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Tri-Diagonal Preconditioner for Toeplitz Systems from Finance

Hong-Kui Pang*, Ying-Ying Zhang and Xiao-Qing Jin

Department of Mathematics, University of Macau, Macao.

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Abstract. We consider a nonsymmetric Toeplitz system which arises in the discretization of a partial integro-differential equation in option pricing problems. The preconditioned conjugate gradient method with a tri-diagonal preconditioner is used to solve this system. Theoretical analysis shows that under certain conditions the tri-diagonal preconditioner leads to a superlinear convergence rate. Numerical results exemplify our theoretical analysis.

AMS subject classifications: 65F10, 65M06, 91B70, 47B35

Key words: European call option, partial integro-differential equation, nonsymmetric Toeplitz system, normalized preconditioned system (matrix), tri-diagonal preconditioner.

1. Introduction

It is well known that the option price for a European call option under Merton's jump diffusion model is determined by the expected value [1,10]

$$v(t,x) \equiv e^{-r(\bar{T}-t)}\mathbf{E}_{\mathbb{Q}}\left[\left(e^{x+L_{\bar{T}-t}}-K\right)^{+}\right], \tag{1.1}$$

where t is the time, x is the logarithmic price, $\mathbb{Q}$ is a risk-neutral measure, r is a risk-free interest rate, $\bar{T}$ is the maturity time, K is the strike price, and $L_{\bar{T}-t}$ is a Lévy process. As an alternative, the option value $v(t,x)$ can also be obtained by solving a partial integro-differential equation (PIDE) [8] as follows:

$$\begin{cases} v_t + \dfrac{\sigma^2}{2}v_{xx} + \left(r - \dfrac{\sigma^2}{2} - \lambda\eta\right)v_x - (r+\lambda)v + \lambda\displaystyle\int_{-\infty}^{\infty} v(t,x+y)\phi(y)\mathrm{d}y = 0, \\ v(\bar{T},x) = H(e^x), \quad \forall\, x \in \mathbb{R}, \end{cases} \tag{1.2}$$

*Corresponding author. *Email address:* ya87402@umac.mo (H.-K. Pang)

where $v(t,x) \in C^{1,2}\left((0,\bar{T}] \times \mathbb{R}\right) \cap C^{0}\left([0,\bar{T}] \times \mathbb{R}\right)$, $\phi(x) = \frac{e^{-(x-\mu_J)^2/2\sigma_J^2}}{\sqrt{2\pi}\sigma_J}$ is the probability density function of the Gaussian distribution, the parameters σ, r, λ, μ_J, σ_J, $\eta = e^{\mu_J+\sigma_J^2/2} - 1$ are constants, and $H(\cdot)$ is the payoff function.

There are many works [1,3,10,11] dealing with numerical solutions of (1.2). Recently Sachs and Strauss [10] eliminated the convection term in this PIDE and discretized the transformed equation implicitly by using finite differences with uniform mesh. The resulting linear system is a dense Toeplitz system $T_n\mathbf{x} = \mathbf{b}$. They solved this system by using the preconditioned conjugate gradient (PCG) method with circulant preconditioners.

In Merton's model, jump sizes are normally distributed with mean μ_J and standard deviation σ_J. With $\mu_J = 0$, discretizing the PIDE without the convection term yields a symmetric Toeplitz system [10,11], while for $\mu_J \neq 0$, the resulting system $T_n\mathbf{x} = \mathbf{b}$ is a nonsymmetric Toeplitz system. In [10,11], only the case of $\mu_J = 0$ was considered. In this paper, we discuss a more general case of $\mu_J \neq 0$. We consider applying the conjugate gradient (CG) method to the following normalized preconditioned system

$$\left(L_n^{-1}T_n\right)^* \left(L_n^{-1}T_n\right)\mathbf{x} = \left(L_n^{-1}T_n\right)^* L_n^{-1}\mathbf{b},$$

where the preconditioner L_n is a tri-diagonal matrix. We show that all the eigenvalues of the normalized preconditioned matrix $\left(L_n^{-1}T_n\right)^* \left(L_n^{-1}T_n\right)$ are clustered around one. Thus the convergence rate of the CG method is superlinear, when applied to solving the normalized preconditioned system. We see from numerical results in Section 4 that the tri-diagonal preconditioner works very well.

2. Discretization of PIDE

For Merton's model, the corresponding PIDE is of the following form on introducing $w(\tau,\xi) \equiv v\left(\bar{T} - \tau, \xi - \zeta\tau\right)$ [10]:

$$\begin{cases} w_\tau - \dfrac{\sigma^2}{2} w_{\xi\xi} + (r+\lambda)w - \lambda \displaystyle\int_{-\infty}^{\infty} w(\tau,z)\phi(z-\xi)\mathrm{d}z = 0, \\ \\ w(0,\xi) = H(e^{\xi}), \quad \forall\ \xi \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $w \in C^{1,2}\left((0,\bar{T}] \times \mathbb{R}\right) \cap C^{0}\left([0,\bar{T}] \times \mathbb{R}\right)$, $\zeta = r - \sigma^2/2 - \lambda\eta$ is a constant, the parameters σ, r, λ, μ_J, σ_J, η and the probability density function of the Gaussian distribution $\phi(x)$ are the same as in (1.2). Hence, the option value $v(t,x)$ in Merton's model can be determined by solving (2.1).

To solve (2.1) numerically, one can use a domain truncation and a finite-difference discretization in space, and the second order backward differentiation formula (BDF2) in time. The domain of ξ is usually chosen to be $\Omega \equiv \left(\xi_-, \xi_+\right)$. For a European call option, the boundary conditions [1] are

$$\begin{cases} w(\tau,\xi) \to 0, & \xi \to -\infty, \\ w(\tau,\xi) \sim Ke^{\xi-\zeta\tau} - Ke^{-r\tau}, & \xi \to +\infty. \end{cases}$$

This motivates the introduction of the function

$$
\begin{aligned}
R\left(\tau,\xi,\xi_+\right) &= \int_{\xi_+}^{+\infty} \left(Ke^{z-\zeta\tau} - Ke^{-r\tau}\right)\phi(z-\xi)\mathrm{d}z \\
&= Ke^{\xi-\zeta\tau+\mu_J+\sigma_J^2/2}\Phi\left(\frac{\xi-\xi_+ +\mu_J+\sigma_J^2}{\sigma_J}\right) - Ke^{-r\tau}\Phi\left(\frac{\xi-\xi_+ +\mu_J}{\sigma_J}\right),
\end{aligned} \tag{2.2}
$$

where $\Phi\left(y\right)$ is the cumulative normal distribution

$$
\Phi\left(y\right) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}\mathrm{d}x. \tag{2.3}
$$

The expression (2.2) is used in the discretization of the integral term of (2.1).

Similar to [1, 10], we consider a uniform mesh in space and in time - i.e.

$$
\begin{cases}
\xi_i = \xi_- + (i-1)h & \text{with } h = \left(\xi_+ - \xi_-\right)/(n+1) = 2\hat{x}/(n+1), \quad i = 1,2,\dots,n+2, \\
\tau_m = mk & \text{with } k = \bar{T}/q, \quad m = 0,1,\dots,q.
\end{cases}
$$

Let $w_i^m \approx w(\tau_m,\xi_i)$ and $\phi_{i,j} \equiv \phi\left(\xi_j - \xi_i\right)$. The integral term in (2.1) is approximated by the composite trapezoidal rule on Ω and the estimate (2.2) on $\mathbb{R}\backslash\Omega$. For the time variable and space variable we use the following approximations:

$$
\begin{aligned}
w_\tau(\tau_m,\xi_i) &\approx \begin{cases}
\left(\frac{3}{2}w_i^m - 2w_i^{m-1} + \frac{1}{2}w_i^{m-2}\right)/k, & m \geq 2, \\
\left(w_i^m - w_i^{m-1}\right)/k, & m = 1,
\end{cases} \\
w_{\xi\xi}\left(\tau_m,\xi_i\right) &\approx \left(w_{i+1}^m - 2w_i^m + w_{i-1}^m\right)/h^2.
\end{aligned}
$$

The initial solution vector is $\mathbf{w}^0 = \left(w_1^0,\dots,w_{n+2}^0\right)^T = \left(H(e^{\xi_1}),\dots,H(e^{\xi_{n+2}})\right)^T$. With the known values w_1^m and w_{n+2}^m from the boundary conditions, we obtain an $n\times n$ linear system with the coefficient matrix T_n which is Toeplitz. For detail of the discretization, we refer to [10].

More precisely, the diagonals of T_n in terms of n and q are given by

$$
\begin{cases}
t_0^{(n)} = \dfrac{\sigma^2\bar{T}(n+1)^2}{4\hat{x}^2 q} + \dfrac{(r+\lambda)\bar{T}}{q} + \dfrac{3}{2} - \dfrac{2\hat{x}\lambda\bar{T}}{\sqrt{2\pi}\sigma_J q(n+1)}e^{-\frac{\mu_J^2}{2\sigma_J^2}}, \\
t_{\pm1}^{(n)} = -\dfrac{\sigma^2\bar{T}(n+1)^2}{8\hat{x}^2 q} - \dfrac{2\hat{x}\lambda\bar{T}}{\sqrt{2\pi}\sigma_J q(n+1)}e^{-\frac{(\mp\frac{2\hat{x}}{n+1}-\mu_J)^2}{2\sigma_J^2}}, \\
t_{\pm j}^{(n)} = -\dfrac{2\hat{x}\lambda\bar{T}}{\sqrt{2\pi}\sigma_J q(n+1)}e^{-\frac{(\mp\frac{j}{n+1}2\hat{x}-\mu_J)^2}{2\sigma_J^2}}, \quad 2 \leq j \leq n-1.
\end{cases} \tag{2.4}
$$

From (2.4), we see that the entries of T_n depend on n. For various grid numbers n, we obtain a family of Toeplitz systems. Hence, we write the resulting systems as

$$T_n^{(n)}\mathbf{w}^m = \mathbf{b}^m, \quad m = 1, \ldots, q, \tag{2.5}$$

where $\mathbf{w}^m = \left(w_2^m, \ldots, w_{n+1}^m\right)^T \in \mathbb{R}^n$ and $\mathbf{b}^m = \left(b_2^m, \ldots, b_{n+1}^m\right)^T \in \mathbb{R}^n$ is the right hand side. Obviously, the coefficient matrix $T_n^{(n)}$ is a nonsymmetric Toeplitz matrix when $\mu_J \neq 0$.

3. Solving Normalized Preconditioned System

We solve (2.5) by applying the CG method to the following normalized preconditioned system

$$\left[(L_n^{(n)})^{-1}T_n^{(n)}\right]^* \left[(L_n^{(n)})^{-1}T_n^{(n)}\right]\mathbf{w}^m = \left[(L_n^{(n)})^{-1}T_n^{(n)}\right]^* (L_n^{(n)})^{-1}\mathbf{b}^m, \tag{3.1}$$

where $L_n^{(n)}$ is a tri-diagonal Toeplitz matrix with diagonals given by

$$\begin{cases} l_0^{(n)} = t_0^{(n)} = \dfrac{\sigma^2 \bar{T}(n+1)^2}{4\hat{x}^2 q} + \dfrac{(r+\lambda)\bar{T}}{q} + \dfrac{3}{2} - \dfrac{2\hat{x}\lambda\bar{T}}{\sqrt{2\pi}\sigma_J q(n+1)} e^{-\frac{\mu_J^2}{2\sigma_J^2}}, \\ l_1^{(n)} = l_{-1}^{(n)} = -\dfrac{\sigma^2\bar{T}(n+1)^2}{8\hat{x}^2 q}, \\ l_j^{(n)} = l_{-j}^{(n)} = 0, \quad 2 \leq j \leq n-1. \end{cases} \tag{3.2}$$

Let $M_n^{(n)}$ be a Toeplitz matrix with diagonals given by

$$\begin{cases} m_0^{(n)} = 0, \\ m_{\pm 1}^{(n)} = -\dfrac{2\hat{x}\lambda\bar{T}}{\sqrt{2\pi}\sigma_J q(n+1)} e^{-\frac{(\mp\frac{2\hat{x}}{n+1} - \mu_J)^2}{2\sigma_J^2}}, \\ m_{\pm j}^{(n)} = -\dfrac{2\hat{x}\lambda\bar{T}}{\sqrt{2\pi}\sigma_J q(n+1)} e^{-\frac{(\mp\frac{j}{n+1}2\hat{x} - \mu_J)^2}{2\sigma_J^2}}, \quad 2 \leq j \leq n-1. \end{cases} \tag{3.3}$$

Then

$$T_n^{(n)} = L_n^{(n)} + M_n^{(n)} \tag{3.4}$$

for every $n \geq 1$. By using a technique (FGF) provided in [11], one can prove the following lemma.

Lemma 3.1. *Let $L_n^{(n)}$ and $M_n^{(n)}$ be the Toeplitz matrix with diagonals given by* (3.2) *and* (3.3) *respectively, $q = \mathcal{O}((n+1)^\alpha)$ with $\alpha > 0$. Then for any $0 < \varepsilon < 1/2$ there exists an $N(\varepsilon) > 0$ such that for all $n > N$ $\|(L_n^{(n)})^{-1}\|_2 \leq 1$ and $\|M_n^{(n)}\|_2 < \varepsilon$.*

We therefore have the following main result.

Theorem 3.1. *Let* $T_n^{(n)}$*and* $L_n^{(n)}$ *be given by* (2.4) *and* (3.2) *respectively,* $q = \mathscr{O}((n+1)^\alpha)$ *with* $\alpha > 0$. *Then for any* $0 < \varepsilon < 1/2$ *there exists an* $N(\varepsilon) > 0$ *such that for all* $n > N$ *all the eigenvalues of* $\left[(L_n^{(n)})^{-1}T_n^{(n)}\right]^* \left[(L_n^{(n)})^{-1}T_n^{(n)}\right]$ *are inside the interval* $[1-\delta,\ 1+\delta]$ *with* $\delta = 2\varepsilon + \varepsilon^2$.

Proof. We have by (3.4),

$$(L_n^{(n)})^{-1}T_n^{(n)} = (L_n^{(n)})^{-1}(L_n^{(n)} + M_n^{(n)}) = I_n + W_n^{(n)},$$

where $W_n^{(n)} = (L_n^{(n)})^{-1}M_n^{(n)}$. By Lemma 3.1, for any $0 < \varepsilon < 1/2$, there exists an $N(\epsilon) > 0$ such that for all $n > N$, $\|(L_n^{(n)})^{-1}\|_2 \le 1$ and $\|M_n^{(n)}\|_2 < \varepsilon$. Then

$$\|W_n^{(n)}\|_2 \le \|(L_n^{(n)})^{-1}\|_2 \cdot \|M_n^{(n)}\|_2 < \varepsilon. \tag{3.5}$$

By (3.5), Weyl's theorem and the fact that

$$\left[(L_n^{(n)})^{-1}T_n^{(n)}\right]^* \left[(L_n^{(n)})^{-1}T_n^{(n)}\right] = I_n + (W_n^{(n)})^* + W_n^{(n)} + (W_n^{(n)})^* W_n^{(n)},$$

we know that all the eigenvalues of the normalized preconditioned matrix

$$\left[(L_n^{(n)})^{-1}T_n^{(n)}\right]^* \left[(L_n^{(n)})^{-1}T_n^{(n)}\right]$$

are inside the interval $[1-\delta,\ 1+\delta]$ with $\delta = 2\varepsilon + \varepsilon^2$. $\square$

Thus, by Corollary 1.11 in [2] or by the Theorem 1.15 in [6], we know that the convergence rate of the CG method when applied to (3.1) is superlinear.

4. Numerical Experiments

In this Section, we give numerical experiments with two preconditioners. All computations are carried out in MATLAB version 2008a on a Dell Inspiron 530 computer with Intel® Core™2 Quad CPU Q6600 @2.40GHz and 2.00 GB of RAM. The analytical expression of the European call option in (1.1) has been found for Merton's model [7]:

$$v(t,x) = \tilde{w}(t, Ke^x) = \tilde{w}(t,s) = \sum_{m=0}^{\infty} \frac{e^{-\lambda(1+\eta)\tau}\left[\lambda(1+\eta)\tau\right]^m}{m!} V_{BS}(\tau, s, K, r_m, \sigma_m), \tag{4.1}$$

where $\tau = \bar{T} - t$, $\sigma_m^2 = \sigma^2 + m\sigma_J^2/\tau$, $\eta = e^{\mu_J + \sigma_J^2/2} - 1$, $r_m = r - \lambda\eta + m \cdot \log(1+\eta)/\tau$ and $V_{BS}(\tau, s, K, r, \sigma) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$ with $d_1 = \frac{\log(s/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$, and Φ given by (2.3). The parameters λ, r, σ, μ_J, σ_J are constants related to Merton's model. We truncate the series (4.1) to 50 terms, which is adequate for the needed accuracy.

In the experiments, we solve the linear system (2.5) by applying the CG method to its corresponding normalized preconditioned system (3.1). The tri-diagonal preconditioner

Table 1: Number of iterations for different preconditioners with $q = \mathcal{O}(n)$.

n	q	l^∞ error	None	Strang	Tri-diagonal
64	5	8.99e-03	28	6	5
128	10	2.28e-03	47	6	5
256	20	5.73e-04	83	7	4
512	40	1.43e-04	152	7	4
1024	80	3.59e-05	283	8	3
2048	160	8.98e-06	533	8	3

given by (3.2) and Strang's circulant preconditioner [2,5,6] are used. The parameters in Merton's model are chosen to be

$$\hat{x} = 5, \quad \lambda = 0.6, \quad \mu_J = -0.6, \quad \sigma_J = 0.5, \quad \bar{T} = 0.5, \quad r = 0.05, \quad \sigma = 0.6, \quad K = 1.$$

The stopping criterion is defined by $\frac{\|\mathbf{r}^{(k)}\|_2}{\|\mathbf{r}^{(0)}\|_2} < 10^{-8}$, where $\mathbf{r}^{(k)}$ is the residual vector after the kth iteration. The initial guess for the normalized preconditioned system is chosen to be the solution from the last time step.

Numbers of iterations for different preconditioners are shown in Table 1. The column "l^∞ error" refers to the infinity norm of the difference between the numerical solution vector and the analytical solution evaluated at the grid points at the final time $\tau = \bar{T}$. "None", "Strang", and "Tri-diagonal" stand for no preconditioner, Strang's preconditioner, and tri-diagonal preconditioner, respectively. The number of iterations in Table 1 is obtained from solving the last system (i.e. the qth system in (2.5)). A careful investigation reveals that it is almost equal to the average numbers of iterations obtained from solving systems (2.5). From Table 1, we see that both Strang's preconditioner and tri-diagonal preconditioner lead to a fast convergence rate, with the tri-diagonal preconditioner proving somewhat more efficient.

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