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Ground States of Two-component Bose-Einstein Condensates with an Internal Atomic Josephson Junction

Weizhu Bao[1,*] and Yongyong Cai[2]

[1] *Department of Mathematics and Center for Computational Science and Engineering, National University of Singapore, Singapore 117543.*
[2] *Department of Mathematics, National University of Singapore, Singapore 117543.*

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Abstract. In this paper, we prove existence and uniqueness results for the ground states of the coupled Gross-Pitaevskii equations for describing two-component Bose-Einstein condensates with an internal atomic Josephson junction, and obtain the limiting behavior of the ground states with large parameters. Efficient and accurate numerical methods based on continuous normalized gradient flow and gradient flow with discrete normalization are presented, for computing the ground states numerically. A modified backward Euler finite difference scheme is proposed to discretize the gradient flows. Numerical results are reported, to demonstrate the efficiency and accuracy of the numerical methods and show the rich phenomena of the ground sates in the problem.

AMS subject classifications: 35Q55, 49J45, 65N06, 65N12, 65Z05, 81-08

Key words: Bose-Einstein condensate, coupled Gross-Pitaevskii equations, two-component, ground state, normalized gradient flow, internal atomic Josephson junction, energy.

1. Introduction

Since the first realization of Bose-Einstein condensates (BEC) in a dilute bosonic gas in 1995 [1,8,15], theoretical studies and numerical methods have been extensively developed for the single-component BEC [4,5,23,29]. Recently, BEC with multiple species have been realized in experiments [17, 18, 25, 26, 28, 32, 34] and some interesting phenomenona absent in single-component BEC were observed in experiments and studied in theory [2, 6, 7, 9, 16, 20, 24]. The simplest multi-component BEC is the binary mixture, which can be used as a model for producing coherent atomic beams (also called atomic laser) [30, 31]. The first experiment for two-component BEC was performed in JILA with $|F=2, m_f=2\rangle$

*Corresponding author. *Email addresses:* `bao@math.nus.edu.sg` (W. Bao), `caiyongyong@nus.edu.sg` (Y. Cai)

and $|1,-1\rangle$ spin states of ^{87}Rb [28]. Since then, extensive experimental and theoretical studies of two-component BEC have been carried out in the last several years [3,10,19,27, 35,38].

At temperature T much smaller than the critical temperature T_c and after proper nondimensionalization and dimension reduction [29,38], a two-component BEC with an internal atomic Josephson junction (or an external driving field) can be well described by the following coupled Gross-Pitaevskii equations (CGPEs) in dimensionless form [29,37,38]:

$$\begin{aligned} i\partial_t\psi_1 &= \left[-\frac{1}{2}\nabla^2 + V(\mathbf{x}) + \delta + (\beta_{11}|\psi_1|^2 + \beta_{12}|\psi_2|^2)\right]\psi_1 + \lambda\psi_2, \\ i\partial_t\psi_2 &= \left[-\frac{1}{2}\nabla^2 + V(\mathbf{x}) + (\beta_{12}|\psi_1|^2 + \beta_{22}|\psi_2|^2)\right]\psi_2 + \lambda\psi_1, \qquad \mathbf{x}\in\mathbb{R}^d. \end{aligned} \tag{1.1}$$

Here, t is time, $\mathbf{x}\in\mathbb{R}^d$ ($d=1,2,3$) is the Cartesian coordinate vector, $\Psi(\mathbf{x},t) := (\psi_1(\mathbf{x},t), \psi_2(\mathbf{x},t))^T$ is the complex-valued macroscopic wave function, $V(\mathbf{x})$ is the real-valued external trapping potential, λ is the effective Rabi frequency to realize the internal atomic Josephson junction (JJ) by a Raman transition, δ is the detuning constant for the Raman transition, and $\beta_{jl} = \beta_{lj} = \frac{4\pi N a_{jl}}{a_0}$ ($j,l=1,2$) are interaction constants with N being the total number of particles in the two-component BEC, a_0 being the dimensionless spatial unit and $a_{jl} = a_{lj}$ ($j,l=1,2$) being the s-wave scattering lengths between the jth and lth component (positive for repulsive interaction and negative for attractive interaction). It is necessary to ensure that the wave function is properly normalized - specifically, we require

$$\|\Psi\|^2 := \|\Psi\|_2^2 = \int_{\mathbb{R}^d}\left[|\psi_1(\mathbf{x},t)|^2 + |\psi_2(\mathbf{x},t)|^2\right]d\mathbf{x} = 1. \tag{1.2}$$

The dimensionless CGPEs (1.1) conserve the total mass or normalization, i.e.

$$N(t) := \|\Psi(\cdot,t)\|^2 = N_1(t) + N_2(t) \equiv \|\Psi(\cdot,0)\|^2 = 1, \qquad t\geq 0, \tag{1.3}$$

with

$$N_j(t) = \|\psi_j(\mathbf{x},t)\|^2 := \|\psi_j(\mathbf{x},t)\|_2^2 = \int_{\mathbb{R}^d}|\psi_j(\mathbf{x},t)|^2\,d\mathbf{x}, \qquad t\geq 0, \qquad j=1,2, \tag{1.4}$$

and the energy

$$E(\Psi) = E_0(\Psi) + 2\lambda\int_{\mathbb{R}^d}\mathrm{Re}(\psi_1\bar{\psi}_2)\,d\mathbf{x}, \tag{1.5}$$

with $\bar{f}$ and $\mathrm{Re}(f)$ denoting the conjugate and real part of a function f, respectively, and

$$\begin{aligned} E_0(\Psi) = \int_{\mathbb{R}^d}\Bigg[&\frac{1}{2}\left(|\nabla\psi_1|^2 + |\nabla\psi_2|^2\right) + V(\mathbf{x})(|\psi_1|^2 + |\psi_2|^2) + \delta|\psi_1|^2 + \frac{1}{2}\beta_{11}|\psi_1|^4 \\ &+ \frac{1}{2}\beta_{22}|\psi_2|^4 + \beta_{12}|\psi_1|^2|\psi_2|^2\Bigg]d\mathbf{x}. \end{aligned} \tag{1.6}$$