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On Perturbation Bounds for the Joint Stationary Distribution of Multivariate Markov Chain Models

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Abstract. Multivariate Markov chain models have previously been proposed in for studying dependent multiple categorical data sequences. For a given multivariate Markov chain model, an important problem is to study its joint stationary distribution. In this paper, we use two techniques to present some perturbation bounds for the joint stationary distribution vector of a multivariate Markov chain with s categorical sequences. Numerical examples demonstrate the stability of the model and the effectiveness of our perturbation bounds.

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Key words: Multivariate Markov chain models, stationary distribution vector, condition number, relative bound.

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1. Introduction and Notations

In many real world problems, there are situations where one would like to consider a number of Markov chains $\{\mathbf{X}_{t,i}\}_{i=1}^{s}$ together at the same time, particularly in the analysis of multiple categorical data sequences. The state of the i-th chain $\mathbf{X}_{t+1,i}$ at time $(t+1)$ often depends not only on $X_{t,i}$ but also on $\{\mathbf{X}_{t,1},\ldots,\mathbf{X}_{t,i-1},\mathbf{X}_{t,i+1},\ldots,\mathbf{X}_{t,s}\}$, resulting in a multivariate Markov chain model. In a conventional model where the multivariate Markov chain has s chains and each chain has the same set of m states, the total number of states is $O(m^s)$. Consequently, one needs to develop simplified multivariate Markov chain models that can capture both the inter-relations and intra-relations among the given chains with a relatively low number of model parameters. A multivariate Markov chain model was proposed for this purpose in Ref. [2], and applied to demand forecasting. Ref. [3] provides

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a detailed survey of multivariate Markov chain models. The purpose of this paper is to propose some perturbation bounds on the joint stationary distribution vector for multivariate Markov chain models. To consider the stability of the joint probability distribution of a multivariate Markov chain, we need to analyse the change of the joint distribution under a small perturbation of the transition matrix, and there are many results on the perturbation theories of Markov chains.

Let us denote the transition probability matrix of a finite irreducible homogeneous Markov chain by P. The stationary distribution vector of P is the unique positive vector $\boldsymbol{\pi}$ satisfying $\pi = P\pi$ and $\sum_j \pi_j = 1$. Suppose that the matrix P is perturbed to the matrix $\tilde{P}$, the transition probability matrix of another finite irreducible homogeneous Markov chain. On denoting the stationary distribution vector of $\tilde{P}$ by $\tilde{\pi}$, the goal is to describe the change $\tilde{\pi} - \pi$ in the stationary distribution in terms of the change $E \equiv \tilde{P} - P$ in the transition probability matrix. For some vector norms, we have

$$\|\tilde{\pi} - \pi\| \le \kappa \|E\|$$

for various different condition numbers κ — e.g. see [5, 7, 8, 10, 11, 13, 15, 16]. However, to the best of our knowledge there is no discussion on perturbation theory for multivariate Markov chain models.

In this paper, we analyse the effects of a small perturbation to the joint stationary distributions of a finite irreducible multivariate Markov chain, when Q is the joint transition probability matrix of such a multivariate Markov chain and

$$\mathbf{\Pi} = (\pi^{(1)T}, \pi^{(2)T}, \dots, \pi^{(s)T})^T$$

is the joint stationary distribution vector satisfying

$$Q\mathbf{\Pi} = \mathbf{\Pi} \quad \text{and} \quad \sum_{i=1}^{m} [\pi^{(j)}]_i = 1, \quad 1 \le j \le s\,.$$

Our goal is to describe the effect on $\tilde{\mathbf{\Pi}}$ when Q is perturbed by a matrix E such that

$$\tilde{Q} = Q + E$$

is the joint transition probability matrix of another irreducible multivariate Markov chain. We first propose perturbation bounds for the joint stationary distribution of a multivariate Markov chain. This is particularly important because the model parameters are different when different estimation methods are employed. Some condition numbers and interesting numerical measures will also be provided. However, while it is theoretically possible to compute condition numbers κ, it is usually expensive — and another possibility is to propose a relative bound that is easy to find without computing $\mathbf{\Pi}$.

The following notation is used throughout this paper:

- for any $\xi \in \mathbb{C}^N$, ξ_i denotes the ith element;
- $\mathbf{1}_l = (1, 1, \dots, 1)^T$ and $\mathbf{0}_l = (0, 0, \dots, 0)^T$ are column vectors with dimension l; and

- for any $M \in \mathbb{C}^{N\times N}$, M_{ij} denotes the element in the ith row and jth column, M_{i*} denotes the ith row of M, and $M_{(i)}$ denotes the submatrix of M on deleting the ith row.

The rest of the paper is organised as follows. In Section 2, we review multivariate Markov chain models, and then propose some properties of the joint transition probability matrix. In Sections 3 and 4, we discuss the perturbation of the joint stationary distribution for the multivariate Markov chain model, and give both absolute and relative perturbation bounds on the joint stationary distribution vector. Numerical examples to demonstrate the effectiveness of our perturbation bounds are presented in Section 5, and brief concluding remarks in Section 6 address some future research issues.

2. A Review of Multivariate Markov Chain Models

A multivariate Markov chain model was proposed by Ching et al. [2, 3] to model the interdependent behaviour of multiple categorical sequences generated by similar sources. When there are s categorical sequences and each has m possible states, it is assumed that the state probability distribution of the j-th sequence at time $t+1$ depends on the state probabilities of all of the sequences (including the j-th) at time t. More precisely, the following relationship is assumed:

$$\mathbf{x}_{t+1}^{(k)} = \sum_{j=1}^{s} \lambda_{kj} P^{(kj)} \mathbf{x}_t^{(j)} \quad \text{for} \quad k = 1, 2, \ldots, s\,,$$

where

$$\lambda_{kj} \geq 0,\ 1 \leq j, k \leq s \quad \text{and} \quad \sum_{j=1}^{s} \lambda_{kj} = 1 \quad \text{for} \quad k = 1, 2, \ldots, s \tag{2.1}$$

and $\mathbf{x}_0^{(j)}$ is the initial probability distribution of the j-th sequence. The state probability distribution of the j-th sequence $\mathbf{x}_{t+1}^{(j)}$ at the time $t+1$ depends on the weighted average of $P^{(jk)}$, which is a one-step transition probability matrix from the states at time t in the j-th sequence to the states in the k-th sequence at time $t+1$. In matrix form, we write

$$\begin{aligned}
\mathbf{X}_{t+1} &\equiv \left(\mathbf{x}_{t+1}^{(1)T},\ \mathbf{x}_{t+1}^{(2)T},\ \ldots,\ \mathbf{x}_{t+1}^{(s)T} \right)^T \\
&= \begin{pmatrix} \lambda_{11}P^{(11)} & \lambda_{12}P^{(12)} & \ldots & \lambda_{1s}P^{(1s)} \\ \lambda_{21}P^{(21)} & \lambda_{22}P^{(22)} & \ldots & \lambda_{2s}P^{(2s)} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{s1}P^{(s1)} & \lambda_{s2}P^{(s2)} & \ldots & \lambda_{ss}P^{(ss)} \end{pmatrix} \begin{pmatrix} \mathbf{x}_t^{(1)} \\ \mathbf{x}_t^{(2)} \\ \vdots \\ \mathbf{x}_t^{(s)} \end{pmatrix} \\
&\equiv Q\mathbf{X}_t\,.
\end{aligned}$$

Let $N = m \times s$ and consider $Q \in \mathbb{R}^{N\times N}$. The requirement in Eq. (2.1) guarantees that when $\mathbf{X}_t^{(i)}$ is a probability vector with sum equal to 1, then $\mathbf{X}_{t+1}^{(i)}$ has the same property. In order to study the model, we first recall the Perron-Frobenius theorem for nonnegative matrices:

Proposition 2.1. [9] *(Perron-Frobenius Theorem) Let M be a non-negative and irreducible square matrix of order N. Then*

(i) *M has a positive real eigenvalue ρ, equal to its spectral radius — i.e. $\rho = \max\{|\lambda|, \lambda \in \sigma(M)\}$, where $\sigma(M)$ denotes the set of eigenvalue of M;*

(ii) *there corresponds an eigenvector $\mathbf{z}$ with real and positive entries such that $M\mathbf{z} = \rho\mathbf{z}$, and $\mathbf{z}$ is unique up to a multiplicative constant; and*

(iii) *ρ is a simple eigenvalue of M.*

Thus although the column sum of Q is not equal to 1 (the column sum of $P^{(jk)}$ is equal to 1), we have the following:

Lemma 2.1. [13,20] *Let $\lambda_{jj}P^{(jj)}$ be irreducible, $j = 1,\dots,s$, and the matrix $\Lambda = [\lambda_{jk}]_{j,k=1}^{s}$ be irreducible. Then Q is irreducible, and 1 is the maximal eigenvalue of Q in modulus. Moreover, if there is an index j such that $P^{(jj)}$ is primitive, then*

$$\lim_{n\to\infty} Q^n = \mathbf{u}\mathbf{v}^T ,$$

where $\mathbf{u}$ and $\mathbf{v}$ are positive N-by-1 vectors.

Further, by using a similar technique as in the proof of Proposition 7.2 of Ref. [3], we obtain:

Corollary 2.1. *Under the assumption of* Lemma 2.1*, there exists a unique vector*

$$\mathbf{\Pi} = \begin{pmatrix} \pi^{(1)T}, & \pi^{(2)T}, & \dots, & \pi^{(s)T} \end{pmatrix}^T$$

where

$$\pi^{(j)} \in \mathbb{R}^m , \quad 1 \le j \le s ,$$

such that

$$Q\mathbf{\Pi} = \mathbf{\Pi} \tag{2.2}$$

and

$$\sum_{i=1}^{m} [\pi^{(j)}]_i = 1 , \quad 1 \le j \le s . \tag{2.3}$$

Lemma 2.2. *Let $\lambda_{jj}P^{(jj)}$ be irreducible ($j = 1,\dots,s$), and let the matrix $\Lambda = [\lambda_{jk}]_{j,k=1}^{s}$ be irreducible. Then there exists a unique nonnegative vector $\mathbf{X}$ such that*

$$\mathbf{X}^T Q = \mathbf{X}^T, \quad \|\mathbf{X}\|_1 = m . \tag{2.4}$$

Furthermore, we have

$$\mathbf{X} = \begin{pmatrix} a_1\mathbf{1}_m^T, & a_2\mathbf{1}_m^T, & \dots, & a_s\mathbf{1}_m^T \end{pmatrix}^T , \tag{2.5}$$

where

$$\sum_{i=1}^{s} a_i = 1 , \quad a_i \ge 0 \quad (i = 1,2,\dots,s) .$$

Proof. Since Λ is irreducible, from Proposition 2.1 there is a unique positive vector $\mathbf{a} = (a_1, a_2, \ldots, a_s)^T$ such that $\mathbf{a}^T\Lambda = \mathbf{a}^T$ and $\sum_{i=1}^{s} a_i = 1$. Let $\mathbf{X}$ be given by (2.5). Since $\mathbf{1}_m^T P^{(jk)} = \mathbf{1}_m^T$ for $j, k = 1, 2, \ldots, m$, it is easy to check that $\mathbf{X}^T Q = \mathbf{X}^T$. □

Lemma 2.3. *If* $\mathbf{\Pi}$ *satisfies Eq. (2.2) and* $\mathbf{X}$ *is given by Eq. (2.5), then*

$$\mathbf{\Pi}^T\mathbf{X} = \mathbf{1}. \tag{2.6}$$

Proof. The result follows immediately from Eq. (2.3). □

The above results are useful in the construction of the multivariate Markov chain model in Ref. [2]. The matrix $P^{(ij)}$ can be estimated by first counting the transition frequencies of states from the sequence (chain) j to the sequence (chain) i, followed by a column normalisation. Under some conditions on λ_{ij}, the model converges to a stationary vector $\mathbf{\Pi}$, which can be estimated by obtaining the proportion of the states occurring in each of the sequences. The model parameters can then be obtained by minimising $||\tilde{\mathbf{\Pi}} - \tilde{Q}\tilde{\mathbf{\Pi}}||$ for some vector norms such as $||.||_1, ||.||_2$ or $||.||_\infty$. Here $\tilde{Q}$ is obtained by replacing all of the $P^{(ij)}$ by their respective estimates $\tilde{P}^{(ij)}$, since the optimal value of $||\tilde{\mathbf{\Pi}} - \tilde{Q}\tilde{\mathbf{\Pi}}||$ can be positive when $\tilde{Q}$ is obtained by replacing $P^{(ij)}$ with $\tilde{P}^{(ij)}$ and λ_{ij} with $\tilde{\lambda}_{ij}$. In that case, the estimated stationary vector $\tilde{\mathbf{\Pi}}$ is not equal to the stationary vector of $\tilde{Q}$, so it is both interesting and important to obtain the perturbation of $\tilde{\mathbf{\Pi}}$.

3. Perturbation Bounds (I)

Suppose the matrix $Q = [\lambda_{jk}P^{(jk)}]$ describes the transitions of a multivariate Markov chain, and

$$\tilde{Q} = [\tilde{\lambda}_{jk}\tilde{P}^{(jk)}] = Q + E$$

is a perturbed matrix of Q. Let matrices Q and $\tilde{Q}$ satisfy the condition of Lemma 2.1. Then there exist vectors $\mathbf{\Pi}$ and $\tilde{\mathbf{\Pi}}$ such that

$$Q\mathbf{\Pi} = \mathbf{\Pi} \quad \text{and} \quad \tilde{Q}\tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}}\,, \tag{3.1}$$

where

$$\sum_{i=1}^{m}[\pi^{(j)}]_i = \sum_{i=1}^{m}[\tilde{\pi}^{(j)}]_i = 1\,, \qquad 1 \le j \le s\,;$$

and there exist vectors $\mathbf{X}$ and $\tilde{\mathbf{X}}$ such that

$$\mathbf{X}^T = \mathbf{X}^T Q \quad \text{and} \quad \tilde{\mathbf{X}}^T = \tilde{\mathbf{X}}^T\tilde{Q}\,, \tag{3.2}$$

where $\mathbf{X}$ is given in (2.5). In this section, we discuss the change $(\tilde{\mathbf{\Pi}} - \mathbf{\Pi})$ in terms of the change $E = \tilde{Q} - Q$. For some norms, we have

$$||\tilde{\mathbf{\Pi}} - \mathbf{\Pi}|| \le \kappa||E||$$

for various different condition numbers κ. For given vectors $\mathbf{\Pi}$ and $\mathbf{X}^T$ as in (2.2) and (2.5) respectively, we set

$$\mathscr{B} = \{B : B\mathbf{\Pi} = \mathbf{0},\ \mathbf{X}^T B = \mathbf{0},\ (B + \mathbf{\Pi}\mathbf{X}^T)^{-1} \text{ exists}\}.$$

First we show that $\mathscr{B} \neq \emptyset$.

Lemma 3.1. *Under the same assumption as* Lemma 2.1*,*

$$\lim_{n\to\infty} Q^n = \mathbf{\Pi}\mathbf{X}^T .$$

Proof. From Lemma 2.1 we have $\lim_{n\to\infty} Q^n = \mathbf{u}\mathbf{v}^T$, where $\mathbf{u}$ and $\mathbf{v}$ are the right and left positive eigenvectors of Q corresponding to the maximal eigenvalue one. From Property 2.1(ii), we can set

$$\mathbf{u} = k\mathbf{\Pi} \qquad \mathbf{v} = l\mathbf{X}\,, \quad k, l > 0\,,$$

so that

$$\lim_{n\to\infty} Q^n = kl\mathbf{\Pi}\mathbf{X}^T\ ; \tag{3.3}$$

and from (3.1)

$$\mathbf{u}\mathbf{v}^T = kl\mathbf{\Pi}\mathbf{X}^T = klQ\mathbf{\Pi}\mathbf{X}^T = klQ^n\mathbf{\Pi}\mathbf{X}^T .$$

Thus from Lemma 2.3 we have

$$\begin{aligned} \mathbf{u}\mathbf{v}^T &= \lim_{n\to\infty} klQ^n\mathbf{\Pi}\mathbf{X}^T \\ &= (\lim_{n\to\infty} Q^n)kl\mathbf{\Pi}\mathbf{X}^T \\ &= kl\mathbf{\Pi}\mathbf{X}^T \cdot kl\mathbf{\Pi}\mathbf{X}^T \\ &= (kl)^2\mathbf{\Pi}\mathbf{X}^T \end{aligned}$$

— i.e. $kl\mathbf{\Pi}\mathbf{X}^T = (kl)^2\mathbf{\Pi}\mathbf{X}^T$, hence $kl = 1$, which together with (3.3) gives the result. □

Lemma 3.2. *The set $\mathscr{B}$ is nonempty.*

Proof. A simple computation gives

$$\begin{aligned} (Q - \mathbf{\Pi}\mathbf{X}^T)^2 &= Q^2 - Q\mathbf{\Pi}\mathbf{X}^T - \mathbf{\Pi}\mathbf{X}^T Q + \mathbf{\Pi}\mathbf{X}^T\mathbf{\Pi}\mathbf{X}^T \\ &= Q^2 - \mathbf{\Pi}\mathbf{X}^T\,, \end{aligned}$$

whence

$$(Q - \mathbf{\Pi}\mathbf{X}^T)^n = Q^n - \mathbf{\Pi}\mathbf{X}^T,$$

and it follows from Lemma 3.1 that

$$\lim_{n\to\infty}(Q - \mathbf{\Pi}\mathbf{X}^T)^n = \lim_{n\to\infty} Q^n - \mathbf{\Pi}\mathbf{X}^T = \mathbf{0}\,.$$

This implies that

$$\rho(Q - \mathbf{\Pi}\mathbf{X}^T) < 1\,, \tag{3.4}$$

so

$$\sum_{k=0}^{\infty}(Q-\mathbf{\Pi}\mathbf{X}^T)^k$$

exists and

$$\sum_{k=0}^{\infty}(Q-\mathbf{\Pi}\mathbf{X}^T)^k=\left(I-(Q-\mathbf{\Pi}\mathbf{X}^T)\right)^{-1}.$$

Letting $A=I-Q$, we have that $(A+\mathbf{\Pi}\mathbf{X}^T)^{-1}$ exists, and hence $\mathbf{X}^TA=\mathbf{0}$ and $A\mathbf{\Pi}=\mathbf{0}$ from Eqs. (2.2) and (2.4). Thus $A\in\mathscr{B}$ — i.e. $\mathscr{B}$ is not empty. □

Lemma 3.3. *$\mathscr{B}$ is a group under the operation of matrix multiplication.*

Proof. (i) (Existence of the identity element) Let $B_0=I-\mathbf{\Pi}\mathbf{X}^T$. Obviously, B_0 is the identify element of $\mathscr{B}$.
(ii) (Existence of inverse) For any $B\in\mathscr{B}$,

$$(B+\mathbf{\Pi}\mathbf{X}^T)^{-1}-\mathbf{\Pi}\mathbf{X}^T\in\mathscr{B}$$

and

$$(B+\mathbf{\Pi}\mathbf{X}^T)(I-\mathbf{\Pi}\mathbf{X}^T)=(I-\mathbf{\Pi}\mathbf{X}^T)(B+\mathbf{\Pi}\mathbf{X}^T)=B,$$

whence

$$\left[(B+\mathbf{\Pi}\mathbf{X}^T)^{-1}-\mathbf{\Pi}\mathbf{X}^T\right]B=B\left[(B+\mathbf{\Pi}\mathbf{X}^T)^{-1}-\mathbf{\Pi}\mathbf{X}^T\right]=I-\mathbf{\Pi}\mathbf{X}^T$$

— i.e. $(B+\mathbf{\Pi}\mathbf{X}^T)^{-1}-\mathbf{\Pi}\mathbf{X}^T$ is the inverse of B in $\mathscr{B}$.
(iii) (Closure) For any $B_1,B_2\in\mathscr{B}$, we have $B_2B_1\in\mathscr{B}$. In fact, it is easy to see that $B_2B_1\mathbf{\Pi}=(\mathbf{X}^TB_2B_1)^T=\mathbf{0}$. Since

$$(B_2B_1+\mathbf{\Pi}\mathbf{X}^T)=(B_2+\mathbf{\Pi}\mathbf{X}^T)(B_1+\mathbf{\Pi}\mathbf{X}^T),$$

we know that $(B_2B_1+\mathbf{\Pi}\mathbf{X}^T)^{-1}$ exists.
(iv) (Associative property) For any $B_1,B_2,B_3\in\mathscr{B}$,

$$(B_1B_2)B_3=B_1(B_2B_3).$$

From Lemma 3.2, $\mathscr{B}$ is not empty. Therefore $\mathscr{B}$ is a group under the operation of matrix multiplication. □

Since $A\in\mathscr{B}$, from the proof of Lemma 3.3 we identify the inverse of A in the group $\mathscr{B}$ as $A^\sharp=(A+\mathbf{\Pi}\mathbf{X}^T)^{-1}-\mathbf{\Pi}\mathbf{X}^T$, which is called the *group inverse* of A.

Theorem 3.1. *Let*

$$Q=[\lambda_{jk}P^{(jk)}]\quad\text{and}\quad\tilde{Q}=[\tilde{\lambda}_{jk}\tilde{P}^{(jk)}]=Q+E,$$

where $\lambda_{jj}P^{(jj)}$ *and* $\tilde{\lambda}_{jj}\tilde{P}^{(jj)}$ *are irreducible (*$j=1,\ldots,s$*), and let the matrices* $\Lambda=[\lambda_{jk}]_{j,k=1}^{s}$ *and* $\tilde{\Lambda}=[\tilde{\lambda}_{jk}]_{j,k=1}^{s}$ *be irreducible. If there are indices* j *and* k *such that* $P^{(jj)}$ *and* $\tilde{P}^{(kk)}$ *are primitive, then* Eqs. (3.1) *and* (3.2) *hold. Furthermore, we have*

$$\tilde{\mathbf{\Pi}}-\mathbf{\Pi}=A^{\sharp}E\tilde{\mathbf{\Pi}}\,, \tag{3.5}$$

$$\tilde{\mathbf{\Pi}}_i-\mathbf{\Pi}_i=(A^{\sharp})_{i*}E\tilde{\mathbf{\Pi}}\,,\quad i=1,2,\ldots,N\,, \tag{3.6}$$

where $A=I-Q$*.*

Proof. From Corollary 2.1 and Lemma 2.2, we have (3.1) and (3.2), and thus $A\mathbf{\Pi}=\mathbf{0}$ and $(A-E)\tilde{\mathbf{\Pi}}=\mathbf{0}$. This implies that

$$A(\tilde{\mathbf{\Pi}}-\mathbf{\Pi})=E\tilde{\mathbf{\Pi}}\,, \tag{3.7}$$

so multiplying both sides of (3.7) by $A^{\sharp}$ on the left we have

$$A^{\sharp}A(\tilde{\mathbf{\Pi}}-\mathbf{\Pi})=A^{\sharp}E\tilde{\mathbf{\Pi}}\,. \tag{3.8}$$

We may show that (3.7) and (3.8) are equivalent, but here we need only show that (3.8) implies (3.7). Since $\tilde{A}\tilde{\mathbf{\Pi}}=\mathbf{0}$ and $\mathbf{X}^TA=\mathbf{0}$, from (3.8) and the group inverse

$$\begin{aligned}A(\tilde{\mathbf{\Pi}}-\mathbf{\Pi})&=AA^{\sharp}A(\tilde{\mathbf{\Pi}}-\mathbf{\Pi})\\&=AA^{\sharp}E\tilde{\mathbf{\Pi}}\\&=(I-\mathbf{\Pi}\mathbf{X}^T)E\tilde{\mathbf{\Pi}}\\&=E\tilde{\mathbf{\Pi}}-\mathbf{\Pi}\mathbf{X}^TE\tilde{\mathbf{\Pi}}\\&=E\tilde{\mathbf{\Pi}}-\mathbf{\Pi}\mathbf{X}^T(A-\tilde{A})\tilde{\mathbf{\Pi}}\\&=E\tilde{\mathbf{\Pi}}\,,\end{aligned}$$

which proves that (3.8) implies (3.7). Since $AA^{\sharp}=I-\mathbf{\Pi}\mathbf{X}^T$, from (3.8)

$$(I-\mathbf{\Pi}\mathbf{X}^T)(\tilde{\mathbf{\Pi}}-\mathbf{\Pi})=A^{\sharp}E\tilde{\mathbf{\Pi}}\,. \tag{3.9}$$

Using the same technique as in the proof of Lemma 2.3, we have $\mathbf{X}^T\tilde{\mathbf{\Pi}}=1$ such that

$$(I-\mathbf{\Pi}\mathbf{X}^T)\tilde{\mathbf{\Pi}}=\tilde{\mathbf{\Pi}}-\mathbf{\Pi} \tag{3.10}$$

and

$$(I-\mathbf{\Pi}\mathbf{X}^T)\mathbf{\Pi}=\mathbf{0}\,. \tag{3.11}$$

From (3.10) and (3.11) we have

$$(I-\mathbf{X}\mathbf{\Pi}^T)(\tilde{\mathbf{\Pi}}-\mathbf{\Pi})=\tilde{\mathbf{\Pi}}-\mathbf{\Pi}\,,$$

which together with (3.9) yields (3.5), and hence (3.6). □

By taking the norm on both sides of (3.5), it is easy to obtain the following corollary:

Corollary 3.1. *In the notation of* Theorem 3.1*, for any operator norm* $||\cdot||$ *we have*

$$\frac{||\tilde{\mathbf{\Pi}}-\mathbf{\Pi}||}{||\tilde{\mathbf{\Pi}}||} \le ||A^{\sharp}E||, \tag{3.12}$$

where $A = I - Q$*.*

Remark 3.1. The 1-norm, 2-norm and ∞-norm are our operator norms of interest, and on writing $\Delta\mathbf{\Pi} = \tilde{\mathbf{\Pi}} - \mathbf{\Pi}$ we have

$$\frac{||\Delta\mathbf{\Pi}||_1}{||\tilde{\mathbf{\Pi}}||_1} \le ||A^{\sharp}E||_1\ , \tag{3.13}$$

$$\frac{||\Delta\mathbf{\Pi}||_\infty}{||\tilde{\mathbf{\Pi}}||_\infty} \le ||A^{\sharp}E||_\infty\ , \tag{3.14}$$

$$\frac{||\Delta\mathbf{\Pi}||_2}{||\tilde{\mathbf{\Pi}}||_2} \le ||A^{\sharp}E||_2\ . \tag{3.15}$$

In particular, on noting that $||\tilde{\mathbf{\Pi}}||_1 = s$ we find the bound (3.13) reduces to an absolute bound — i.e.

$$||\Delta\mathbf{\Pi}||_1 \le s||A^{\sharp}E||_1 \le s||A^{\sharp}||_1||E||_1 \equiv \kappa_1||E||_1. \tag{3.16}$$

Remark 3.2. For the perturbation matrix $E = (\Delta\lambda_{jk}P^{(jk)})$, if there is no perturbation on $P^{(jk)}$ we have

$$||E||_1 = ||(\Delta\lambda_{jk}P^{(jk)})||_1 = ||\Delta\Lambda||_1\ ,$$

whence from (3.16)

$$||\tilde{\mathbf{\Pi}}-\mathbf{\Pi}||_1 \le \kappa_1||\Delta\Lambda||_1\ .$$

The above bound is useful, because different model parameters λ_{ij} in Ref. [2] can be obtained by different methods.

Remark 3.3. In Ref. [18], Wei presented the perturbation bound for the singular linear system as follows. Let A be a singular matrix with index one. If $||A^{\sharp}E|| < 1$, then for any solution y to $(A+E)y = b+\Delta b$ there is a solution x to $Ax = b$ such that

$$\frac{||y-x||}{||x||} \le \frac{||A^{\sharp}||||E||}{1-||A^{\sharp}E||} + \frac{||A||||A^{\sharp}||||\Delta b||}{(1-||A^{\sharp}E||)||b||}\ .$$

The multivariate Markov model and its perturbed model can be rewritten as

$$A\mathbf{\Pi} = 0, (A+E)\tilde{\mathbf{\Pi}} = 0\ .$$

Thus on taking $\Delta b = 0$, by Wei's technique we also obtain the following bound:

$$\frac{||\tilde{\mathbf{\Pi}}-\mathbf{\Pi}||}{||\mathbf{\Pi}||} \le \frac{||A^{\sharp}||||E||}{1-||A^{\sharp}E||}\ . \tag{3.17}$$

However, for the 1-norm the bound in (3.17) is not as sharp as that in (3.12), which also holds without the assumption $||A^{\sharp}E|| < 1$.

Corollary 3.2. *Under the same assumption as in* Theorem 3.1*, we have*

$$\kappa_1 \leq \frac{s\|I - \mathbf{\Pi}\mathbf{X}^T\|_1}{1 - \|Q - \mathbf{\Pi}\mathbf{X}^T\|_1} .$$

Proof. On noting that

$$\begin{aligned} A^\sharp &= (A + \mathbf{\Pi}\mathbf{X}^T)^{-1} - \mathbf{\Pi}\mathbf{X}^T \\ &= (A + \mathbf{\Pi}\mathbf{X}^T)^{-1}(I - \mathbf{\Pi}\mathbf{X}^T) \\ &= (I - (Q - \mathbf{\Pi}\mathbf{X}^T))^{-1}(I - \mathbf{\Pi}\mathbf{X}^T) , \end{aligned}$$

from (3.4), we have

$$\|(I - (Q - \mathbf{\Pi}\mathbf{X}^T))^{-1}\| \leq \frac{1}{1 - \|Q - \mathbf{\Pi}\mathbf{X}^T\|}$$

and hence

$$\|A^\sharp\|_1 \leq \frac{\|I - \mathbf{\Pi}\mathbf{X}^T\|_1}{1 - \|Q - \mathbf{\Pi}\mathbf{X}^T\|_1} .$$

□

Our next perturbation bound is derived from Eqs. (3.5) and (3.6), by making use of the following results.

Lemma 3.4. [8, 11] *For any vector* $\mathbf{d}$ *and for any vector* $\mathbf{c}$ *such that* $\mathbf{c}^T\mathbf{1} = 0$*,*

$$|\mathbf{d}^T\mathbf{c}| \leq \|\mathbf{c}\|_1 \left(\frac{\mathbf{d}_{max} - \mathbf{d}_{min}}{2}\right) . \tag{3.18}$$

Theorem 3.2. *Under the same assumption as* Theorem 3.1*,*

$$\begin{aligned} \|\tilde{\mathbf{\Pi}} - \mathbf{\Pi}\|_\infty &\leq s\|E\|_1 \max_j \left\{\frac{\max_i (A^\sharp)_{ij} - \min_i (A^\sharp)_{ij}}{2}\right\} \\ &\equiv \kappa_2 \|E\|_1 . \end{aligned}$$

Proof. Let $\mathbf{c} = E\tilde{\mathbf{\Pi}}$ and $\mathbf{d} = (A^\sharp)_{k*}$, $k = 1, \ldots, N$. Since $E^T\mathbf{1}_N = \mathbf{0}$, we have $\mathbf{c}^T\mathbf{1}_N = 0$, and hence from Theorem 3.1 and Lemma 3.4

$$\begin{aligned} |\tilde{\pi}_k - \pi_k| &= |(A^\sharp)_{k*} E\tilde{\mathbf{\Pi}}| \\ &\leq \|E\tilde{\mathbf{\Pi}}\|_1 \left(\frac{\max_i (A^\sharp)_{ki} - \min_i (A^\sharp)_{ki}}{2}\right) \\ &\leq \|\tilde{\mathbf{\Pi}}^T\|_1 \|E\|_1 \left(\frac{\max_i (A^\sharp)_{ki} - \min_i (A^\sharp)_{ki}}{2}\right) \end{aligned}$$

such that

$$\|\tilde{\mathbf{\Pi}} - \mathbf{\Pi}\|_\infty \le s\|E\|_1 \max_j \left\{ \left(\frac{\max_i (A^\sharp)_{ij} - \min_i (A^\sharp)_{ij}}{2} \right) \right\} .$$

□

From Theorem 3.2, we also have:

Theorem 3.3. *Under the same assumption as* Theorem 3.1*,*

$$\|\tilde{\mathbf{\Pi}} - \mathbf{\Pi}\|_\infty \le s \max_{ij} |(A^\sharp)_{ij}| \|E\|_1 \equiv \kappa_3 \|E\|_1 .$$

Remark 3.4. When $s = 1$, the multivariate Markov chain reduces to the standard Markov chain — and then the bounds in (3.16) and in Theorem 3.2 reduce to the corresponding bounds in Refs. [15] and [11], respectively.

4. Perturbation Bounds (II)

Although we have found some perturbation bounds for the joint stationary distribution vector of the multivariate Markov chain, it is difficult to compute $A^\sharp$ and therefore also worthwhile to look for other bounds. In this section, we discuss the variation of the joint stationary probability distribution vector further.

First of all, we have:

Lemma 4.1. [12] *Let $\tilde{A} = I - \tilde{Q}$, where $\tilde{Q}$ is given by* Theorem 3.1. *The matrix $\begin{pmatrix} \tilde{A}_{(i)} \\ \mathbf{1}_N^T \end{pmatrix}$ is nonsingular for $i = 1, 2, \ldots, N$.*

Next, we present a perturbation bound of the joint stationary distribution of the chain:

Theorem 4.1. *Under the same assumption of* Theorem 3.1*, let*

$$\Delta\mathbf{\Pi} = \tilde{\mathbf{\Pi}} - \mathbf{\Pi}.$$

Then for any operator norm $\|\cdot\|$, we have

$$\frac{\|\Delta\mathbf{\Pi}\|}{\|\mathbf{\Pi}\|} \le \min_{1\le i\le N} \|\widetilde{\mathscr{A}}_{(i)}^{-1} \mathscr{E}_{(i)}\| , \tag{4.1}$$

where

$$\widetilde{\mathscr{A}} = \begin{pmatrix} \tilde{A} \\ \mathbf{1}_N^T \end{pmatrix}, \qquad \mathscr{E} = \begin{pmatrix} E \\ \mathbf{0}_N^T \end{pmatrix}.$$

Proof. From Corollary 2.1, the following systems of linear equations have unique solutions $\mathbf{\Pi}$ and $\tilde{\mathbf{\Pi}}$, respectively:

$$\begin{cases} Q\mathbf{\Pi} = \mathbf{\Pi} \,, \\ \sum_{i=1}^{m} [\pi^{(j)}]_i = 1, \ 1 \le j \le s \,, \end{cases} \tag{4.2}$$

$$\begin{cases} \tilde{Q}\tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}} \,, \\ \sum_{i=1}^{m} [\tilde{\pi}^{(j)}]_i = 1, \ 1 \le j \le s \,. \end{cases} \tag{4.3}$$

Clearly,

$$\tilde{A}\Delta\mathbf{\Pi} = -\tilde{A}\mathbf{\Pi} = -(A+E)\mathbf{\Pi} = -E\mathbf{\Pi} \tag{4.4}$$

and

$$\sum_{i=1}^{m} [\Delta\pi^{(j)}]_i = 0, \ 1 \le j \le s \,, \tag{4.5}$$

and on combining (4.4) and (4.5) we have

$$\begin{pmatrix} \tilde{A} \\ \mathbf{1}_N^T \end{pmatrix} \Delta\mathbf{\Pi} = -\begin{pmatrix} E \\ \mathbf{0}_N^T \end{pmatrix} \mathbf{\Pi} \,. \tag{4.6}$$

Letting $\widetilde{\mathscr{A}} = \begin{pmatrix} \tilde{A} \\ \mathbf{1}_N^T \end{pmatrix}$ and $\mathscr{E} = \begin{pmatrix} E \\ \mathbf{0}_N^T \end{pmatrix}$, we then obtain

$$\widetilde{\mathscr{A}}_{(i)}\Delta\mathbf{\Pi} = -\mathscr{E}_{(i)}\mathbf{\Pi}, \ 1 \le i \le N \,, \quad \text{such that} \quad \Delta\mathbf{\Pi} = \widetilde{\mathscr{A}}_{(i)}^{-1}\mathscr{E}_{(i)}\mathbf{\Pi} \,.$$

On taking the norm on both sides of the above equality, we then have

$$\|\Delta\mathbf{\Pi}\| = \|\widetilde{\mathscr{A}}_{(i)}^{-1}\mathscr{E}_{(i)}\mathbf{\Pi}\| \,,$$

and hence $\|\widetilde{\mathscr{A}}_{(i)}^{-1}\mathscr{E}_{(i)}\mathbf{\Pi}\| \le \|\widetilde{\mathscr{A}}_{(i)}^{-1}\mathscr{E}_{(i)}\|\|\mathbf{\Pi}\|$, which together with the above equality gives the desired bound (4.1). ☐

We also have the following corollary from Theorem 4.1:

Corollary 4.1. *In the notation of* Theorem 4.1*, we have*

$$\frac{\|\Delta\mathbf{\Pi}\|_1}{\|\mathbf{\Pi}\|_1} \le \min_{1\le i\le N} \left\{ \|\widetilde{\mathscr{A}}_{(i)}^{-1}\mathscr{E}_{(i)}\|_1 \right\} \,, \tag{4.7}$$

$$\frac{\|\Delta\mathbf{\Pi}\|_\infty}{\|\mathbf{\Pi}\|_\infty} \le \min_{1\le i\le N} \left\{ \|\widetilde{\mathscr{A}}_{(i)}^{-1}\mathscr{E}_{(i)}\|_\infty \right\} \,, \tag{4.8}$$

$$\frac{\|\Delta\mathbf{\Pi}\|_2}{\|\mathbf{\Pi}\|_2} \le \min_{1\le i\le N} \left\{ \|\widetilde{\mathscr{A}}_{(i)}^{-1}\mathscr{E}_{(i)}\|_2 \right\} \,. \tag{4.9}$$

Remark 4.1. It is notable that A can be regarded as the perturbed matrix of $\tilde{A}$, and hence

$$\frac{||\Delta\mathbf{\Pi}||_1}{||\tilde{\mathbf{\Pi}}||_1} \le \min_{1\le i\le N} \left\{||\mathscr{A}_{(i)}^{-1}\mathscr{E}_{(i)}||_1\right\}, \tag{4.10}$$

$$\frac{||\Delta\mathbf{\Pi}||_\infty}{||\tilde{\mathbf{\Pi}}||_\infty} \le \min_{1\le i\le N} \left\{||\mathscr{A}_{(i)}^{-1}\mathscr{E}_{(i)}||_\infty\right\}, \tag{4.11}$$

$$\frac{||\Delta\mathbf{\Pi}||_2}{||\tilde{\mathbf{\Pi}}||_2} \le \min_{1\le i\le N} \left\{||\mathscr{A}_{(i)}^{-1}\mathscr{E}_{(i)}||_2\right\}, \tag{4.12}$$

where

$$\mathscr{A} = \begin{pmatrix} A \\ \mathbf{1}_N^T \end{pmatrix}.$$

Since $||\mathbf{\Pi}||_1 = s$, the relative bound (4.7) reduces to the absolute one, so we may compare (4.10), (4.11) and (4.12) with (3.13), (3.14) and (3.15), respectively.

From a numerical example, we found that neither of these Sections 3 and 4 results are superior. We used Example 1 in Section 5, with the perturbation matrix E a sparse random matrix generated by the MATLAB function '$sprand$' when the density is set equal to 0.1, and then multiplied by 10^{-8}. We tested four times and obtained the results shown in Table 1, where we omit the factor 10^{-7} in all of the results due to space constraints.

Table 1: Numerical comparison of the bounds in Sections 3 and 4.

Times	(3.13)		(4.10)	(3.14)		(4.11)	(3.15)		(4.12)
1	0.1864	$>$	0.1360	0.1821	$>$	0.1300	0.1157	$>$	0.0843
2	0.1707	$<$	0.1932	0.1303	$<$	0.1853	0.0967	$<$	0.1098
3	0.2417	$>$	0.1695	0.2916	$<$	0.3065	0.1588	$<$	0.1735
4	0.1051	$<$	0.1123	0.2081	$>$	0.1857	0.0973	$<$	0.1093

Remark 4.2. We have the following simple bounds from (4.10), (4.11) and (4.12):

$$\frac{||\Delta\mathbf{\Pi}||_1}{||\tilde{\mathbf{\Pi}}||_1} \le \min_{1\le i\le N} ||\mathscr{A}_{(i)}^{-1}||_1||E||_1, \tag{4.13}$$

$$\frac{||\Delta\mathbf{\Pi}||_\infty}{||\tilde{\mathbf{\Pi}}||_\infty} \le \min_{1\le i\le N} ||\mathscr{A}_{(i)}^{-1}||_\infty||E||_\infty, \tag{4.14}$$

$$\frac{||\Delta\mathbf{\Pi}||_2}{||\tilde{\mathbf{\Pi}}||_2} \le \min_{1\le i\le N} ||\mathscr{A}_{(i)}^{-1}||_2||E||_2. \tag{4.15}$$

Finally, it is notable that for our bounds we needed to only compute $\min_{1\le i\le N} ||\mathscr{A}_{(i)}^{-1}||$, and it is clearly more difficult to compute the group inverse of a matrix than the inverse of a matrix.

5. Numerical Examples

In this section, we use two multivariate Markov chain models for different applications given in Refs. [2] and [13], to illustrate the results of our perturbation approach. All the runs were done using MATLAB 7.9.0 on a computer with a 2.66GHZ CPU and a 3.48GB memory.

Example 1. For the following two categorical data sequences from Ref. [2],

$$S_1 = \{4,3,1,3,4,4,3,3,1,2,3,4\} \quad \text{and} \quad S_2 = \{1,2,3,4,1,4,4,3,3,1,3,1\},$$

the transition probability matrices are given by

$$P^{(11)} = \begin{pmatrix} 0 & 0 & \frac{2}{5} & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{5} & \frac{2}{3} \\ 0 & 0 & \frac{2}{5} & \frac{1}{3} \end{pmatrix}, \qquad P^{(12)} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{3}{5} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{2}{3} \end{pmatrix},$$

$$P^{(21)} = \begin{pmatrix} 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ \frac{2}{3} & 0 & \frac{1}{4} & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{1}{4} & \frac{1}{3} \end{pmatrix}, \qquad P^{(22)} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{4} & \frac{1}{3} \end{pmatrix}.$$

On solving the corresponding linear programming problems, we obtained

$$\Lambda = \begin{pmatrix} 0.5000 & 0.5000 \\ 0.8858 & 0.1142 \end{pmatrix}.$$

Example 2. This example was taken from the two criterion networks constructed in Ref. [13], by considering papers that belong to the category of “Information Search and Retrieval” and “Computing Methodologies” respectively. We collected the papers from both conferences, for which reference lists are provided in DBLP — more precisely, we collected papers from 1999 to 2010 for KDD, and papers in 2000 and from 2002 to 2009 for CIKM. We considered 317 papers belonging to the category “Information Search and Retrieval” and 320 belonging to “Computing Methodologies”. There were 56 common papers that appeared in both networks. The parameter matrix Λ was estimated to be

$$\Lambda = \begin{pmatrix} 0.8678 & 0.1322 \\ 0.1437 & 0.8562 \end{pmatrix}.$$

We describe the absolute perturbation bounds given in (3.16) and Theorem 3.2 by κ_1 and κ_2 in Table 2.

In Figs. 1 and 2, we show the change in the relative bounds (4.7)–(4.9) with respect to the change of $\|E\|_1$, $\|E\|_\infty$, and $\|E\|_2$ for Examples 1 and 2. The results are consistent

Table 2: The κ_1 and κ_2 of Examples 1 and 2.

	Example 1	Example 2
κ_1	6.1662	20.7352
κ_2	2.6784	7.9666

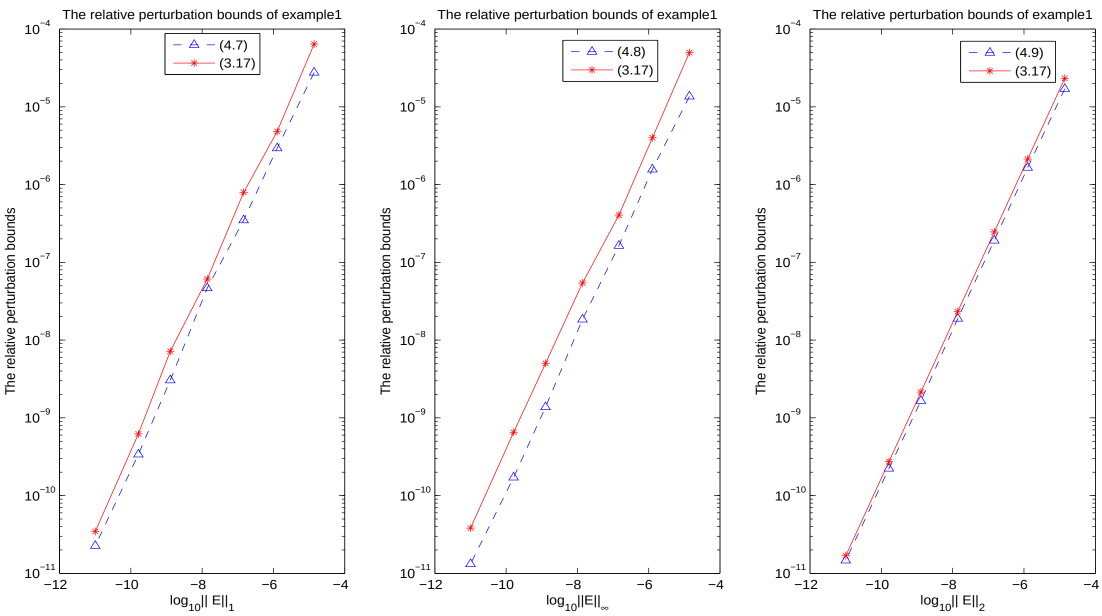

Figure 1: The relationship between $\|E\|_1$, $\|E\|_\infty$, and $\|E\|_2$ and relative perturbation bounds (4.10)–(4.12) of Π of Example 1.

with the relative perturbation bounds given in (4.7)–(4.9), and the (3.17) relative bounds obtained from Wei's technique [18]. We observe that the relative bounds depend almost linearly on $log_{10}\|E\|_*$, where $* = 1, 2, \infty$; and that that our bounds are better, except for the bound in Example 2 with the 2-norm.

6. Concluding Remarks

In this paper, we considered the absolute and relative perturbation for the joint stationary distribution vector of multivariate Markov chain models. We give some perturbation bounds, and numerical calculations demonstrate the effectiveness of our bounds. In our future research, we intend to extend our method to derive perturbation bounds for the high-order Markov chain model [3] and the high-order multivariate Markov chain model [4].

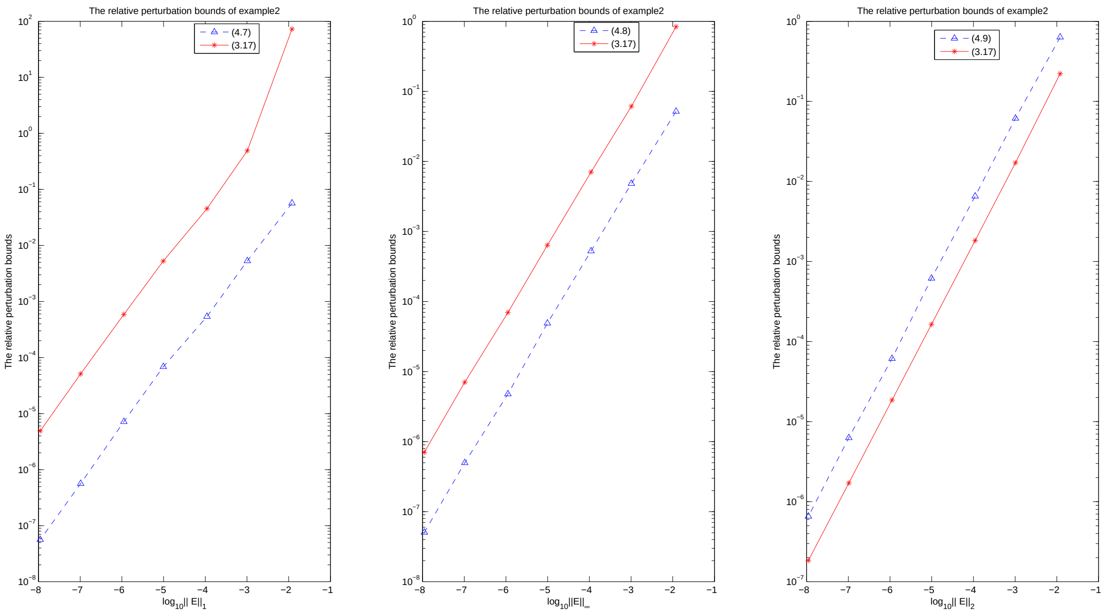

Figure 2: The relationship between $\|E\|_2$ and relative perturbation bound of Π of Example 2.

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