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Tri-Diagonal Preconditioner for Toeplitz Systems from Finance

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Abstract. We consider a nonsymmetric Toeplitz system which arises in the discretization of a partial integro-differential equation in option pricing problems. The preconditioned conjugate gradient method with a tri-diagonal preconditioner is used to solve this system. Theoretical analysis shows that under certain conditions the tri-diagonal preconditioner leads to a superlinear convergence rate. Numerical results exemplify our theoretical analysis.

AMS subject classifications: 65F10, 65M06, 91B70, 47B35

Key words: European call option, partial integro-differential equation, nonsymmetric Toeplitz system, normalized preconditioned system (matrix), tri-diagonal preconditioner.

1. Introduction

It is well known that the option price for a European call option under Merton's jump diffusion model is determined by the expected value [1, 10]

$$v(t,x) \equiv e^{-r(\bar{T}-t)}\mathbf{E}_{\mathbb{Q}}\left[\left(e^{x+L_{\bar{T}-t}}-K\right)^{+}\right], \tag{1.1}$$

where t is the time, x is the logarithmic price, $\mathbb{Q}$ is a risk-neutral measure, r is a risk-free interest rate, $\bar{T}$ is the maturity time, K is the strike price, and $L_{\bar{T}-t}$ is a Lévy process. As an alternative, the option value $v(t,x)$ can also be obtained by solving a partial integro-differential equation (PIDE) [8] as follows:

$$\begin{cases} v_t+\dfrac{\sigma^2}{2}v_{xx}+\left(r-\dfrac{\sigma^2}{2}-\lambda\eta\right)v_x-(r+\lambda)v+\lambda\displaystyle\int_{-\infty}^{\infty}v(t,x+y)\phi(y)\mathrm{d}y=0, \\ v(\bar{T},x)=H(e^x), \quad \forall\ x\in\mathbb{R}, \end{cases} \tag{1.2}$$

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where $v(t,x) \in C^{1,2}\left((0,\bar{T}] \times \mathbb{R}\right) \cap C^0\left([0,\bar{T}] \times \mathbb{R}\right)$, $\phi(x) = \frac{e^{-(x-\mu_J)^2/2\sigma_J^2}}{\sqrt{2\pi}\sigma_J}$ is the probability density function of the Gaussian distribution, the parameters σ, r, λ, μ_J, σ_J, $\eta = e^{\mu_J+\sigma_J^2/2} - 1$ are constants, and $H(\cdot)$ is the payoff function.

There are many works [1,3,10,11] dealing with numerical solutions of (1.2). Recently Sachs and Strauss [10] eliminated the convection term in this PIDE and discretized the transformed equation implicitly by using finite differences with uniform mesh. The resulting linear system is a dense Toeplitz system $T_n\mathbf{x} = \mathbf{b}$. They solved this system by using the preconditioned conjugate gradient (PCG) method with circulant preconditioners.

In Merton's model, jump sizes are normally distributed with mean μ_J and standard deviation σ_J. With $\mu_J = 0$, discretizing the PIDE without the convection term yields a symmetric Toeplitz system [10,11], while for $\mu_J \neq 0$, the resulting system $T_n\mathbf{x} = \mathbf{b}$ is a nonsymmetric Toeplitz system. In [10,11], only the case of $\mu_J = 0$ was considered. In this paper, we discuss a more general case of $\mu_J \neq 0$. We consider applying the conjugate gradient (CG) method to the following normalized preconditioned system

$$\left(L_n^{-1}T_n\right)^*\left(L_n^{-1}T_n\right)\mathbf{x} = \left(L_n^{-1}T_n\right)^* L_n^{-1}\mathbf{b},$$

where the preconditioner L_n is a tri-diagonal matrix. We show that all the eigenvalues of the normalized preconditioned matrix $\left(L_n^{-1}T_n\right)^*\left(L_n^{-1}T_n\right)$ are clustered around one. Thus the convergence rate of the CG method is superlinear, when applied to solving the normalized preconditioned system. We see from numerical results in Section 4 that the tri-diagonal preconditioner works very well.

2. Discretization of PIDE

For Merton's model, the corresponding PIDE is of the following form on introducing $w(\tau,\xi) \equiv v\left(\bar{T} - \tau, \xi - \zeta\tau\right)$ [10]:

$$\begin{cases} w_\tau - \dfrac{\sigma^2}{2}w_{\xi\xi} + (r+\lambda)w - \lambda\displaystyle\int_{-\infty}^{\infty} w(\tau,z)\phi(z-\xi)\mathrm{d}z = 0, \\ w(0,\xi) = H(e^{\xi}), \quad \forall\ \xi \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $w \in C^{1,2}\left((0,\bar{T}] \times \mathbb{R}\right) \cap C^0\left([0,\bar{T}] \times \mathbb{R}\right)$, $\zeta = r - \sigma^2/2 - \lambda\eta$ is a constant, the parameters σ, r, λ, μ_J, σ_J, η and the probability density function of the Gaussian distribution $\phi(x)$ are the same as in (1.2). Hence, the option value $v(t,x)$ in Merton's model can be determined by solving (2.1).

To solve (2.1) numerically, one can use a domain truncation and a finite-difference discretization in space, and the second order backward differentiation formula (BDF2) in time. The domain of ξ is usually chosen to be $\Omega \equiv \left(\xi_-, \xi_+\right)$. For a European call option, the boundary conditions [1] are

$$\begin{cases} w(\tau,\xi) \to 0, & \xi \to -\infty, \\ w(\tau,\xi) \sim Ke^{\xi-\zeta\tau} - Ke^{-r\tau}, & \xi \to +\infty. \end{cases}$$