Sinc Nyström Method for Singularly Perturbed Love's Integral Equation

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Abstract. An efficient numerical method is proposed for the solution of Love's integral equation

$$f(x) + \frac{1}{\pi} \int_{-1}^{1} \frac{c}{(x-y)^2 + c^2} f(y) dy = 1, \quad x \in [-1, 1]$$

where $c > 0$ is a small parameter, by using a sinc Nyström method based on a double exponential transformation. The method is derived using the property that the solution $f(x)$ of Love's integral equation satisfies $f(x) \to 0.5$ for $x \in (-1, 1)$ when the parameter $c \to 0$. Numerical results show that the proposed method is very efficient.

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1. Introduction

We consider numerical methods for the solution of Love's integral equation

$$f(x) + \frac{1}{\pi} \int_{-1}^{1} \frac{c}{(x-y)^2 + c^2} f(y) dy = 1, \quad x \in [-1, 1], \tag{1.1}$$

where $c > 0$ is a small parameter. This integral equation arises in determining the capacity of a circular plate condenser, and it has been shown to possess a unique, continuous, real and even solution [5].

Different numerical methods for the solution of (1.1) have been proposed by several authors. The equation with $c = 1$ was considered in Refs. [2–4, 17, 18]. Agida & Kumar proposed a solution scheme for $c \geq 1$, based on Boubaker polynomials [1]. For $c < 1$,

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there are numerical difficulties [6, 10–12]. Numerical results have been presented by Pastore [10] for small $c > 0$ — viz. $c \in [10^{-4}, 10^{-2}]$. In this article, we consider even smaller values — i.e. $c \leq 10^{-7}$.

We derive our method by exploiting the property that for $x \in (-1, 1)$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{c}{(x - y)^2 + c^2} f(y) dy \to f(x)$$

when $c \to 0$ — i.e. the solution of (1.1) is nearly equal to $1/2$ for $x \in (-1, 1)$ [6]. We discretise the integral equation by using a DE-sinc quadrature, which is a sinc quadrature based on a double exponential (DE) transformation. The DE transformation was first proposed by Takahasi and Mori [16] for an efficient evaluation of integrals of analytic functions with singularities at end-points, and it is useful not only for numerical integrations but also for various kinds of sinc numerical methods [13, 15]. Ref. [8] provides a review.

The outline of the remainder of this article is as follows. In Section 2, we summarise some basic results for sinc approximations and DE transformations, and a DE-sinc quadrature that is then applied to Love's equation (1.1) in Section 3. Numerical results in Section 4 illustrate the efficiency and accuracy of the proposed numerical scheme.

2. A DE-sinc Quadrature

There are some basic results for sinc numerical methods based on double exponential transformations, or socalled DE-sinc numerical methods. In particular, we introduce a DE-sinc quadrature. Let us first mention some familiar related notation and concepts:

- The set of all integers, the set of all real numbers, and the set of all complex numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively;
- x and z denote the real and complex variables, respectively; and
- D_d is the strip region of width $2d$ ($d > 0$) defined by

$$D_d = \{\zeta \in \mathbb{C} : |\mathrm{Im}\, \zeta| < d\}.$$

The sinc function is defined by

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Let $h > 0$ denote the mesh size in the sinc approximation, and let

$$S_{k,h}(x) \equiv \mathrm{sinc}(x/h - k), \quad k \in \mathbb{Z}$$

denote the sinc bases corresponding to h.

A function f defined in $(-\infty, \infty)$ is said to decay double exponentially if there exist positive constants α and M such that

$$\left|f(x)\right| \le M \exp(-\alpha \exp|x|), \quad x \in (-\infty, +\infty).$$

Let φ be a one-to-one transformation that maps $(-\infty, \infty)$ into (a, b) with $\varphi(-\infty) = a$ and $\varphi(\infty) = b$ — and moreover, consider a conformal map on the strip D_d. A function g defined in a finite interval is said to decay double exponentially with respect to the conformal map φ if there exist positive constants α and M such that

$$\left|g(\varphi(t))\varphi'(t)\right| \le M \exp(-\alpha \exp|t|), \quad t \in (-\infty, +\infty). \tag{2.1}$$

A conformal map φ satisfying (2.1) is called a double exponential transformation.

We now introduce the sinc approximation of functions defined in the whole real line. If $f(z)$ is an analytic function defined on D_d satisfying

$$\lim_{x \to \pm\infty} f(x) = 0,$$

then $f(x)$ can be expressed as

$$f(x) = \sum_{k \in \mathbb{Z}} f(kh)S_{k,h}(x) + E_{\text{sinc}}(h), \quad x \in \mathbb{R} \tag{2.2}$$

where $E_{\text{sinc}}(h)$ is the error in the approximation. It is known that under certain additional mild analytical conditions the error $E_{\text{sinc}}(h)$ decays exponentially as h tends to zero (cf. [7] and references therein) — i.e.

$$E_{\text{sinc}}(h) = O\left(\exp\left(-\frac{\pi d}{h}\right)\right) \quad \text{as } h \to 0.$$

Letting $f_h(x) = \sum_{k \in \mathbb{Z}} f(kh)S_{k,h}(x)$, we note that $S_{k,h}(j) = \delta_{k-j}$ where

$$\delta_j \equiv \begin{cases} 1, & j = 0, \\ 0, & j \neq 0 \end{cases}$$

is the Kronecker delta. It follows that

$$f_h(jh) = f(jh),$$

so $f_h(x)$ is an interpolation of $f(x)$ at the grid points $\{kh : k \in \mathbb{Z}\}$. From (2.2), $f_h(x)$ converges to $f(x)$ as $h \to 0$ — and since $\int_{-\infty}^{\infty} \text{sinc}(x)dx = 1$, we have $\int_{-\infty}^{\infty} S_{k,h}(x)dx = h$ and hence the sinc quadrature for integrals in the whole line — viz.

$$\int_{-\infty}^{\infty} f(x)dx \approx \int_{-\infty}^{\infty} \left\{ \sum_{k=-n}^{n} f(kh)S_{k,h}(x) \right\} dx = h \sum_{k=-n}^{n} f(kh), \tag{2.3}$$

which is the truncated trapezoidal rule in the interval $(-\infty, +\infty)$.

We next consider the sinc approximation of functions defined in finite intervals. Let $g(x)$ be a function defined in (a, b) and φ be a double exponential transformation, and then define $G(x) = g(\varphi(x))$ — i.e. $g(x) = G\left(\varphi^{-1}(x)\right)$. From (2.2) we obtain the following sinc expansion for $g(x)$ over (a, b):

$$g(x) = G\left(\varphi^{-1}(x)\right) = \sum_{k \in \mathbb{Z}} G(kh) S_{k,h}\left(\varphi^{-1}(x)\right) + E_{\text{sinc}}(h)$$

$$= \sum_{k \in \mathbb{Z}} g(\varphi(kh)) S_{k,h}\left(\varphi^{-1}(x)\right) + E_{\text{sinc}}(h) .$$

Thus we obtain an approximation of $g(x)$ as follows:

$$g(x) \approx \sum_{k=-n}^{n} g(x_k) S_{k,h}\left(\varphi^{-1}(x)\right) , \quad x \in (a, b),$$

where $x_k = \varphi(kh)$, $k \in \mathbb{Z}$ are the sinc points of the interval (a, b). By using the sinc quadrature (2.3), we derive a sinc quadrature for integrals in the interval $[a, b]$ — viz.

$$\int_a^b f(x)dx = \int_{-\infty}^{\infty} f(\varphi(t))\varphi'(t)dt \approx h \sum_{k=-n}^{n} f(x_k)\varphi'(kh). \tag{2.4}$$

Since this quadrature is obtained by using the sinc quadrature based on the DE transformation, it is called a DE-sinc quadrature.

In this article, we use the DE transformation [16]

$$x = \varphi_{DE}(t) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh t\right) + \frac{b+a}{2} ,$$

since it is optimal in some sense and the corresponding sinc method gives very fast convergence [14]. We note that the weights can easily be obtained from

$$\varphi'_{DE}(t) = \frac{b-a}{2}\frac{\frac{\pi}{2}\cosh t}{\cosh^2\left(\frac{\pi}{2}\sinh t\right)}. \tag{2.5}$$

The following theorem guarantees the exponential convergence of the DE-sinc quadrature (2.4).

Theorem 2.1. ([9, Theorem 3.1]) *Suppose there exist $\alpha > 0$, $M > 0$ and $0 < d < \pi/2$, such that for $z \in D_d$*

$$|f(z)| \leq M\left|(z-a)^{\alpha-1}(b-z)^{\alpha-1}\right|.$$

Let n be a positive integer greater than $\alpha/(4d)$ and h be selected by the formula

$$h = \frac{\log(4dn/\alpha)}{n} . \tag{2.6}$$

Then there exists a constant $\tilde{M}$ independent of n such that

$$\left|\int_a^b f(x)dx - h\sum_{k=-n}^{n} f(\varphi_{DE}(kh))\varphi'_{DE}(kh)\right| \leq \tilde{M}\exp\left(-\frac{2\pi dn}{\log(4dn/\alpha)}\right).$$

3. Numerical Methods for Love's Integral Equation

We now proceed to the numerical method for the solution of Love's integral equation (1.1) when c is very small. Let $\omega = 1/c$, $s = \omega x$, $t = \omega y$ and $\tilde{f}(t) = f(t/\omega)$. Equation (1.1) can be written

$$\tilde{f}(s) + \frac{1}{\pi} \int_{-\omega}^{\omega} \kappa(s,t)\tilde{f}(t)dt = 1, \quad s \in [-\omega, \omega], \tag{3.1}$$

where

$$\kappa(s,t) = \frac{1}{(t-s)^2 + 1}. \tag{3.2}$$

If c is close to 0 then ω is very large, which presents the difficulty in solving the equation numerically that the integrating range $[-\omega, \omega]$ is then very large. Moreover, the solution of the equation has singularities at both ends — cf. Fig, 2 in Section 4. For $s \in (-\omega, \omega)$, we have [6]

$$\frac{1}{\pi} \int_{-\omega}^{\omega} \kappa(s,t)\tilde{f}(t)dt \to \tilde{f}(s) \quad \text{as} \quad \omega \to \infty, \tag{3.3}$$

such that the solution $\tilde{f}(s)$ of (3.1) is nearly equal to 1/2. Motivated by (3.3), we introduce a new function $g(s) = \tilde{f}(s) - 1/2$ that satisfies

$$g(s) \approx 0, \quad s \in (-\omega, \omega)$$

when ω is very large. Substituting $g(s) + 1/2$ for $\tilde{f}(s)$ in (3.1), we obtain

$$g(s) + \frac{1}{\pi} \int_{-\omega}^{\omega} \kappa(s,t)g(t)dt = p(s), \quad s \in [-\omega, \omega], \tag{3.4}$$

where

$$p(s) = \frac{1}{2} - \frac{1}{2\pi}[\arctan(\omega + s) + \arctan(\omega - s)]. \tag{3.5}$$

Since $g(t) \approx 0$ for $t \in (-\omega, \omega)$ and $0 \le \kappa(s,t) \le 1$,

$$\kappa(s,t)g(t) \approx 0, \quad t \in (-\omega, \omega), \ s \in [-\omega, \omega],$$

which is crucial to obtain highly accurate numerical solutions for (3.4). (One can imagine that if the integrand is exactly equal to 0, then all quadrature rules are error free.) Applying the DE-sinc quadrature (2.4) to the integral operator in (3.4), we obtain

$$\int_{-\omega}^{\omega} \kappa(s,t)g(t)dt \approx h \sum_{j=-n}^{n} \kappa(s,s_j)g(s_j)\varphi'_{DE}(jh), \tag{3.6}$$

where h is defined by (2.6) and

$$s_j = \varphi_{DE}(jh) = \omega \tanh\left(\frac{\pi}{2}\sinh(jh)\right), \quad j = 0, \pm 1, \ldots, \pm n.$$

Then applying (3.6) to (3.4), we obtain the following approximate equation for $g(s)$:

$$g(s) + \frac{h}{\pi} \sum_{k=-n}^{n} \kappa(s,s_k)\varphi'_{DE}(kh)g_k = p(s)\,, \tag{3.7}$$

where g_k is an approximate of $g(s_k)$, $k = 0, \pm 1, \ldots, \pm n$. Substituting s_j and g_j for s and $g(s)$ respectively, we obtain the discretised linear system

$$g_j + \frac{h}{\pi} \sum_{k=-n}^{n} \kappa(s_j,s_k)\varphi'_{DE}(kh)g_k = p(s_j)\,, \quad j = 0, \pm 1, \ldots, \pm n\,,$$

where $\kappa(\cdot,\cdot)$, $\varphi'_{DE}(\cdot)$ and $p(\cdot)$ are given by (3.2), (2.5), and (3.5), respectively.

Let $\mathbf{g} = \left[g_{-n}, \ldots, g_{-1}, g_0, g_1, \ldots, g_n\right]^T$, $\mathbf{p} = \left[p(s_{-n}), \ldots, p(s_{-1}), p(s_0), p(s_1), \ldots, p(s_n)\right]^T$, and $K = \left[\kappa(s_j,s_k)\varphi'_{DE}(kh)\right]_{j,k=-n}^{n}$, we have

$$\left(I + \frac{h}{\pi}K\right)\mathbf{g} = \mathbf{p}. \tag{3.8}$$

Using $\mathbf{g}$ and (3.7), we can obtain a Nyström approximation for $g(s)$:

$$\begin{aligned} g_n(s) &= p(s) - \frac{h}{\pi} \sum_{k=-n}^{n} \kappa(s,s_k)\varphi'_{DE}(kh)g_k \\ &= \frac{1}{2} - \frac{1}{2\pi}\left[\arctan(w+s) + \arctan(w-s)\right] - \frac{h}{\pi}\sum_{k=-n}^{n}\kappa(s,s_k)\varphi'_{DE}(kh)g_k\,. \end{aligned}$$

Finally, from $f(x) = \tilde{f}(\omega x) = g(\omega x) + \frac{1}{2}$, we obtain an approximation for $f(x)$ — viz.

$$\begin{aligned} f(x) \approx f_n(x) &= 1 - \frac{1}{2\pi}\left[\arctan(\omega + \omega x) + \arctan(\omega - \omega x)\right] \\ &\quad - \frac{h}{\pi}\sum_{k=-n}^{n}\kappa(\omega x,s_k)\varphi'_{DE}(kh)g_k\,, \quad x \in [-1,1]\,. \end{aligned} \tag{3.9}$$

4. Numerical Results

Pastore first proposed numerical methods suitable for Love's equation (1.1) with very small c, and the numerical results showed that to obtain an accurate numerical solution for smaller c more quadrature points are required [10]. For instance, 1400 quadrature points were required for $c = 10^{-2}$, and 130000 quadrature points were required for $c = 10^{-4}$.

In the DE-sinc method, it is important to choose a suitable mesh size h. If the parameters α and d are known, we can use the formula (2.6) to determine h. However, when h is given by (2.6) there are many sinc points ($x_k = \varphi_{DE}(kh)$, where $k = 0, \pm 1, \ldots, \pm n$) that repeat at the ends of the interval. For example, under 15-digit accuracy with $\alpha = 1$

Table 1: Approximate solution values for Love's integral equation with $c = 10^{-7}$ using the DE-sinc method.

n	128	256	512
$f_n(0)$	0.500000000000272	0.500000000000543	0.500000000001086
$f_n(0.01)$	0.50000032152511	0.50000032152511	0.50000032152510
$f_n(0.5)$	0.500000042441312	0.500000042441312	0.500000042441311
$f_n(0.99)$	0.500001599534110	0.500001599532126	0.500001599528743
$f_n(1)$	0.707158198329308	0.707118916862578	0.707109622751810

Table 2: Estimated errors at points 0, 0.01, 0.5, 0.99, and 1 ($c = 10^{-7}$).

n	128	256
$\left| f_{2n}(0) - f_n(0) \right|$	2.71e–13	5.43e–13
$\left| f_{2n}(0.01) - f_n(0.01) \right|$	0	1.0e–15
$\left| f_{2n}(0.5) - f_n(0.5) \right|$	0	1.0e–15
$\left| f_{2n}(0.99) - f_n(0.99) \right|$	1.98e–12	3.38e–12
$\left| f_{2n}(1) - f_n(1) \right|$	3.93e–5	9.29e–6

and $d = \pi/6$, for $n = 512$ there are 57 sinc points at the left (right) end of the interval for $n = 128$ and 283 sinc points at the left (right) end of the interval. In our numerical tests, we set $h = 3/n$ in order to avoid repeated quadrature points — e.g. for the case $[a, b] = [-1, 1]$, $h = 3/n$ with $n = 512$, the first five sinc points are -0.999999999999957, -0.999999999999948, -0.999999999999938, -0.999999999999926, and -0.999999999999911.

In the following examples, we show numerical results for $c = 10^{-7}$, 10^{-8} and 10^{-9}, respectively. We apply Gaussian elimination to (3.8) to obtain **g**, and apply (3.9) to obtain approximate values of $f(x)$.

Example 4.1. The case $c = 10^{-7}$ ($\omega = 10^7$).

Approximate values of $f(x)$ at the points 0, 0.1, 0.5, 0.99 and 1 are shown in Table 1 for $n = 128, 256, 512$. (We recall that the number of unknowns in the linear system (3.8) is $N = 2n + 1$.) To estimate the accuracy of the approximate values, we computed the values of $|f_{2n}(x) - f_n(x)|$ presented in Table 2.

From Table 2, we evidently obtain approximate function values with accuracy $O(10^{-12})$ by using several hundreds of quadrature points for interior points. As to $f_n(\pm 1)$, the accuracy is about $O(n^{-2})$. In order to show clearly how the maximum error of the numerical solution depends on n, we plotted the maximum errors against n and n^{-2} with $n = 64$, 128, ... and 1024, respectively — cf. Figs. 1(a) and (b). From Fig. 1(b), we see that the maximum error depends linearly on n^{-2}. The numerical solution shown in Fig. 2 changes quickly at the ends of the interval, and a close-up in Fig. 2(b) shows that the solution is
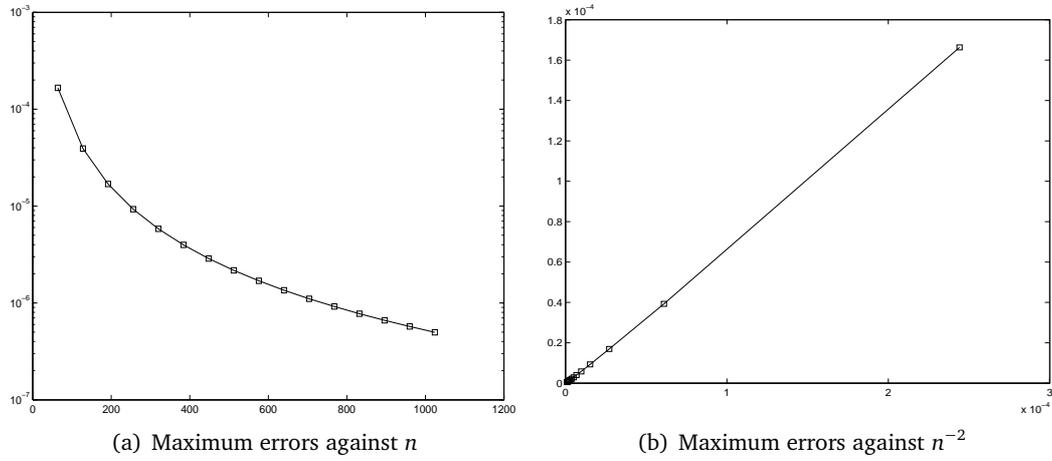
(a) Maximum errors against n (b) Maximum errors against n^{-2}

Figure 1: The maximum errors in the numerical solutions of Love's integral equation with $c = 10^{-7}$.

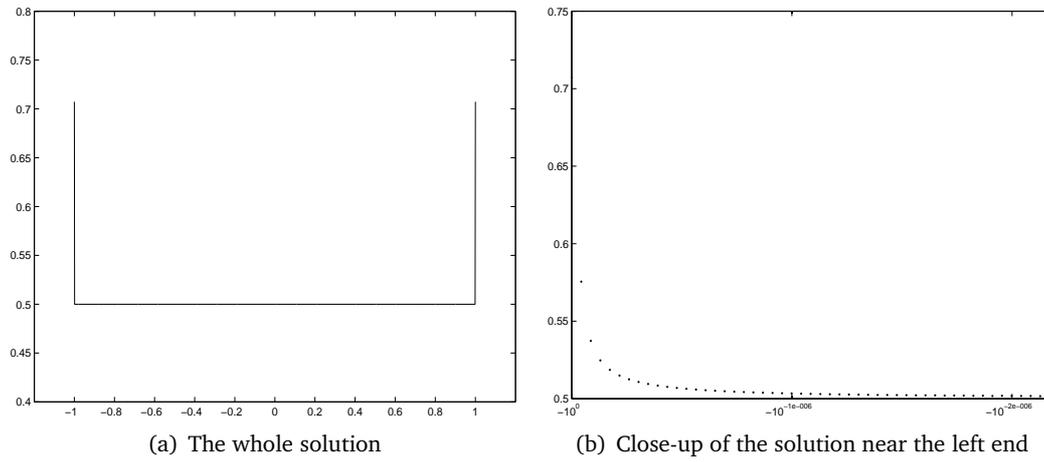(a) The whole solution   (b) Close-up of the solution near the left end

Figure 2: The solution of Love's integral equation with $c = 10^{-7}$.

smooth.

Example 4.2. The case $c = 10^{-8}$ ($\omega = 10^{8}$).

The values of $f_n(x)$ and $|f_{2n}(x) - f_n(x)|$ at the points 0, 0.1, 0.5, 0.99 and 1 are shown in Tables 3 and 4 respectively.

We see from Table 4 that we obtain a numerical solution of accuracy $O(10^{-14})$, on using several hundred interior quadrature points. As to $f_n(\pm 1)$, we again find that the accuracy is about $O(n^{-2})$.

Example 4.3. The case $c = 10^{-9}$ ($\omega = 10^{9}$).

Approximate solution values at the points 0, 0.1, 0.5, 0.99 and 1 are shown in Table 5. From Table 6 we see that the numerical solution is obtained to machine accuracy by using several hundred quadrature points. As to $f_n(\pm 1)$, the accuracy is again about $O(n^{-2})$.

Table 3: Approximate solution values for Love's integral equation with $c = 10^{-8}$ using the DE-sinc method.

n	128	256	512
$f_n(0)$	0.500000000000003	0.500000000000005	0.500000000000011
$f_n(0.01)$	0.500000003215251	0.500000003215251	0.500000003215251
$f_n(0.5)$	0.500000004244132	0.500000004244132	0.500000004244132
$f_n(0.99)$	0.500000159954593	0.500000159954574	0.500000159954554
$f_n(1)$	0.707175370184665	0.707123323055632	0.707110859164324

Table 4: Estimated errors at points 0, 0.01, 0.5, 0.99, and 1 ($c = 10^{-8}$).

n	128	256
$\left| f_{2n}(0) - f_n(0) \right|$	2.00e–15	6.00e–15
$\left| f_{2n}(0.01) - f_n(0.01) \right|$	0	0
$\left| f_{2n}(0.5) - f_n(0.5) \right|$	0	0
$\left| f_{2n}(0.99) - f_n(0.99) \right|$	1.90e–14	2.008e–14
$\left| f_{2n}(1) - f_n(1) \right|$	5.20e–5	1.25e–5

Table 5: Approximate solution values for Love's integral equation with $c = 10^{-9}$ using the DE-sinc method.

n	128	256	512
$f_n(0)$	0.500000000000000	0.500000000000000	0.500000000000000
$f_n(0.01)$	0.500000000321525	0.500000000321525	0.500000000321525
$f_n(0.5)$	0.500000000424413	0.500000000424413	0.500000000424413
$f_n(0.99)$	0.500000015995470	0.500000015995470	0.500000015995470
$f_n(1)$	0.707196155789079	0.707129879624316	0.707113967957589

Acknowledgments

Table 6: Estimated errors at points 0, 0.01, 0.5, 0.99, and 1 ($c = 10^{-9}$).

n	128	256
$\left\|f_{2n}(0) - f_n(0)\right\|$	0	0
$\left\|f_{2n}(0.01) - f_n(0.01)\right\|$	0	0
$\left\|f_{2n}(0.5) - f_n(0.5)\right\|$	0	0
$\left\|f_{2n}(0.99) - f_n(0.99)\right\|$	0	0
$\left\|f_{2n}(1) - f_n(1)\right\|$	6.63e–5	1.59e–5

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