^{-1} z_n, \quad P_{\gamma_n(t_2),\gamma_n(t_3)} \circ P_{\gamma_n(t_1),\gamma_n(t_2)} = P_{\gamma_n(t_1),\gamma_n(t_3)}. \tag{3.12}$$

Remark 3.3. Denote

$$r(p) := \mathrm{Exp}_p^{-1}\Big(\Pi_C\big(\mathrm{Exp}_p(-V(p))\big)\Big). \tag{3.13}$$

According to [25, Proposition 2.5], $p \in C$ is a solution of problem (1.2) if and only if $r(p) = 0_p$. If $\mathrm{dist}(w_n, z_n) = 0$, then it follows from (3.6) that

$$w_n = z_n = \Pi_C\left(\mathrm{Exp}_{w_n}\big(-V(w_n)\big)\right) \in C.$$

In this case, w_n is a solution of problem (1.2).

If $\mathrm{dist}(w_n, z_n) \ne 0$, according to [25, Proposition 4.1], there exists a nonnegative integer j such that

$$-\Big\langle V\big(\gamma_n(\widehat{\eta}_n \cdot \eta^j)\big), \gamma_n'\big(\widehat{\eta}_n \cdot \eta^j\big)\Big\rangle \ge \delta \cdot \mathrm{dist}^2(w_n, z_n).$$

Since $\mu_n \ge 0$, the above inequality implies that the backtracking line search (3.7) termites in finite steps. Thus η_n is well-defined.

Remark 3.4. The solution set of problem (1.2) is denoted by $\mathscr{S}$, i.e.

$$\mathscr{S} := \big\{x \in C \mid \langle V(x), \mathrm{Exp}_x^{-1} y\rangle \ge 0 \quad \text{for all} \quad y \in C\big\}. \tag{3.14}$$

Since both w_n and z_n belong to C, we obtain $y_n \in C$. By (3.14), for any point $x \in \mathscr{S}$, we have $\langle V(x), \mathrm{Exp}_x^{-1} y_n \rangle \geq 0$. Since the vector field V is pseudomonotone, it follows from (1.1) that $\langle \mathrm{Exp}_{y_n}^{-1} x, V(y_n) \rangle \leq 0$. Thus the solution set $\mathscr{S}$ is contained in H_n. Then we have

$$\mathscr{S} \subseteq H_n \cap C \quad \text{for all} \quad n \geq 0. \tag{3.15}$$

By the definitions of h_n, H_n and $\Pi_{H_n} w_n$, we obtain

$$h_n(\Pi_{H_n} w_n) = \left\langle V(y_n), \mathrm{Exp}_{y_n}^{-1}(\Pi_{H_n} w_n) \right\rangle \leq 0. \tag{3.16}$$

According to (4.9) in Lemma 4.3, if $\mathrm{dist}(w_n, z_n) \neq 0$, then

$$h_n(w_n) \geq \eta_n \delta \cdot \mathrm{dist}^2(w_n, z_n) - \eta_n \mu_n.$$

In particular, if $\mu_n < \delta \cdot \mathrm{dist}^2(w_n, z_n)$, then we have $h_n(w_n) > 0$, which implies that $w_n \notin H_n$. In this case, it holds that

$$\Pi_{H_n} w_n \in \partial H_n := \left\{ x \in \mathscr{H} \mid h_n(x) = 0 \right\} \subset H_n, \quad h_n(\Pi_{H_n} w_n) = 0.$$

In this case, the boundary ∂H_n of H_n separates w_n from the solution set $\mathscr{S}$ and the point $\Pi_{H_n} w_n$ is closer to $\mathscr{S}$ than w_n.

The Euclidean space $\mathbb{R}^n$ is a special kind of Hadamard manifold with sectional curvature equal to 0. We consider the application of Algorithm 3.2 for solving pseudomonotone equation on the Euclidean space $\mathbb{R}^n$. For the Euclidean space $\mathbb{R}^n$, it holds that

$$\mathrm{Exp}_{\mathbf{x}} \xi_{\mathbf{x}} = \mathbf{x} + \xi_{\mathbf{x}}, \quad \mathrm{Exp}_{\mathbf{x}}^{-1} \mathbf{y} = \mathbf{y} - \mathbf{x}, \quad \mathrm{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2, \tag{3.17}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\xi_{\mathbf{x}} \in T_{\mathbf{x}} \mathbb{R}^n$, $\mathrm{Id}_{\mathbb{R}^n}$ denotes the identity map of $\mathbb{R}^n$, and $\| \cdot \|_2$ denotes the 2-norm of $\mathbb{R}^n$. In this case, the definition of pseudomonotone in (1.1) reduces to the concept proposed in [13]. That is $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a pseudomonotone map if

$$\langle F(\mathbf{x}_1), \mathbf{x}_2 - \mathbf{x}_1 \rangle \geq 0 \implies \langle F(\mathbf{x}_2), \mathbf{x}_1 - \mathbf{x}_2 \rangle \leq 0 \quad \text{for all} \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of $\mathbb{R}^n$, i.e. $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a pseudomonotone map and $C \subset \mathbb{R}^n$ be a closed and convex set. The variational inequality problem for F and C is to find a point $\mathbf{x} \in C$ such that

$$\langle F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0 \quad \text{for all} \quad \mathbf{y} \in C. \tag{3.18}$$

Based on (3.17), if $F(\mathbf{x}) \neq \mathbf{0}_n$, then

$$H_{\mathbf{y}} := \left\{ \mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x} - \mathbf{y}, F(\mathbf{y}) \rangle \leq 0 \right\}$$

is a closed halfspace of $\mathbb{R}^n$. If $\mathbf{w} \notin H_{\mathbf{y}}$, then the projection of $\mathbf{w}$ onto $H_{\mathbf{y}}$ has the following explicit form:

$$\Pi_{H_{\mathbf{y}}} \mathbf{w} = \mathbf{y} - \frac{\langle F(\mathbf{w}), \mathbf{y} - \mathbf{w} \rangle}{\|F(\mathbf{w})\|_2^2} F(\mathbf{w}). \tag{3.19}$$

According to (3.17) and (3.19), the application of Algorithm 3.2 for solving the variational inequality (3.18) can be stated in the following form.

Algorithm 3.3 Inertial Halpern-Type Algorithm in $\mathbb{R}^n$

1: Choose three sequences $\{a_n\}$, $\{\epsilon_n\}$, and $\{\mu_n\}$ satisfying conditions (C1)-(C4), and two constants $\eta \in (0,1)$ and $\delta \in (0,1/2)$. Give a prescribed point $\mathbf{u} \in \mathbb{R}^n$. Select arbitrary points $\mathbf{x}_{-1}, \mathbf{x}_0 \in C$ and $\theta \in [0,1)$. Set $n := 0$.

2: Choose θ_n such that $0 \le \theta_n \le \overline{\theta}_n$, where

$$\overline{\theta}_n := \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{\|\mathbf{x}_n - \mathbf{x}_{n-1}\|_2}\right\}, & \mathbf{x}_n \neq \mathbf{x}_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

3: Compute

$$\mathbf{w}_n := \Pi_C\big(\mathbf{x}_n + \theta_n(\mathbf{x}_n - \mathbf{x}_{n-1})\big), \quad \mathbf{z}_n := \Pi_C\big(\mathbf{w}_n - F(\mathbf{w}_n)\big).$$

4: If $\mathbf{w}_n = \mathbf{z}_n$, then stop. Otherwise, compute

$$\eta_n := \max\big\{\eta^j\big\}, \quad j = 0,1,2,\ldots$$

such that

$$-\big\langle F\big(\mathbf{w}_n + \eta_n(\mathbf{z}_n - \mathbf{w}_n)\big), \mathbf{z}_n - \mathbf{w}_n\big\rangle \ge \delta \cdot \|\mathbf{z}_n - \mathbf{w}_n\|_2^2 - \mu_n.$$

Compute

$$\mathbf{y}_n := \mathbf{w}_n + \eta_n(\mathbf{z}_n - \mathbf{w}_n).$$

5: Compute

$$\begin{aligned} \Pi_{H_n}\mathbf{w}_n &:= \mathbf{y}_n - \frac{\langle F(\mathbf{w}_n), \mathbf{y}_n - \mathbf{w}_n\rangle}{\|F(\mathbf{w}_n)\|_2^2} F(\mathbf{w}_n), \\ \mathbf{x}_{n+1} &:= \Pi_C\big(\alpha_n \mathbf{u} + (1 - a_n)\Pi_{H_n}\mathbf{w}_n\big). \end{aligned}$$

6: Replace n by $n+1$ and go to step 2.

Remark 3.5. Specially, since the sectional curvature of $\mathbb{R}^n$ is equal to 0, the parameter τ_n is chosen as ∞ for $n \ge 0$. This algorithm is different from the existing inertial methods for solving variational inequalities in $\mathbb{R}^n$ [24,28,34,40].

4. Convergence Analysis

In this section, we establish the global convergence of Algorithm 3.2. To prove its global convergence, we need the following conditions:

(C5) The vector field $V : \mathscr{H} \to T\mathscr{H}$ is continuous and pseudomonotone.

(C6) The sets $\{H_n\}$ are closed convex subsets of $\mathscr{H}$.

(C7) The solution set $\mathscr{S}$ is nonempty.

Specially, if the Hadamard manifold $\mathscr{H}$ is of constant curvature, then $\{H_n\}$ are closed convex subsets of $\mathscr{H}$ [11, Corollary 3.1]. Next, we give some lemmas needed to prove the global convergence of Algorithm 3.2. First, we can prove that $\{(\theta_n/a_n)\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\|\}$ and $\{\theta_n\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\|\}$ are infinitesimal sequences.

Lemma 4.1. *If* $\{x_n\}$ *is the iterative sequence generated by Algorithm* 3.2*, then*

$$\lim_{n\to\infty}\theta_n\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\| = \lim_{n\to\infty}\frac{\theta_n}{a_n}\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\| = 0. \tag{4.1}$$

Proof. If $x_{n-1}=x_n$, then $\mathrm{Exp}_{x_n}^{-1}x_{n-1}=0_{x_n}$. By (3.4), we have $\overline{\theta}_n=\theta$. Since $0\le\theta_n\le\overline{\theta}_n$, we obtain

$$0\le\theta_n\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|\le\overline{\theta}_n\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|=\theta\cdot 0=0. \tag{4.2}$$

If $x_{n-1}\neq x_n$, then it follows from (3.4) that

$$\begin{aligned}0&\le\theta_n\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|\le\overline{\theta}_n\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|\\&=\min\left\{\theta,\frac{\epsilon_n}{\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\|}\right\}\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|\le\epsilon_n.\end{aligned} \tag{4.3}$$

Taking into account conditions (C1)-(C4) and the inequalities (4.2), (4.3), we obtain

$$\begin{aligned}&0\le\lim_{n\to\infty}\theta_n\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|\le\lim_{n\to\infty}\epsilon_n=0,\\&0\le\lim_{n\to\infty}\frac{\theta_n}{a_n}\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|\le\lim_{n\to\infty}\frac{\epsilon_n}{a_n}=0.\end{aligned}$$

This completes the proof. □

For the boundedness of iterative sequences generated by Algorithm 3.2, we have the following result.

Lemma 4.2. *Let* $\{x_n\}$ *be an iterative sequence generated by Algorithm* 3.2*. If conditions (C1)-(C7) hold, then the iterative sequences* $\{x_n\},\{w_n\},\{z_n\}$*, and* $\{y_n\}$ *are all bounded.*

Proof. Given a point $x_*\in\mathscr{S}$, it follows from (3.15) that $x_*\in H_n\cap C$. Thus we obtain $x_*=\Pi_C x_*=\Pi_{H_n}x_*$. Using (2.2), (2.6), (3.5), (3.10), and the triangle inequality, we have

$$\begin{aligned}\mathrm{dist}(\Pi_{H_n}w_n,x_*)&=\mathrm{dist}(\Pi_{H_n}w_n,\Pi_{H_n}x_*)\\&\le\mathrm{dist}(w_n,x_*)\\&=\mathrm{dist}\big(\Pi_C\big(\mathrm{Exp}_{x_n}(-\theta_n\mathrm{Exp}_{x_n}^{-1}x_{n-1})\big),\Pi_C x_*\big)\\&\le\mathrm{dist}\big(\mathrm{Exp}_{x_n}(-\theta_n\mathrm{Exp}_{x_n}^{-1}x_{n-1}),x_*\big)\\&\le\mathrm{dist}\big(\mathrm{Exp}_{x_n}(-\theta_n\mathrm{Exp}_{x_n}^{-1}x_{n-1}),x_n\big)+\mathrm{dist}(x_n,x_*)\\&=\theta_n\left\|\mathrm{Exp}_{x_n}^{-1}x_{n-1}\right\|+\mathrm{dist}(x_n,x_*).\end{aligned} \tag{4.4}$$

Let $\triangle(u, x_*, \Pi_{H_n} w_n)$ denote the geodesic triangle in $\mathscr{H}$. It follows from (2.8), (3.10), and (4.4) that

$$\begin{aligned}\operatorname{dist}(x_{n+1}, x_*) &= \operatorname{dist}\Big(\Pi_C\Big(\operatorname{Exp}_u\big((1-a_n)\operatorname{Exp}_u^{-1}(\Pi_{H_n} w_n)\big)\Big), \Pi_C x_*\Big)\\ &\le \operatorname{dist}\Big(\operatorname{Exp}_u\big((1-a_n)\operatorname{Exp}_u^{-1}(\Pi_{H_n} w_n)\big), x_*\Big)\\ &\le a_n \operatorname{dist}(u, x_*) + (1-a_n)\operatorname{dist}(\Pi_{H_n} w_n, x_*)\\ &\le a_n \operatorname{dist}(u, x_*) + (1-a_n)\big(\theta_n \|\operatorname{Exp}_{x_n}^{-1} x_{n-1}\| + \operatorname{dist}(x_n, x_*)\big)\\ &= (1-a_n)\operatorname{dist}(x_n, x_*) + a_n\left[\operatorname{dist}(u, x_*) + (1-a_n)\frac{\theta_n}{a_n}\left\|\operatorname{Exp}_{x_n}^{-1} x_{n-1}\right\|\right].\end{aligned}$$

Based on (4.1) and the condition $a_n \in (0,1)$, the sequence $\{(1-a_n)(\theta_n/a_n)\|\operatorname{Exp}_{x_n}^{-1} x_{n-1}\|\}$ is bounded. Setting

$$M := 2\max\left\{\operatorname{dist}(u, x_*), \sup_{n\ge 1}(1-a_n)\frac{\theta_n}{a_n}\left\|\operatorname{Exp}_{x_n}^{-1} x_{n-1}\right\|\right\},$$

we write

$$\operatorname{dist}(x_{n+1}, x_*) \le (1-a_n)\operatorname{dist}(x_n, x_*) + a_n M. \tag{4.5}$$

By the first conclusion of Lemma 2.5, there exists a constant $\widetilde{M}_1 \ge 0$ such that

$$\operatorname{dist}(x_n, x_*) < \widetilde{M}_1 \quad \text{for all} \quad n \ge 0. \tag{4.6}$$

Thus, $\{x_n\}$ is a bounded sequence. Since $\{x_n\}$ is bounded, it follows from (4.1), (4.4), and (4.6) that

$$\begin{aligned}\limsup_{n\to\infty}\operatorname{dist}(w_n, x_*) &\le \lim_{n\to\infty}\theta_n\left\|\operatorname{Exp}_{x_n}^{-1} x_{n-1}\right\| + \limsup_{n\to\infty}\operatorname{dist}(x_n, x_*)\\ &\le 0 + \widetilde{M}_1 = \widetilde{M}_1.\end{aligned} \tag{4.7}$$

Thus the sequence $\{w_n\}$ is bounded.

According to (2.2), (2.6), (3.6), and the triangle inequality, we have

$$\begin{aligned}\operatorname{dist}(z_n, x_*) &= \operatorname{dist}\Big(\Pi_C\big(\operatorname{Exp}_{w_n}(-V(w_n))\big), x_*\Big)\\ &\le \operatorname{dist}\Big(\Pi_C\big(\operatorname{Exp}_{w_n}(-V(w_n))\big), \Pi_C w_n\Big) + \operatorname{dist}(\Pi_C w_n, x_*)\\ &= \operatorname{dist}\Big(\Pi_C\big(\operatorname{Exp}_{w_n}(-V(w_n))\big), \Pi_C w_n\Big) + \operatorname{dist}(\Pi_C w_n, \Pi_C x_*)\\ &\le \operatorname{dist}\big(\operatorname{Exp}_{w_n}(-V(w_n)), w_n\big) + \operatorname{dist}(w_n, x_*)\\ &= \|V(w_n)\| + \operatorname{dist}(w_n, x_*).\end{aligned}$$

Since $V : \mathscr{H} \to T\mathscr{H}$ is continuous, there exists a constant $\widetilde{M}_2 > 0$ such that $\|V(w_n)\| \le \widetilde{M}_2$ for all $n \ge 0$. It follows from (4.7) and the above inequality that

$$\limsup_{n\to\infty}\operatorname{dist}(z_n, x_*) \le \limsup_{n\to\infty}\|V(w_n)\| + \limsup_{n\to\infty}\operatorname{dist}(w_n, x_*) \le \widetilde{M}_2 + \widetilde{M}_1. \tag{4.8}$$

Thus the sequence $\{z_n\}$ is also bounded.

Based on (2.8), (3.8), and (3.9), we obtain

$$\begin{aligned}\operatorname{dist}(y_n, x_*) &= \operatorname{dist}\big(\operatorname{Exp}_{w_n}(\eta_n \operatorname{Exp}_{w_n}^{-1} z_n), x_*\big)\\ &\le (1-\eta_n)\operatorname{dist}(w_n, x_*) + \eta_n \operatorname{dist}(z_n, x_*)\\ &\le \operatorname{dist}(w_n, x_*) + \operatorname{dist}(z_n, x_*).\end{aligned}$$

This and the inequalities (4.7), (4.8) yield

$$\limsup_{n\to\infty} \operatorname{dist}(y_n, x_*) \le \limsup_{n\to\infty} \operatorname{dist}(w_n, x_*) + \limsup_{n\to\infty} \operatorname{dist}(z_n, x_*) \le \widetilde{M}_2 + 2\widetilde{M}_1.$$

Thus the sequence $\{y_n\}$ is bounded, which finishes the proof. $\square$

For the value of the function h_n defined by (3.11) at $w_n \in \mathscr{H}$, we have the following estimations.

Lemma 4.3. *Let $\{x_n\}$ be an iterative sequence generated by Algorithm* 3.2. *Suppose that conditions (C1)-(C7) hold. If $w_n \neq z_n$, then*

$$h_n(w_n) \ge \eta_n \delta \cdot \operatorname{dist}^2(w_n, z_n) - \eta_n \mu_n. \tag{4.9}$$

In addition, there exists a constant $\widehat{M}_1 > 0$ such that

$$h_n(w_n) \le \widehat{M}_1 \cdot \operatorname{dist}(w_n, \Pi_{H_n} w_n) \quad \textit{for all} \quad n \ge 0. \tag{4.10}$$

Proof. If $w_n \neq z_n$, then (3.7)-(3.9), (3.11), and (3.12) give

$$\begin{aligned}h_n(w_n) &= \big\langle V(y_n), \operatorname{Exp}_{y_n}^{-1} w_n \big\rangle\\ &= -\big\langle V(y_n), P_{w_n,y_n} \operatorname{Exp}_{w_n}^{-1}(y_n) \big\rangle\\ &= -\big\langle V(y_n), P_{w_n,y_n} \operatorname{Exp}_{w_n}^{-1}\big(\operatorname{Exp}_{w_n}(\eta_n \operatorname{Exp}_{w_n}^{-1} z_n)\big) \big\rangle\\ &= -\big\langle V(y_n), P_{w_n,y_n}(\eta_n \operatorname{Exp}_{w_n}^{-1} z_n) \big\rangle\\ &= -\eta_n \big\langle V(\gamma_n(\eta_n)), \gamma_n'(\eta_n) \big\rangle\\ &\ge \eta_n \delta \cdot \operatorname{dist}^2(w_n, z_n) - \eta_n \mu_n.\end{aligned}$$

Thus, the inequalities in (4.9) hold. On the other hand, since the sequence $\{y_n\}$ is bounded and $V : \mathscr{H} \to T\mathscr{H}$ is continuous, there exists a positive constant $\widetilde{M}_3 > 0$ such that

$$\|V(y_n)\| \le \widetilde{M}_3 \quad \text{for all} \quad n \ge 0. \tag{4.11}$$

According to (2.6), (3.2), and (3.7)-(3.9), we obtain

$$\begin{aligned}\operatorname{dist}(w_n, y_n) &= \operatorname{dist}\big(w_n, \operatorname{Exp}_{w_n}(\eta_n \operatorname{Exp}_{w_n}^{-1} z_n)\big)\\ &= \big\|\eta_n \operatorname{Exp}_{w_n}^{-1} z_n\big\| \le \widehat{\eta}_n \big\|\operatorname{Exp}_{w_n}^{-1} z_n\big\|\\ &\le \min\left\{\eta, \frac{\tau_n}{\operatorname{dist}(w_n, z_n)}\right\} \operatorname{dist}(w_n, z_n)\\ &\le \tau_n \le \frac{1}{4\sqrt{\kappa}}.\end{aligned}$$

By (2.11) and the above inequality, it holds that

$$\begin{aligned}
&\left\|\mathrm{Exp}_{y_n}^{-1} w_n - \mathrm{Exp}_{y_n}^{-1}(\Pi_{H_n} w_n)\right\| \\
&\leq \sqrt{1+2\kappa\cdot \mathrm{dist}^2(y_n, w_n)}\cdot \mathrm{dist}(w_n, \Pi_{H_n} w_n) \\
&\leq \frac{3\sqrt{2}}{4}\cdot \mathrm{dist}(w_n, \Pi_{H_n} w_n).
\end{aligned}$$

By (2.2), (3.10), (3.16), (4.11), the Cauchy-Swartz inequality, and the above inequality, we have

$$\begin{aligned}
h_n(w_n) &\leq h_n(w_n) - h_n(\Pi_{H_n} w_n) \\
&= \left\langle V(y_n), \mathrm{Exp}_{y_n}^{-1} w_n\right\rangle - \left\langle V(y_n), \mathrm{Exp}_{y_n}^{-1}(\Pi_{H_n} w_n)\right\rangle \\
&= \left\langle V(y_n), \mathrm{Exp}_{y_n}^{-1} w_n - \mathrm{Exp}_{y_n}^{-1}(\Pi_{H_n} w_n)\right\rangle \\
&\leq \|V(y_n)\|\cdot \left\|\mathrm{Exp}_{y_n}^{-1} w_n - \mathrm{Exp}_{y_n}^{-1}(\Pi_{H_n} w_n)\right\| \\
&\leq \frac{3\sqrt{2}\widetilde{M}_3}{4}\cdot \mathrm{dist}(w_n, \Pi_{H_n} w_n) \equiv \widehat{M}_1\cdot \mathrm{dist}(w_n, \Pi_{H_n} w_n),
\end{aligned}$$

where $\widehat{M}_1 := 3\sqrt{2}\widetilde{M}_3/4$. This completes the proof. □

The convergence of a subsequence of $\{\mathrm{dist}(w_n, \Pi_{H_n} w_n)\}$ to zero implies that the corresponding subsequence $\{\mathrm{dist}(w_n, z_n)\}$ is an infinitesimal sequence.

Lemma 4.4. *Let $\{x_n\}$ be an iterative sequence generated by Algorithm* 3.2*. Suppose that conditions (C1)-(C7) hold. If $\{w_{n_k}\}$ is a subsequence of $\{w_n\}$ such that*

$$\lim_{k\to\infty} \mathrm{dist}(w_{n_k}, \Pi_{H_{n_k}} w_{n_k}) = 0,$$

then

$$\lim_{k\to\infty} \mathrm{dist}(w_{n_k}, z_{n_k}) = 0. \tag{4.12}$$

Proof. Without loss of generality, we assume that

$$\lim_{n\to\infty} \mathrm{dist}(w_n, \Pi_{H_n} w_n) = 0.$$

Since $\{w_n\}$ and $\{z_n\}$ are both bounded, the sequence $\{\mathrm{dist}(z_n, w_n)\}$ is also bounded. Therefore, there exists a constant $b \geq 0$ such that

$$\limsup_{n\to\infty} \mathrm{dist}^2(z_n, w_n) = b. \tag{4.13}$$

According to (4.9) and (4.10), we have

$$\eta_n \mathrm{dist}^2(w_n, z_n) \leq \frac{\widehat{M}_1}{\delta}\mathrm{dist}(w_n, \Pi_{H_n} w_n) + \frac{1}{\delta}\eta_n\mu_n \quad \text{for all} \quad n \geq 0. \tag{4.14}$$

If $\liminf_{n\to\infty}\eta_n \neq 0$, then $\eta_n > 0$ for all n sufficiently large. Since $\lim_{n\to\infty}\operatorname{dist}(w_n, \Pi_{H_n} w_n) = 0$, it follows from (3.3), (4.13), and (4.14) that

$$
\begin{aligned}
b &= \limsup_{n\to\infty} \operatorname{dist}^2(w_n, z_n) \\
&= \limsup_{n\to\infty} \frac{1}{\eta_n}\eta_n \operatorname{dist}^2(w_n, z_n) \\
&\le \limsup_{n\to\infty} \frac{1}{\eta_n}\frac{\widehat{M}_1}{\delta}\operatorname{dist}(w_n, \Pi_{H_n} w_n) + \limsup_{n\to\infty}\frac{1}{\delta}\mu_n \\
&\le \limsup_{n\to\infty} \frac{\widehat{M}_1}{\delta}\operatorname{dist}(w_n, \Pi_{H_n} w_n)\frac{1}{\liminf_{n\to\infty}\eta_n} + 0 = 0.
\end{aligned}
$$

By the above inequality, the equality in (4.12) holds.

If $\liminf_{n\to\infty}\eta_n = 0$, then there is a subsequence $\{\eta_{n_k}\}$ of η_n such that $\lim_{k\to\infty}\eta_{n_k} = 0$. Define

$$
\overline{y}_n := \gamma_n\big(\eta^{-1}\eta_n\big) = \operatorname{Exp}_{w_n}\big(\eta^{-1}\eta_n \operatorname{Exp}_{w_n}^{-1} z_n\big), \quad p_{n_k} := \operatorname{Exp}_{w_{n_k}}\big(-V(w_{n_k})\big). \tag{4.15}
$$

It follows from (3.6) and (3.12) that

$$
P_{w_{n_k},p_{n_k}} V(w_{n_k}) = \operatorname{Exp}_{p_{n_k}}^{-1} w_{n_k}, \quad z_{n_k} = \Pi_C p_{n_k}. \tag{4.16}
$$

Since $\lim_{k\to\infty}\eta_{n_k} = 0$, it follows from (2.6), (4.13), and (4.15) that

$$
\begin{aligned}
0 &\le \limsup_{k\to\infty}\operatorname{dist}(\overline{y}_{n_k}, w_{n_k}) \\
&= \limsup_{k\to\infty}\operatorname{dist}\Big(\operatorname{Exp}_{w_{n_k}}\big(\eta^{-1}\eta_{n_k}\operatorname{Exp}_{w_{n_k}}^{-1}(z_{n_k})\big), w_{n_k}\Big) \\
&= \limsup_{k\to\infty}\eta^{-1}\eta_{n_k}\operatorname{dist}\Big(\operatorname{Exp}_{w_{n_k}}\big(\operatorname{Exp}_{w_{n_k}}^{-1}(z_{n_k})\big), w_{n_k}\Big) \\
&\le \eta^{-1}\cdot\lim_{k\to\infty}\eta_{n_k}\cdot\limsup_{k\to\infty}\operatorname{dist}(z_{n_k}, w_{n_k}) = \eta^{-1}\cdot 0\cdot b = 0.
\end{aligned}
$$

Since $V : \mathscr{H} \to T\mathscr{H}$ is continuous, $\{y_n\}$ and $\{w_n\}$ are bounded, it follows that

$$
\lim_{k\to\infty}\left\|P_{\overline{y}_{n_k},w_{n_k}} V(\overline{y}_{n_k}) - V(w_{n_k})\right\| = 0. \tag{4.17}
$$

Since $\lim_{k\to\infty}\eta_{n_k} = 0$, the inequality in (3.7) is not satisfied by $\eta^{-1}\eta_{n_k}$ for all n_k sufficiently large. According to (3.7), (3.8), (3.12), and (4.15), for all n_k sufficiently large it holds

$$
\begin{aligned}
&\delta\cdot\operatorname{dist}^2(w_{n_k}, z_{n_k}) - \mu_{n_k} > -\big\langle V(\gamma_{n_k}(\eta^{-1}\eta_{n_k})), \gamma'_{n_k}(\eta^{-1}\eta_{n_k})\big\rangle \\
&= -\Big\langle V(\overline{y}_{n_k}), P_{w_{n_k},\overline{y}_{n_k}}\operatorname{Exp}_{w_{n_k}}^{-1} z_{n_k}\Big\rangle = \Big\langle P_{\overline{y}_{n_k},w_{n_k}} V(\overline{y}_{n_k}), -\operatorname{Exp}_{w_{n_k}}^{-1} z_{n_k}\Big\rangle \\
&= \Big\langle P_{\overline{y}_{n_k},w_{n_k}} V(\overline{y}_{n_k}) - V(w_{n_k}), -\operatorname{Exp}_{w_{n_k}}^{-1} z_{n_k}\Big\rangle + \Big\langle V(w_{n_k}), -\operatorname{Exp}_{w_{n_k}}^{-1} z_{n_k}\Big\rangle \\
&= \Big\langle P_{\overline{y}_{n_k},w_{n_k}} V(\overline{y}_{n_k}) - V(w_{n_k}), -\operatorname{Exp}_{w_{n_k}}^{-1} z_{n_k}\Big\rangle + \Big\langle \operatorname{Exp}_{w_{n_k}}^{-1} p_{n_k}, \operatorname{Exp}_{w_{n_k}}^{-1} z_{n_k}\Big\rangle.
\end{aligned} \tag{4.18}
$$

Based on (2.9), we know that in the geodesic triangle $\triangle(p_{n_k}, z_{n_k}, w_{n_k})$,

$$\mathrm{dist}^2(w_{n_k}, z_{n_k}) + \mathrm{dist}^2(w_{n_k}, p_{n_k}) - 2\big\langle \mathrm{Exp}_{w_{n_k}}^{-1} z_{n_k}, \mathrm{Exp}_{w_{n_k}}^{-1} p_{n_k} \big\rangle \le \mathrm{dist}^2(z_{n_k}, p_{n_k}).$$

Combining (4.18) and the above inequality yields

$$\begin{aligned}
&\mathrm{dist}^2(w_{n_k}, p_{n_k}) - \mathrm{dist}^2(z_{n_k}, p_{n_k}) \\
&\le (2\delta - 1) \cdot \mathrm{dist}^2(w_{n_k}, z_{n_k}) - 2\mu_{n_k} + 2\big\langle P_{\overline{y}_{n_k}, w_{n_k}} V(\overline{y}_{n_k}) - V(w_{n_k}), \mathrm{Exp}_{w_{n_k}}^{-1} z_{n_k} \big\rangle \\
&\le (2\delta - 1) \cdot \mathrm{dist}^2(w_{n_k}, z_{n_k}) - 2\mu_{n_k} + 2\big\| P_{\overline{y}_{n_k}, w_{n_k}} V(\overline{y}_{n_k}) - V(w_{n_k}) \big\| \cdot \big\| \mathrm{Exp}_{w_{n_k}}^{-1} z_{n_k} \big\|.
\end{aligned}$$

Based on (2.6), (4.13), (4.17), and the above inequality, we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \big(\mathrm{dist}^2(w_{n_k}, p_{n_k}) - \mathrm{dist}^2(z_{n_k}, p_{n_k}) \big) \\
&\le (2\delta - 1) \cdot \limsup_{k\to\infty} \mathrm{dist}^2(w_{n_k}, z_{n_k}) - 2 \limsup_{k\to\infty} \mu_{n_k} \\
&\quad + 2 \limsup_{k\to\infty} \big\| P_{\overline{y}_{n_k}, w_{n_k}} V(\overline{y}_{n_k}) - V(w_{n_k}) \big\| \cdot \mathrm{dist}(w_{n_k}, z_{n_k}) \\
&= (2\delta - 1) \cdot b - 2 \cdot 0 + 2 \cdot 0 \cdot \sqrt{b} = (2\delta - 1) \cdot b \le 0.
\end{aligned}$$

By the above inequality, if $b > 0$, then

$$\mathrm{dist}^2(w_{n_k}, p_{n_k}) < \mathrm{dist}^2(z_{n_k}, p_{n_k})$$

for k sufficiently large, which contradicts to the convexity of C and the fact that z_{n_k} is the projection of p_{n_k} on C. Hence, $b = 0$, and the proof is complete. □

The limit of a convergent subsequence of $\{x_n\}$ belongs to the solution set if the corresponding subsequence of $\{\mathrm{dist}(w_n, \Pi_{H_n} w_n)\}$ is an infinitesimal sequence.

Lemma 4.5. *Let $\{x_n\}$ be an iterative sequence generated by Algorithm* 3.2. *Suppose that conditions (C1)-(C7) hold. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging to a point $\widehat{x} \in \mathscr{H}$. If $\lim_{k\to\infty} \mathrm{dist}(w_{n_k}, \Pi_{H_{n_k}} w_{n_k}) = 0$, then $\widehat{x} \in \mathscr{S}$.*

Proof. Since $x_n \in C$, it follows from (2.6) and (3.5) that

$$\begin{aligned}
\mathrm{dist}(w_n, x_n) &= \mathrm{dist}\big(\Pi_C\big(\mathrm{Exp}_{x_n}(-\theta_n \mathrm{Exp}_{x_n}^{-1} x_{n-1})\big), \Pi_C x_n\big) \\
&\le \mathrm{dist}\big(\mathrm{Exp}_{x_n}\big(-\theta_n \mathrm{Exp}_{x_n}^{-1} x_{n-1}\big), x_n\big) \\
&= \theta_n \big\| \mathrm{Exp}_{x_n}^{-1} x_{n-1} \big\|.
\end{aligned} \tag{4.19}$$

Take the limit on both sides of (4.19). According to (4.1), we get $\lim_{n\to\infty} \mathrm{dist}(w_n, x_n) = 0$. Since $\lim_{n\to\infty} \mathrm{dist}(w_n, \Pi_{H_n} w_n) = 0$, the assumptions in Lemma 4.4 are satisfied. Thus we obtain $\lim_{n\to\infty} \mathrm{dist}(w_n, z_n) = 0$. Since $\lim_{k\to\infty} x_{n_k} = \widehat{x}$, it holds that

$$\lim_{k\to\infty} w_{n_k} = \lim_{k\to\infty} \Pi_{H_{n_k}} w_{n_k} = \lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} x_{n_k} = \widehat{x} \in C. \tag{4.20}$$

Let p_{n_k} be defined by (4.15). Denote $\widehat{p} := \lim_{k\to\infty} p_{n_k}$. According to Lemma 2.4 and (4.20), we have $\widehat{p} = \mathrm{Exp}_{\widehat{x}}(-V(\widehat{x}))$. By (2.1), for any point $x \in C$ it holds

$$0 \geq \left\langle \mathrm{Exp}_{z_{n_k}}^{-1} p_{n_k}, \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle = \left\langle \mathrm{Exp}_{z_{n_k}}^{-1} p_{n_k} + P_{w_{n_k}, z_{n_k}} V(w_{n_k}) - P_{w_{n_k}, z_{n_k}} V(w_{n_k}), \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle.$$

Thus, we obtain

$$\left\langle \mathrm{Exp}_{z_{n_k}}^{-1} p_{n_k} + P_{w_{n_k}, z_{n_k}} V(w_{n_k}), \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \leq \left\langle P_{w_{n_k}, z_{n_k}} V(w_{n_k}), \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle.$$

Using (2.7), (3.12), (4.16), and the above equality, we have

$$\begin{aligned}
&\left\langle V(w_{n_k}), P_{z_{n_k}, w_{n_k}} \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&= \left\langle P_{w_{n_k}, z_{n_k}} V(w_{n_k}), \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&\geq \left\langle \mathrm{Exp}_{z_{n_k}}^{-1} p_{n_k} + P_{w_{n_k}, z_{n_k}} V(w_{n_k}), \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&= \left\langle -\mathrm{Exp}_{p_{n_k}}^{-1} z_{n_k} + P_{w_{n_k}, p_{n_k}} V(w_{n_k}), P_{z_{n_k}, p_{n_k}} \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&= \left\langle -\mathrm{Exp}_{p_{n_k}}^{-1} z_{n_k} + \mathrm{Exp}_{p_{n_k}}^{-1} w_{n_k}, P_{z_{n_k}, p_{n_k}} \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle.
\end{aligned}$$

Taking the limit on both sides of the above inequality, it follows from Lemma 2.4 and (4.20) that

$$\begin{aligned}
\left\langle V(\widehat{x}), \mathrm{Exp}_{\widehat{x}}^{-1} x \right\rangle &= \left\langle \lim_{k\to\infty} V(w_{n_k}), \lim_{k\to\infty} P_{z_{n_k}, w_{n_k}} \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&= \lim_{k\to\infty} \left\langle V(w_{n_k}), P_{z_{n_k}, w_{n_k}} \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&\geq \lim_{k\to\infty} \left\langle -\mathrm{Exp}_{p_{n_k}}^{-1} z_{n_k} + \mathrm{Exp}_{p_{n_k}}^{-1} w_{n_k}, P_{z_{n_k}, p_{n_k}} \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&= \left\langle -\lim_{k\to\infty} \mathrm{Exp}_{p_{n_k}}^{-1} z_{n_k} + \lim_{k\to\infty} \mathrm{Exp}_{p_{n_k}}^{-1} w_{n_k}, \lim_{k\to\infty} P_{z_{n_k}, p_{n_k}} \mathrm{Exp}_{z_{n_k}}^{-1} x \right\rangle \\
&= \left\langle -\mathrm{Exp}_{\widehat{p}}^{-1} \widehat{z} + \mathrm{Exp}_{\widehat{p}}^{-1} \widehat{z}, P_{\widehat{x}, \widehat{p}} \mathrm{Exp}_{\widehat{x}}^{-1} x \right\rangle = 0.
\end{aligned}$$

By the arbitrariness of $x \in C$, we have $\widehat{x} \in \mathscr{S}$. The proof is complete. $\square$

For the distance sequences $\{\mathrm{dist}^2(w_n, x_*)\}$, $\{\mathrm{dist}^2(x_n, x_*)\}$, and $\{\mathrm{dist}^2(w_{n-1}, x_n)\}$, we obtain the following special relationship.

Lemma 4.6. *Let $\{x_n\}$ be an iterative sequence generated by Algorithm 3.2. Suppose that conditions (C1)-(C7) hold. Given a point $x_* \in \mathscr{S}$, there exists a positive constant $\widehat{M}_2 > 0$ such that*

$$\begin{aligned}
\mathrm{dist}^2(w_n, x_*) \leq \mathrm{dist}^2(x_n, x_*) &+ \theta_n 2\widehat{M}_2 \cdot \mathrm{dist}^2(x_n, x_{n-1}) \\
&+ \theta_n\big(\mathrm{dist}^2(x_n, x_*) - \mathrm{dist}^2(x_{n-1}, x_*)\big).
\end{aligned} \tag{4.21}$$

Proof. Let x_* be a point in $\mathscr{S}$. Since the sequence $\{x_n\}$ is bounded, $\sup_{n\geq 0} \mathrm{dist}(x_n, x_*) < \infty$. Set

$$\widehat{M}_2 := \sup_{n\geq 0} \frac{\sqrt{\kappa}\mathrm{dist}(x_*, x_n)}{\tanh(\sqrt{\kappa}\mathrm{dist}(x_*, x_n))}.$$

Since $\lim_{t\to 0} t/\tanh t = 1$ and $t/\tanh t$ increases monotonically on $(0,\infty)$, we obtain

$$1 \le \frac{\sqrt{\kappa}\,\mathrm{dist}(x_*, x_n)}{\tanh(\sqrt{\kappa}\,\mathrm{dist}(x_*, x_n))} \le \widehat{M}_2 < +\infty \quad \text{for all} \quad n \ge 0. \tag{4.22}$$

Denote $q_n := \mathrm{Exp}_{x_n}(-\theta_n \mathrm{Exp}_{x_n}^{-1} x_{n-1})$. By (2.6) and (3.5), we have

$$\mathrm{dist}(x_n, q_n) = \theta_n \cdot \mathrm{dist}(x_{n-1}, x_n), \quad w_n = \Pi_C q_n. \tag{4.23}$$

Let $\triangle(x_{n-1}, x_n, x_*)$ and $\triangle(x_n, q_n, x_*)$ be geodesic triangles in $\mathscr{H}$. Let α be the angle at x_n in the geodesic triangle $\triangle(x_n, q_n, x_*)$. Then the angle at x_n in the geodesic triangle $\triangle(x_{n-1}, x_n, x_*)$ is $\pi - \alpha$. Based on (2.10), (4.22), and (4.23), in the geodesic triangle $\triangle(x_n, q_n, x_*)$ we have

$$\begin{aligned}
\mathrm{dist}^2(x_*, q_n) &\le \frac{\sqrt{\kappa}\,\mathrm{dist}(x_*, x_n)}{\tanh(\sqrt{\kappa}\,\mathrm{dist}(x_*, x_n))}\mathrm{dist}^2(x_n, q_n) + \mathrm{dist}^2(x_*, x_n) \\
&\quad - 2\mathrm{dist}(x_n, q_n)\mathrm{dist}(x_*, x_n)\cos\alpha \\
&\le \widehat{M}_2 \cdot \mathrm{dist}^2(x_n, q_n) + \mathrm{dist}^2(x_*, x_n) \\
&\quad - 2\mathrm{dist}(x_n, q_n)\mathrm{dist}(x_*, x_n)\cos\alpha \\
&= \widehat{M}_2 \cdot \theta_n^2 \cdot \mathrm{dist}^2(x_{n-1}, x_n) + \mathrm{dist}^2(x_*, x_n) \\
&\quad + 2\theta_n \cdot \mathrm{dist}(x_{n-1}, x_n)\mathrm{dist}(x_*, x_n)\cos(\pi - \alpha).
\end{aligned} \tag{4.24}$$

According to (2.10), (4.22), and (4.23), in the geodesic triangle $\triangle(x_{n-1}, x_n, \overline{x})$ we have

$$\begin{aligned}
\mathrm{dist}^2(x_*, x_{n-1}) &\le \frac{\sqrt{\kappa}\,\mathrm{dist}(x_*, x_n)}{\tanh(\sqrt{\kappa}\,\mathrm{dist}(x_*, x_n))}\mathrm{dist}^2(x_{n-1}, x_n) + \mathrm{dist}^2(x_*, x_n) \\
&\quad - 2\mathrm{dist}(x_{n-1}, x_n)\mathrm{dist}(x_*, x_n)\cos(\pi - \alpha) \\
&\le \widehat{M}_2 \cdot \mathrm{dist}^2(x_{n-1}, x_n) + \mathrm{dist}^2(x_*, x_n) \\
&\quad - 2\mathrm{dist}(x_{n-1}, x_n)\mathrm{dist}(x_*, x_n)\cos(\pi - \alpha).
\end{aligned}$$

Combining (4.24) and the above inequality, we obtain

$$\begin{aligned}
\mathrm{dist}^2(x_*, q_n) &\le \widehat{M}_2 \theta_n^2 \cdot \mathrm{dist}^2(x_{n-1}, x_n) + \mathrm{dist}^2(x_*, x_n) \\
&\quad + \theta_n \big(\widehat{M}_2 \cdot \mathrm{dist}^2(x_{n-1}, x_n) + \mathrm{dist}^2(x_*, x_n) - \mathrm{dist}^2(x_*, x_{n-1})\big) \\
&= \mathrm{dist}^2(x_n, x_*) + \theta_n\big(\widehat{M}_2\theta_n + \widehat{M}_2\big) \cdot \mathrm{dist}^2(x_n, x_{n-1}) \\
&\quad + \theta_n\big(\mathrm{dist}^2(x_n, x_*) - \mathrm{dist}^2(x_{n-1}, x_*)\big) \\
&\le \mathrm{dist}^2(x_n, x_*) + 2\theta_n\widehat{M}_2 \cdot \mathrm{dist}^2(x_n, x_{n-1}) \\
&\quad + \theta_n\big(\mathrm{dist}^2(x_n, x_*) - \mathrm{dist}^2(x_{n-1}, x_*)\big).
\end{aligned} \tag{4.25}$$

Since $x_* \in \mathscr{S} \subseteq C$, we have $\Pi_C x_* = x_*$. By (2.2) and (4.23), we have

$$\mathrm{dist}(w_n, x_*) = \mathrm{dist}(\Pi_C q_n, \Pi_C x_*) \le \mathrm{dist}(q_n, x_*).$$

The inequality in (4.21) follows from (4.25) and the above inequality. This completes the proof. $\square$

By using the above lemmas and conclusions, we obtain the following global convergence property of Algorithm 3.2.

Theorem 4.1. *Let $\{x_n\}$ be an iterative sequence generated by Algorithm* 3.2*. Suppose that conditions (C1)-(C7) hold. Then the sequence $\{x_n\}$ converges to $\overline{x} := \Pi_{\mathscr{S}} u$.*

Proof. Since $\{x_n\}$ is bounded, there exists a positive constant $\widehat{M}_3 > 0$ such that

$$\operatorname{dist}(x_n, \overline{x}) \leq \widehat{M}_3 \quad \text{for all} \quad n \geq 0. \tag{4.26}$$

By using (4.21), (4.26), and the triangle inequality, we have

$$\begin{aligned}
\operatorname{dist}^2(w_n, \overline{x}) \leq\ & \operatorname{dist}^2(x_n, \overline{x}) + \theta_n 2\widehat{M}_2 \cdot \operatorname{dist}^2(x_n, x_{n-1}) \\
& + \theta_n \big(\operatorname{dist}(x_n, \overline{x}) - \operatorname{dist}(x_{n-1}, \overline{x})\big) \cdot \big(\operatorname{dist}(x_n, \overline{x}) + \operatorname{dist}(x_{n-1}, \overline{x})\big) \\
\leq\ & \operatorname{dist}^2(x_n, \overline{x}) + \theta_n 2\widehat{M}_2 \cdot \operatorname{dist}^2(x_n, x_{n-1}) + \theta_n 2\widehat{M}_3 \cdot \operatorname{dist}(x_n, x_{n-1}) \\
=\ & \operatorname{dist}^2(x_n, \overline{x}) + \theta_n \operatorname{dist}(x_n, x_{n-1}) \cdot \big(2\widehat{M}_2 \cdot \operatorname{dist}(x_n, x_{n-1}) + 2\widehat{M}_3\big).
\end{aligned} \tag{4.27}$$

Let $\triangle(u, \Pi_{H_n} w_n, \overline{x})$ be a geodesic triangle in $\mathscr{H}$ and $\triangle(u', (\Pi_{H_n} w_n)', \overline{x}')$ be its comparison triangle in $\mathbb{R}^2$. Let β_{n+1} be the angle at $\overline{x}$ in $\triangle(u, \Pi_{H_n} w_n, \overline{x})$ and β'_{n+1} be the angle at $\overline{x}'$ in $\triangle(u', (\Pi_{H_n} w_n)', \overline{x}')$. By (2.4), it holds $\beta_{n+1} \leq \beta'_{n+1}$, thus $\cos \beta'_{n+1} \leq \cos \beta_{n+1}$ for $n \geq 0$. According to (2.3) and (2.6), we have

$$\begin{aligned}
& \big\langle \operatorname{Exp}^{-1}_{\overline{x}} u, \operatorname{Exp}^{-1}_{\overline{x}} \big(\Pi_{H_n} w_n\big)\big\rangle \\
=\ & \big\|\operatorname{Exp}^{-1}_{\overline{x}} u\big\| \cdot \big\|\operatorname{Exp}^{-1}_{\overline{x}} \big(\Pi_{H_n} w_n\big)\big\| \cos \beta_{n+1} \\
=\ & \operatorname{dist}(\overline{x}, u) \cdot \operatorname{dist}(\overline{x}, \Pi_{H_n} w_n) \cos \beta_{n+1} \\
=\ & \|\overline{x}' - u'\|_2 \cdot \big\|\overline{x}' - \big(\Pi_{H_n} w_n\big)'\big\|_2 \cos \beta_{n+1} \\
\geq\ & \|\overline{x}' - u'\|_2 \cdot \big\|\overline{x}' - \big(\Pi_{H_n} w_n\big)'\big\|_2 \cos \beta'_{n+1} \\
=\ & \big\langle \overline{x}' - u', \overline{x}' - \big(\Pi_{H_n} w_n\big)'\big\rangle.
\end{aligned} \tag{4.28}$$

Based on (2.1) and (2.9), we obtain

$$\begin{aligned}
\operatorname{dist}^2\big(\Pi_{H_n} w_n, \overline{x}\big) =\ & \operatorname{dist}^2(w_n, \overline{x}) - \operatorname{dist}^2(\Pi_{H_n} w_n, w_n) \\
& + 2\big\langle \operatorname{Exp}^{-1}_{\Pi_{H_n} w_n} \overline{x}, \operatorname{Exp}^{-1}_{\Pi_{H_n} w_n} w_n \big\rangle \\
\leq\ & \operatorname{dist}^2(w_n, \overline{x}) - \operatorname{dist}^2(\Pi_{H_n} w_n, w_n).
\end{aligned}$$

Taking into account (2.2)-(2.6), (3.15), (4.27), (4.28), and the above inequality, we have

$$\begin{aligned}
\operatorname{dist}^2(x_{n+1}, \overline{x}) &= \operatorname{dist}^2\Big(\Pi_C\big(\operatorname{Exp}_u\big((1 - a_n)\operatorname{Exp}^{-1}_u(\Pi_{H_n} w_n)\big)\big), \overline{x}\Big) \\
&= \operatorname{dist}^2\Big(\Pi_C\big(\operatorname{Exp}_u\big((1 - a_n)\operatorname{Exp}^{-1}_u(\Pi_{H_n} w_n)\big)\big), \Pi_C \overline{x}\Big)
\end{aligned}$$

$$\begin{aligned}
&\le \mathrm{dist}^2\Big(\mathrm{Exp}_u\big((1-a_n)\mathrm{Exp}_u^{-1}(\Pi_{H_n} w_n)\big), \overline{x}\Big)\\
&\le \Big\|\big(\mathrm{Exp}_u\big((1-a_n)\mathrm{Exp}_u^{-1}(\Pi_{H_n} w_n)\big)' - \overline{x}'\Big\|^2\\
&= \big\|a_n u' + (1-a_n)(\Pi_{H_n} w_n)' - \overline{x}'\big\|_2^2\\
&= \big\|a_n(u' - \overline{x}') + (1-a_n)\big((\Pi_{H_n} w_n)' - \overline{x}'\big)\big\|_2^2\\
&= (1-a_n)^2\big\|(\Pi_{H_n} w_n)' - \overline{x}'\big\|_2^2 + a_n^2\|u' - \overline{x}'\|_2^2\\
&\quad + 2a_n(1-a_n)\big\langle \overline{x}' - u', \overline{x}' - (\Pi_{H_n} w_n)'\big\rangle\\
&\le (1-a_n)^2\mathrm{dist}^2(\Pi_{H_n} w_n, \overline{x}) + a_n^2\mathrm{dist}^2(u, \overline{x})\\
&\quad + 2a_n(1-a_n)\big\langle \mathrm{Exp}_{\overline{x}}^{-1}u, \mathrm{Exp}_{\overline{x}}^{-1}(\Pi_{H_n} w_n)\big\rangle\\
&\le (1-a_n)^2\big[\mathrm{dist}^2(w_n, \overline{x}) - \mathrm{dist}^2(\Pi_{H_n} w_n, w_n)\big] + a_n^2\mathrm{dist}^2(u, \overline{x})\\
&\quad + 2a_n(1-a_n)\big\langle \mathrm{Exp}_{\overline{x}}^{-1}u, \mathrm{Exp}_{\overline{x}}^{-1}(\Pi_{H_n} w_n)\big\rangle\\
&\le (1-a_n)^2\Big[\mathrm{dist}^2(x_n, \overline{x}) + \theta_n \mathrm{dist}(x_n, x_{n-1}) \cdot (2\widehat{M}_2 \cdot \mathrm{dist}(x_n, x_{n-1}) + 2\widehat{M}_3)\Big]\\
&\quad - (1-a_n)^2\mathrm{dist}^2(\Pi_{H_n} w_n, w_n) + a_n^2\mathrm{dist}^2(u, \overline{x})\\
&\quad + 2a_n(1-a_n)\big\langle \mathrm{Exp}_{\overline{x}}^{-1}u, \mathrm{Exp}_{\overline{x}}^{-1}(\Pi_{H_n} w_n)\big\rangle\\
&\le (1-a_n)\mathrm{dist}^2(x_n, \overline{x}) + \theta_n \mathrm{dist}(x_n, x_{n-1}) \cdot \big(2\widehat{M}_2 \cdot \mathrm{dist}(x_n, x_{n-1}) + 2\widehat{M}_3\big)\\
&\quad + a_n^2\mathrm{dist}^2(u, \overline{x}) - (1-a_n)^2\mathrm{dist}^2(\Pi_{H_n} w_n, w_n)\\
&\quad + 2a_n(1-a_n)\big\langle \mathrm{Exp}_{\overline{x}}^{-1}u, \mathrm{Exp}_{\overline{x}}^{-1}(\Pi_{H_n} w_n)\big\rangle, \qquad (4.29)
\end{aligned}$$

where the condition $0 < 1 - a_n < 1$ is used.

For the sake of simplicity, for each $n \ge 0$, let

$$\begin{aligned}
&\Phi_n := \mathrm{dist}^2(x_n, \overline{x}),\\
&\zeta_n := \frac{\theta_n}{a_n}\mathrm{dist}(x_n, x_{n-1}) \cdot \big(2\widehat{M}_2 \mathrm{dist}(x_n, x_{n-1}) + 2\widehat{M}_3\big)\\
&\qquad + a_n \mathrm{dist}^2(u, \overline{x}) + 2(1-a_n)\big\langle \mathrm{Exp}_{\overline{x}}^{-1}u, \mathrm{Exp}_{\overline{x}}^{-1}(\Pi_{H_n} w_n)\big\rangle,\\
&\vartheta_n := (1-a_n)^2\mathrm{dist}^2(\Pi_{H_n} w_n, w_n),\\
&\pi_n := a_n \zeta_n.
\end{aligned} \qquad (4.30)$$

Conditions (C1)-(C4) give $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. By (4.4), the sequence $\{\Pi_{H_n} w_n\}$ is bounded, so that $\{\langle \mathrm{Exp}_{\overline{x}}^{-1}u, \mathrm{Exp}_{\overline{x}}^{-1}(\Pi_{H_n} w_n)\rangle\}$ is also bounded. Since $\lim_{n\to\infty} a_n = 0$, it follows from (4.1) that

$$\begin{aligned}
\lim_{n\to\infty} \pi_n &= \lim_{n\to\infty} \theta_n \mathrm{dist}(x_n, x_{n-1}) \cdot \big(2\widehat{M}_2 \mathrm{dist}(x_n, x_{n-1}) + 2\widehat{M}_3\big)\\
&\quad + \lim_{n\to\infty} a_n^2 \mathrm{dist}^2(u, \overline{x}) + \lim_{n\to\infty} 2a_n(1-a_n)\big\langle \mathrm{Exp}_{\overline{x}}^{-1}u, \mathrm{Exp}_{\overline{x}}^{-1}(\Pi_{H_n} w_n)\big\rangle = 0.
\end{aligned}$$

According to Lemma 2.6, to complete the proof, it remains to show that for any subsequence $\{\vartheta_{n_k}\}$ of $\{\vartheta_n\}$, if $\lim_{k\to\infty} \vartheta_{n_k} = 0$ then $\limsup_{k\to\infty} \zeta_{n_k} \le 0$.

If $\lim_{k\to\infty} \vartheta_{n_k} = 0$, then we have

$$\lim_{k\to\infty} \operatorname{dist}(\Pi_{H_{n_k}} w_{n_k}, w_{n_k}) = 0. \tag{4.31}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\lim_{j\to\infty} x_{n_{k_j}} = \widehat{x}$ and

$$\limsup_{k\to\infty} \left\langle \operatorname{Exp}_{\overline{x}}^{-1} u, \operatorname{Exp}_{\overline{x}}^{-1}(\Pi_{H_{n_k}} w_{n_k})\right\rangle = \lim_{j\to\infty} \left\langle \operatorname{Exp}_{\overline{x}}^{-1} u, \operatorname{Exp}_{\overline{x}}^{-1}(\Pi_{H_{n_{k_j}}} w_{n_{k_j}})\right\rangle. \tag{4.32}$$

By (4.31) and Lemma 4.5, we have $\widehat{x} \in \mathscr{S}$. By (4.20), we have $\lim_{k\to\infty} \Pi_{H_{n_k}} w_{n_k} = \widehat{x}$. In addition, $\overline{x} = \Pi_{\mathscr{S}} u \in \mathscr{S}$. Based on (2.1), Lemma 2.4, and (4.32), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \left\langle \operatorname{Exp}_{\overline{x}}^{-1} u, \operatorname{Exp}_{\overline{x}}^{-1}(\Pi_{H_{n_k}} w_{n_k})\right\rangle \\
&= \lim_{j\to\infty} \left\langle \operatorname{Exp}_{\overline{x}}^{-1} u, \operatorname{Exp}_{\overline{x}}^{-1}(\Pi_{H_{n_{k_j}}} w_{n_{k_j}})\right\rangle \\
&= \left\langle \lim_{j\to\infty} \operatorname{Exp}_{\overline{x}}^{-1} u, \lim_{j\to\infty} \operatorname{Exp}_{\overline{x}}^{-1}(\Pi_{H_{n_{k_j}}} w_{n_{k_j}})\right\rangle \\
&= \left\langle \operatorname{Exp}_{\overline{x}}^{-1} u, \operatorname{Exp}_{\overline{x}}^{-1} \widehat{x}\right\rangle \le 0.
\end{aligned} \tag{4.33}$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} a_n = 0$, it follows from (4.1) that

$$\lim_{n\to\infty} \frac{\theta_n}{a_n} \operatorname{dist}(x_n, x_{n-1}) \cdot \left(2\widehat{M}_2 \operatorname{dist}(x_n, x_{n-1}) + 2\widehat{M}_3\right) + a_n \operatorname{dist}^2(u, \overline{x}) = 0. \tag{4.34}$$

Combining with (4.33) and (4.34), we obtain $\limsup_{k\to\infty} \zeta_{n_k} \le 0$. By Lemma 2.6, we have $\lim_{n\to\infty} \Phi_n = 0$, which implies that $\lim_{n\to\infty} x_n = \overline{x}$. This completes the proof. □

5. Numerical Experiments

In this section, numerical performance of Algorithm 3.2 for solving problem (1.2) is reported. To show the efficiency of Algorithm 3.2, we compare it with the Korpelevich method [25]. For simplicity, we write `Korpelevich` for the Korpelevich method. All numerical tests are carried out using MATLAB R2010a on a Lenovo Laptop Intel(R) Core(TM) i7-8550U with a 1.80 GHz CPU and 16-GB RAM.

For Algorithm 3.2, we set

$$\begin{aligned}
&\eta = 0.5, \qquad a_k = \frac{10^{-4}}{10^2 k + 1}, \quad \theta = \frac{1}{5}, \qquad \theta_k = \overline{\theta}_k, \\
&\epsilon_k = \frac{10^2}{(k+1)^2}, \quad \mu_k = \frac{1}{k+1}, \qquad \delta = 10^{-4}.
\end{aligned}$$

For the Korpelevich method, we set $\beta_k = 1$ and $\delta = 10^{-4}$. For comparison purposes, we repeat our experiments over 10 different randomly generated problems. Below, we write `CT`, `IT`, `NF`, and `Res` for the average total computing time in seconds, average number of iterations, average number of function evaluations, and average residual $\|r(w_k)\|$ in Algorithm 3.2 or $\|r(x_k)\|$ for the Korpelevich method at the final iterates of the corresponding algorithms, accordingly. Moreover, we write `Res0.` for the averaged residual $\|r(w_0)\|$ in Algorithm 3.2 or $\|r(x_0)\|$ for the Korpelevich method at the initial iterates of the corresponding algorithms. The stopping criteria for Algorithm 3.2 and Korpelevich's method are set to

$$\|r(w_k)\| \leq 10^{-6}, \quad \|r(x_k)\| \leq 10^{-6}.$$

Example 5.1 (cf. Tang & Huang [25]). Let

$$\mathbb{H}^n := \left\{p = [p_1, \ldots, p_n, p_{n+1}]^T \in \mathbb{R}^{n+1} \mid p_{n+1} > 0 \text{ and } \langle p, p\rangle = -1\right\},$$

where the metric of $\mathbb{H}^n$ is induced from the Lorentz metric of $\mathbb{R}^{n+1}$

$$\langle p, q\rangle := p_1 q_1 + \cdots + p_n q_n - p_{n+1} q_{n+1} \quad \text{for all} \quad p, q \in \mathbb{R}^{n+1}.$$

The sectional curvature of $\mathbb{H}^n$ is equal to -1 at each point.

The vector field

$$V : \mathbb{H}^2 \to T\mathbb{H}^2 : p \mapsto \left(p_1 p_3, p_2 p_3, p_3^2 - 1\right)^T \tag{5.1}$$

is a monotone on $\mathbb{H}^2$. Let

$$C := \left\{p = [p_1, p_2, p_3]^T \in \mathbb{H}^2 \mid 1 \leq p_3 \leq 2\right\}.$$

The set C is a closed convex subset of $\mathbb{H}^2$. We consider the variational inequality problem associated with V and C. This problem has a unique solution $[0, 0, 1]^T$.

For Algorithm 3.2, we set $\tau_k = 1/4 - 1/(k+1)$ for $k \geq 0$. The starting points for Korpelevich's method and Algorithm 3.2 are randomly generated by using the MATLAB built-in function `randn`

$$c = \texttt{randn}(2,1), \quad x_0 = \left[c, \sqrt{c^T c + 1}\right]^T, \quad x_{-1} = x_0.$$

The prescribed point u for Algorithm 3.2 is randomly generated by using the MATLAB built-in function `randn`

$$v = \texttt{randn}(2,1), \quad u = \left[v, \sqrt{v^T v + 1}\right]^T.$$

For Example 5.1, the averaged Riemannian distance dist$(x_n, [0,0,1]^T)$ at the initial and final iterates of the corresponding algorithms are denoted by '`Rdist0.`' and '`Rdist.`', respectively.

Numerical results for Example 5.1 are given in Table 1. From these numerical results, we observe that Algorithm 3.2 performs better than Korpelevich's method in terms of computational time and iteration numbers. In addition, the convergence histories of Korpelevich's method and Algorithm 3.2 for Example 5.1 are given in Fig. 1. The left subfigure depicts the Riemannian distance the residual versus the number of iterations, and the right subfigure depicts the logarithm of the Riemannian distance dist$(x_n, [0,0,1]^T)$ versus the number of iterations.

Table 1: Numerical results for Example 5.1.

Algorithm	CT	IT	NF	Res0	Res	Rdist0	Rdist
Korpelevich	0.1728 s	19.8	40.6	1.1362	7.7459×10^{-7}	0.9476	7.6493×10^{-7}
Algorithm 3.2	0.0750 s	5.5	6.5	1.1362	3.8806×10^{-7}	0.9476	3.5788×10^{-7}

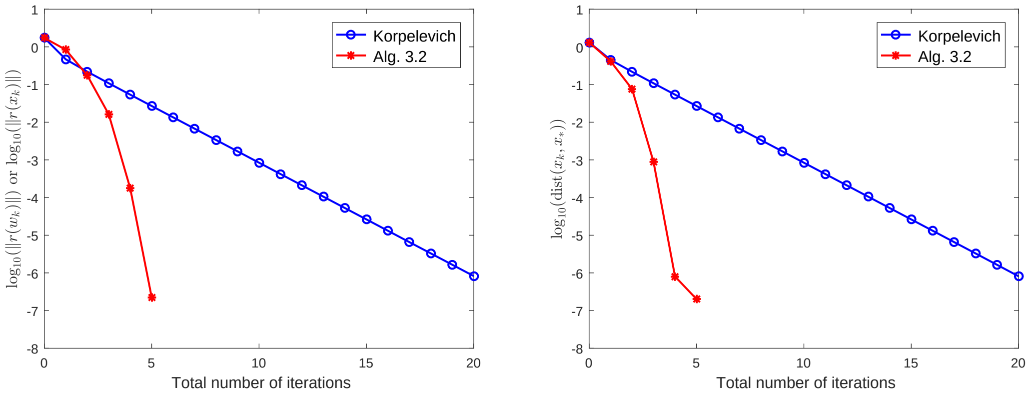


Figure 1: Convergence history of one test.

Example 5.2 (cf. Ansari & Babu [2]). Let $\mathbb{R}_{++} := \{x \in \mathbb{R} \mid x > 0\}$ be endowed with the following Riemannian metric:

$$\langle u, v \rangle := \frac{u * v}{x^2} \quad \text{for all} \quad u, v \in T_x\mathbb{R}_{++}.$$

The sectional curvature of $\mathbb{R}_{++}$ is equal to 0 at each point. The vector field

$$V : \mathbb{R}_{++} \to T\mathbb{R}_{++} : x \mapsto x \ln x \tag{5.2}$$

is a monotone on $\mathbb{R}_{++}$. Let

$$C := \{x \in \mathbb{R}_{++} \mid x \geq 0.5\}.$$

The set C is a closed convex subset of $\mathbb{R}_{++}$. We consider the variational inequality problem associated with V and C. The solution set of this problem is $\{1\}$.

For Algorithm 3.2, we set $\tau_k = 10^6$. The starting points for Korpelevich's method and Algorithm 3.2 are randomly generated by the MATLAB built-in function `rand`

$$x_0 = 6 + \texttt{rand}(1), \quad x_{-1} = x_0.$$

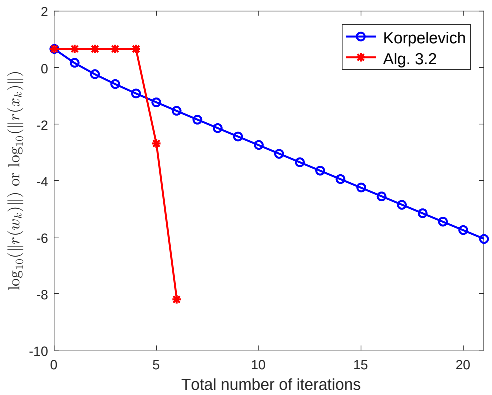
The prescribed point u for Algorithm 3.2 is randomly generated by the MATLAB built-in function `rand`

$$u = 16 + \texttt{rand}(1).$$

For Example 5.2, the averaged Riemannian distance $\mathrm{dist}(x_n, 1)$ at the initial and final iterates of the corresponding algorithms are denoted by '`Rdist0.`' and '`Rdist.`', respectively.

Table 2: Numerical results for Example 5.2.

Algorithm	CT	IT	NF	Res0	Res	Rdist0	Rdist
Korpelevich	0.0002 s	21.0	43.0	4.7767	8.9255×10^{-7}	1.8718	8.9255×10^{-7}
Algorithm 3.2	0.0001 s	6.2	7.2	4.7767	7.7680×10^{-9}	1.8718	9.2703×10^{-9}



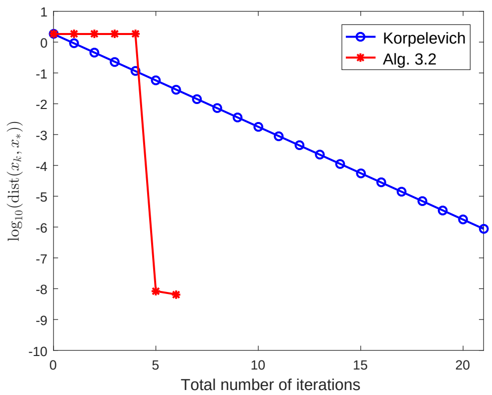


Figure 2: Convergence history of one test.

Numerical results for Example 5.2 are given in Table 2. The convergence histories of Korpelevich's method and Algorithm 3.2 for Example 5.2 are given in Fig. 2. From Example 5.2, Algorithm 3.2 also performs better than Korpelevich's method.

6. Concluding Remarks

The problem of solving variational inequality problems for univalued pseudomonotone vector field on Hadamard manifolds is concerned in this paper. To solve this problem, a special inertial Halpern-type algorithm is proposed. The global convergence of this new method is established under some mild assumptions. Specially, the lower boundedness of the sectional curvature of the underlying Hadamard manifold is required. In the future research, we will consider the generalization of this method for solving variational inequality problems for set-valued vector fields on general Riemannian manifolds.

Acknowledgments

We are very grateful to the two anonymous referees for their valuable comments on the paper, which have considerably improved the paper.

The research of the first author is supported by the Zhejiang Provincial Natural Science Foundation of China (LY21A010004) and by the National Natural Science Foundation of

China (11701514). The research of the second author is supported by the research grants MYRG2019-00042-FST and CPG2023-00023-FST from the University of Macau. The research of the third author is supported by the Zhejiang Provincial Natural Science Foundation of China (LY21A010010) and the National Natural Science Foundation of China (11601112).

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