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Optimal Production Control in Stochastic Manufacturing Systems with Degenerate Demand

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Abstract. The paper studies the production inventory problem of minimizing the expected discounted present value of production cost control in manufacturing systems with degenerate stochastic demand. We have developed the optimal inventory production control problem by deriving the dynamics of the inventory-demand ratio that evolves according to a stochastic neoclassical differential equation through Ito's Lemma. We have also established the Riccati based solution of the reduced (one- dimensional) HJB equation corresponding to production inventory control problem through the technique of dynamic programming principle. Finally, the optimal control is shown to exist from the optimality conditions in the HJB equation.

AMS subject classifications: 49N05, 60H10, 90B30, 90B05

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1. Introduction

Let us consider the dynamics of the state equation which says that the inventory at time t is increased by the production rate and decreased by the demand rate, and can be written as

$$dI(t) = [k(t) - D(t)]dt, \quad I(0) = I, \ \ I > 0, \tag{1.1}$$

and the demand equation with the production rate is described by the Brownian motion

$$dD(t) = AD(t)dt + \sigma D(t)dw(t), \quad D(0) = D, \ \ D > 0. \tag{1.2}$$

Here $I(t)$ is the inventory level for production rate at time t (state variable), $D(t)$ is the demand rate at time t, A is a non-zero constant, σ is the non-zero constant diffusion coefficient, $k(t) \geq 0$ represents the production rate at time t (control variable), w_t is

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a one-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathscr{F}, P)$ endowed with the natural filtration $\mathscr{F}_t$ generated by $\sigma(w(s), s \leq t)$, $I(0)$ is the initial value of inventory level and $D(0)$ is the initial value of the demand rate.

In this paper, the optimization problem involves minimizing the expected discounted cost control

$$J(k(t)) = E\left[\int_0^\infty e^{-\beta t}\left\{f(I(t)) + k(t)^2\right\}dt\right] \tag{1.3}$$

over $k \in \mathscr{K}$ for all $t \geq 0$ where, $\mathscr{K}$ denotes the class of all admissible controls of production processes, $\beta > 0$ is the constant non-negative discount rate.

We assume f is a continuous, non-negative, convex function satisfying the polynomial growth condition such that

$$0 \leq f(I(t)) \leq L(1 + |I(t)|^n), \quad I(t) \in \mathbf{R}, \quad \forall t \geq 0, \quad n \in \mathbf{N}_+ \tag{1.4}$$

for some constant $L > 0$.

By the Principle of Optimality, it is natural that u solves the general (two-dimensional) Hamilton Jacobi-Bellman (HJB) equation

$$\begin{aligned} &-\beta u(I(t), D(t)) + \frac{1}{2}\sigma^2 D(t)^2 u_{DD} + AD(t)u_D - D(t)u_I + F^*(u_I) + h(I(t)) = 0, \\ &u(0, D) = 0, \quad I(t) > 0, D(t) > 0, \end{aligned} \tag{1.5}$$

where $F^*(u_I) = \min_{k(t)\geq 0}\{k(t)^2 + k(t)u_I\}$, u_I, u_D, u_{DD} are partial derivatives of $u(I(t), D(t))$ with respect to $I(t)$ and $D(t)$, and $F^*(I)$ is the Legendre transform of $F(I)$, - i.e. $F^*(I(t)) = \min_{k\geq 0}\{k^2(t) + k(t)I(t)\}$.

There exists a $v \in (0, \infty)$ such that $u(I(t), D(t)) = D(t)^2 v(I(t)/D(t)), D(t) > 0$. Since

$$\begin{aligned} u_I &= D(t)v'(I(t)/D(t)), \\ u_D &= 2D(t)v(I(t)/D(t)) - I(t)v'(I(t)/D(t)), \\ u_{DD} &= 2v(I(t)/D(t)) - 2(I(t)/D(t))v'(I(t)/D(t)) + (I(t)/D(t))^2 v''(I(t)/D(t)), \end{aligned}$$

on setting $z(t) = I(t)/D(t)$ and substituting into (1.5) yields, we have

$$\begin{aligned} &-\tilde{\beta}v(z(t)) + \frac{1}{2}\sigma^2 z^2(t)v''(z(t)) + \tilde{A}z(t)v'(z(t)) + F^*(v'(z(t))) + f(z(t)) = 0, \\ &v(0) = 0, \quad z(t) > 0, \forall t \geq 0 \end{aligned} \tag{1.6}$$

where $\tilde{\beta} = -\beta + \sigma^2 + 2A, \tilde{A} = -A + \sigma^2$, $F^*(v'(z(t))) = \min_{p(t)\geq 0}\{(p(t)+1)^2 + p(t)v'(z(t))\}$, and $F^*(z(t))$ is the Legendre transform of $F(z(t))$ - i.e. $F^*(z(t)) = \min_{p(t)\geq 0}\{(p(t)+1)^2 + p(t)z(t)\}$.

The control problem of production planning in manufacturing systems with discount rate has been studied by many authors - e.g. Fleming, Sethi and Soner (1987), Sethi and Zhang (1994). The Bellman equation associated with the production inventory control problems is quite different, and it is treated by Bensoussan et al. (1984) for the one-dimensional manufacturing systems with an unbounded control region. This type of optimization problem has been studied also by Morimoto and Kawaguchi (2002) for renewable resources, Baten and Morimoto (2005) for degenerate diffusions, Baten and Sobhan (2007) for one-sector neoclassical growth model with the Constant Elasticity of Substitution (CES) function. The optimality can be shown by an extension of the results given in Morimoto and Okada (1999) and Sethi et al. (1997).

The main purpose of the paper is to give a synthesis of optimal production control with an application to production control theory, via the first derivatives of the function v to the reduced HJB equation. We apply the dynamic programming principle of Bellman (1957) to this inventory-production control problem, to establish the solution form of the value function to the reduced (one-dimensional) HJB equation (1.6).

This paper is organized as follows. In Section 2, we develop the optimal inventory production control model by deriving the dynamics of the inventory-demand ratio that evolves according to stochastic neoclassical differential equation through Itô′s Lemma. In Section 3, we give the solution form of the value function from the reduced (one-dimensional) HJB equation. In Section 4, an optimal cost control is shown to exist from the optimality conditions in the HJB equation. Finally, Section 5 contains some concluding remarks.

2. Development of the Reduced Optimal Production Control Model

In order to develop the reduced optimal inventory production control model, the dynamics of the state equation of inventory level (1.1) can be reduced to a one-dimensional process by working in intensive (per capita) variables. Define $z(t) \equiv I(t)/D(t)$, inventory-demand ratio, and $p(t) \equiv k(t)/D(t)$, per capita production.

To determine the stochastic differential equation for the inventory-demand ratio, $z(t) \equiv I(t)/D(t)$, we apply Itô's Lemma as follows:

$$z(t) = I(t)/D(t) \equiv G(D,t), \quad \partial G(D,t)/\partial D(t) = -I(t)/D(t)^2 = -z(t)/D(t),$$
$$\partial^2 G(D,t)/\partial D(t)^2 = 2I(t)/D(t)^3 = 2z(t)/D(t)^2,$$
$$\partial G(D,t)/\partial t = \dot{I(t)}/D(t) = (k(t) - D(t))/D(t) = k(t)/D(t) - 1 = p(t) - 1.$$

From Itô's Lemma,

$$dz(t) = \frac{\partial G(D,t)}{\partial D(t)} dD(t) + \frac{\partial G(D,t)}{\partial t} dt + \frac{1}{2}\frac{\partial^2 G(D,t))}{\partial D(t)^2}(dD(t))^2. \tag{2.1}$$

From (1.1), we have that $(dD(t))^2 = \sigma^2(D(t))^2 dt$. Substituting the above expressions into (2.1), we have the inventory-demand ratio at time t governed by the stochastic neoclassical differential equation for demand

$$\begin{aligned} dz(t) = & -\frac{z(t)}{D(t)}[AD(t)dt + \sigma D(t)dw(t)] + [p(t) - 1]dt + z(t)\sigma^2 dt \\ \le & [\tilde{A}z(t) + p(t)]dt - \sigma z(t)dw(t), \quad z(0) = z,\ z > 0. \end{aligned}$$

where $\tilde{A} = -A + \sigma^2$.

The inventory production control problem becomes

$$\tilde{J}(p(t) = E\left[\int_0^\infty e^{-\tilde{\beta}t}\left\{f(z(t)) + (p(t)+1)^2\right\}dt\right]; \quad p(t) \geq 0 \tag{2.2}$$

subject to the degenerate stochastic differential equation

$$dz(t) = [\tilde{A}z(t) + p(t)]dt - \sigma z(t)dw(t), \quad z(0) = z,\ z > 0. \tag{2.3}$$

To find the solution form of the value function from the HJB equation (1.6), we refer to Prato (1984) and Prato and Ichikawa (1990) for the degenerate linear optimal control problems related to the Riccati equation in the case where $f(z(t)) = Lz^2(t)$, $L > 0$. Then the objective function (2.2) becomes

$$\tilde{J}(p(t)) = E\left[\int_0^\infty e^{-\tilde{\beta}t}\left\{z^2(t) + (p(t)+1)^2\right\}dt\right], \tag{2.4}$$

subject to degenerate stochastic differential equation (2.3).

In order to ensure the integrability of $\tilde{J}(p(t))$, we can assume that for $n \geq 2$

$$-\tilde{\beta} + 2n\widetilde{A} + n(2n-1)\sigma^2 < 0. \tag{2.5}$$

This condition (2.5) is needed for the integrability of $z(t)$ or $\tilde{J}(p(t))$.

3. Value Function

3.1. Value function solution to the HJB equation

The value function can be defined as a function whose value is the minimum value of the objective function of the production inventory control problem (2.2) and (2.4) for the manufacturing system - i.e.

$$\begin{aligned} V(z) &= \inf_{p(t)\geq 0} E\left[\int_0^\infty e^{-\tilde{\beta}t}\left\{z^2(t) + (p(t)+1)^2\right\}dt\right] \\ &= \inf_{p(t)\geq 0} \tilde{J}(p(t),\ \forall t \geq 0. \end{aligned} \tag{3.1}$$

In order obtain the solution form of the value function to the HJB equation (1.6), we apply the dynamic programming principle of optimality to minimize the expression inside the bracket of (1.6) [i.e. $F^*(v'(z)) = \min_{p(t)\geq 0}\{(p(t)+1)^2 + p(t)v'(z)\}$], and taking the derivative on the expression inside the bracket with respect to $p(t)$ as well as setting it to zero. Then this procedure yields

$$p^*(t) = -\frac{v'(z(t))}{2} - 1. \tag{3.2}$$

Substituting (3.2) into (1.6) yields

$$-\tilde{\beta}v(z(t))+\frac{1}{2}\sigma^2 z^2(t)v''(z(t))+\tilde{A}z(t)v'(z(t))-\frac{(v'(z(t)))^2}{4}-v'(z(t))+z^2(t)=0 \quad (3.3)$$

known as the HJB equation. This is a partial differential equation which has a solution form

$$v(z(t))=b(t)z^2(t). \quad (3.4)$$

Then

$$v'(z(t))=2b(t)z(t), \qquad v''(z(t))=2b(t). \quad (3.5)$$

Substituting (3.4) and (3.5) into (3.3) yields

$$(1-\tilde{\rho}b+\sigma^2 b-b^2+2b\tilde{A})z^2(t)-2bz(t)=0. \quad (3.6)$$

Since (3.6) must hold for any value of $z(t)$, we must have the Riccati equation

$$b^2-b(2\tilde{A}+\sigma^2+\tilde{\beta})-1=0,$$

from which we obtain

$$b=\frac{-(2\tilde{A}+\sigma^2+\tilde{\beta})\pm\sqrt{(2\tilde{A}+\sigma^2+\tilde{\beta})^2+4}}{2}.$$

Thus (3.4) is a solution form of the value function from the HJB equation (3.3).

4. An Application to Production Control Theory

In this Section, we shall study the production control problem to minimize the cost (3.1) over the class $\mathscr{K}_p$ of all progressively measurable processes $p(t)$ such that $0\leq p(t)\leq k(t)$ and

$$\lim_{t\to\infty}\frac{1}{t}E\left[e^{-\tilde{\beta}t}|z(t)|^n\right]=0 \quad (4.1)$$

for the response $z(t)$ to $p(t)$.

Let us consider the stochastic differential equation

$$dz^*(t)=[\tilde{A}z^*(t)+\psi_p(v'(z^*(t)))]dt-\sigma z^*(t)dw_t, \quad z^*(0)=z,\ z>0, \quad (4.2)$$

where $\psi_p(v'(z))=\arg\min F(v'(z))=-\frac{(v'(z(t)))^2}{4}-v'(z(t))$, i.e.,

$$\psi_p(v'(z))=\begin{cases} p-1 & \text{if } v'(z)\leq -2p \\ -\frac{v'(z)}{2}-1 & \text{if } -2p<v'(z)\leq 0 \\ -1 & \text{if } 0<v'(z) \end{cases} \quad (4.3)$$

and $\psi_p(v'(z)):=-\frac{v'(z)}{2}-1$ is the minimizer of $\min_{p\geq 0}\{(p(t)+1)^2+p(t)v'((z))\}$.

Lemma 4.1. *Under (2.5) and for each $n \in \mathbf{N}_+$, there exists $L > 0$ such that*

$$E\left[e^{-\tilde{\beta}t}|z^*(t)|^{2n}\right] \leq L(1+t).$$

Proof: Since we have by Itô′s formula

$$d(z^*(t))^2 = 2z^*(t)\left[(\tilde{A}z^*(t) + \psi_p(v'(z^*(t))))dt - \sigma z^*(t)dw_t\right] + \sigma^2(z^*(t))^2dt,$$

and

$$d(\tilde{z}(t))^2 = 2\tilde{z}(t)\left[\tilde{A}\tilde{z}(t)dt - \sigma\tilde{z}(t)dw_t\right] + \sigma^2(\tilde{z}(t))^2dt.$$

Now we can see $(z^*(t))^2 \leq (\tilde{z}(t))^2$ by the comparison theorem (Ikeda and Watanabe 1981). Since the explosion time $\sigma = \inf\{t : |z^*(t)| = \infty\}$, we have $\infty = (z^*_\sigma)^2 \leq (\tilde{z}_\sigma)^2$. Hence $\sigma = \infty$. By the monotonicity of $\psi(v'(z^*(t)))$, the uniqueness of (4.2) holds. Thus we conclude that (4.2) admits a unique strong solution $(z(t)^*)$ [cf. Ikeda and Watanabe 1981, Chap. 4, Thm. 1.1] with

$$E\left[|z^*(t)|^{2n}\right] < \infty. \tag{4.4}$$

Using Itô′s formula, by (2.5) and taking expectation on both sides we have

$$\begin{aligned} E\left[e^{-\tilde{\beta}t}|z^*(t)|^{2n}\right] \leq & |z|^{2n} + 2nE\left[\int_0^t e^{-\tilde{\beta}s}\psi_p(v'(z^*(s)))|z^*(s)|^{2n-1}\mathrm{sgn}(z^*(s))ds\right] \\ = & |z|^{2n} + E\left[\int_0^t e^{-\tilde{\beta}s}\mathbf{Z}^{(\mathbf{n})}(s)ds\right], \end{aligned} \tag{4.5}$$

where $\mathbf{Z}^{(\mathbf{n})}(\mathbf{s}) = 2n\psi_p(v'(z^*(s)))|z^*(s)|^{2n-1}\mathrm{sgn}(z^*(s))$.

By (4.3) it is easily seen that $z\psi_p(v'(z)) \leq (p-1)|z|$ if $|z| \geq c$, for sufficiently large $c > 0$. Clearly

$$\sup_s E\left[e^{-\tilde{\beta}s}\mathbf{Z}^{(\mathbf{n})}(s)1_{(|z^*(s)|<c)}\right] < \infty.$$

Also

$$E\left[e^{-\tilde{\beta}s}\mathbf{Z}^{(\mathbf{n})}(s)1_{(|z^*(s)|\geq c)}\right] \leq E\left[e^{-\tilde{\beta}s}(p-1)|z|1_{(|z^*(s)|\geq c)}\right].$$

In addition, from (4.4) we see that the right-hand side of equation (4.5) is bounded from above. This completes the proof.

Theorem 4.1. *We assume (1.4) and (2.5). Then the optimal cost control p^*_t is given by*

$$p^*(t) = \psi_p(v'(z^*(t))),$$

and the minimum value by

$$\tilde{J}(p^*(t)) = v(z),$$

where $z^*(t)$ is defined by (4.2).

Proof: We first note by (4.3) that $F_p^*(z(t)) = \min_{p\geq 0}\{(p(t)+1)^2 + p(t)z(t)\}$, and the minimum is attained by $\psi_p(z(t))$. We apply Itô's formula for convex functions (Karatzas and Shreve 1991, p. 219) and taking expectation on both sides to obtain

$$E\left[e^{-\tilde{\beta}t}v(z^*(t))\right] = v(z) + E\left[\int_0^t e^{-\tilde{\beta}s}(-\tilde{\beta}v(z^*(s)) + \tilde{A}z^*(s)v'(z^*(s)) + p_s^* v'(z^*(s)) + \frac{1}{2}\sigma^2(z^*(s))^2 v''(z^*(s)))ds\right].$$

By virtue of (1.6)

$$E\left[e^{-\tilde{\beta}t}v(z^*(t))\right] = v(\hat{z}) - E\left[\int_0^t e^{-\tilde{\beta}s}\left\{f(z^*(s)) + (p^*(s)+1)^2\right\}ds\right]. \tag{4.6}$$

Choose $n \in \mathbf{N}_+$ such that $2n > m$. By (1.4) and Lemma 4.1, we have

$$\begin{aligned}\frac{1}{t}E\left[e^{-\tilde{\beta}t}v(z^*(t))\right] &\leq \frac{L}{t}\left(1 + E\left[e^{-\tilde{\beta}mt}|z^*(t)|^m\right]\right)\\ &\leq \frac{L}{t}\left(1 + E\left(\left[e^{-2n\tilde{\beta}t}|z^*(t)|^{2n}\right]\right)^{\frac{m}{2n}}\right)\\ &\leq \frac{L}{t}\left(1 + (L(1+t))^{\frac{m}{2n}}\right) < \infty\end{aligned}$$

which implies

$$\liminf_{t\to\infty}\frac{1}{t}E\left[e^{-\tilde{\beta}t}v(z^*(t))\right] = 0. \tag{4.7}$$

Hence $z^*(t)$ satisfies (4.1). Applying the relation (4.7) to (4.6) we obtain

$$E\left[\int_0^\infty e^{-\tilde{\beta}s}\left\{f(z^*(s)) + (p^*(s)+1)^2\right\}ds\right] \leq v(z)$$

from which $\tilde{J}(p^*(t)) \leq v(z(t))$. By (1.4), we have $\tilde{J}(p^*(t)) \leq v(z(t)) < \infty$, hence

$$p^* = (p^*(t)) \in \mathscr{K}_p.$$

Clearly $F_p^*(z(t)) \leq (p(t)+1)^2 + p(t)z(t)$ for every $p(t) \in \mathscr{K}_p$. Again following the same construction as above and by (1.6), we have

$$E\left[e^{-\tilde{\beta}t}v(z(t))\right] \geq v(z) - E\left[\int_0^t e^{-\tilde{\beta}s}\left\{f(z(s)) + (p(s)+1)^2\right\}ds\right].$$

Now, by (4.1)

$$E\left[\int_0^\infty e^{-\tilde{\beta}s}\left\{f(z(s)) + (p(s)+1)^2\right\}ds\right] \geq v(z).$$

Thus we deduce $J(p^*(t)) \geq v(z)$. The proof is complete.

5. Concluding Remarks

In this paper, we have studied the production control problem of minimizing the expected discounted value of cost control in stochastic manufacturing systems with degenerate demand. We have shown that there exists a solution form of the value function to the reduced (one-dimensional) Hamilton Jacobi-Bellman Equation. We have analyzed the optimal policy, which is characterized as a minimizing selector of the HJB equation.

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