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On Condition Numbers for the Weighted Moore-Penrose Inverse and the Weighted Least Squares Problem involving Kronecker Products

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Abstract. We establish some explicit expressions for norm-wise, mixed and componentwise condition numbers for the weighted Moore-Penrose inverse of a matrix $A \otimes B$ and more general matrix function compositions involving Kronecker products. The condition number for the weighted least squares problem (WLS) involving a Kronecker product is also discussed.

AMS subject classifications: 65F10

Key words: (Weighted) Moore-Penrose inverse, weighted least squares, Kronecker product, condition number.

1. Introduction

Consider the weighted least squares problem (WLS) involving Kronecker products [6, 25]

$$\min_{\mathbf{v}} \|(A \otimes B)\mathbf{v} - \mathbf{c}\|_C , \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $A \otimes B \in \mathbb{R}^{mp\times nq}_{nq}$, $\mathbf{c} \in \mathbb{R}^{mp}$, $C = M \otimes P$, $M \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{p\times p}$ are two symmetric positive definite matrices, with $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{m\times n}_r$ respectively denoting the set of all $m \times n$ real matrices and the set of all $m \times n$ real matrices with rank r, and $\mathbb{R}^m = \mathbb{R}^{m\times 1}$. The solution of (1.1) is relevant to the weighted Moore-Penrose inverse involving a Kronecker product. Kronecker products are widely used in system and control theory [7, 8, 26], signal processing [9], image processing [23], computing Markov chains [16], and play an important role in computing the solution of Sylvester matrix equations [14].

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Here we study the condition numbers of the weighted Moore-Penrose inverse and the WLS problem (1.1) involving Kronecker products, important for sensitivity in some computational problems as first discussed by Rice [20]. To take into account the relative scaling of data components or possible sparseness, two kinds of condition numbers have increasingly been considered — viz. mixed condition numbers and component-wise condition numbers [11]. Mixed condition numbers measure errors in the output with norms but the input perturbation component-wise, and component-wise condition numbers measure both the error in the output and the perturbation in the input component-wise.

There are some earlier publications on the condition numbers of the weighted Moore-Penrose inverse involving a Kronecker product and the WLS problem (1.1). Perturbation analysis for the LS problem is discussed in Refs. [2, 3, 5, 21]) for example, and related results on mixed and component-wise condition numbers of the WLS problem in Ref. [17]. Recently, Diao *et al*. [10] presented explicit expression for condition numbers for the linear least squares problem involving Kronecker products.

The rest of this paper is organized as follows. In Section 2, some basic notation and preliminaries are provided. In Section 3, we investigate the norm-wise, mixed, and component-wise condition numbers for the weighted Moore-Penrose inverse involving Kronecker products. In Section 4, we discuss the condition numbers for the associated WLS problem (1.1), and in Section 5 we report some numerical comparisons.

2. Preliminaries

For $A \in \mathbb{R}^{m\times n}$, we denote the transpose of A by A^T, the rank of A by rank(A), and the identity matrix of order n by I_n, respectively. The symbols $\|\cdot\|_F$ and $\|\cdot\|_2$ stand for the Frobenius norm and the spectral norm (or the Euclidean vector norm). For a vector $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\|\mathbf{a}\|_\infty$ denotes the infinity norm and $D_{\mathbf{a}} = \mathrm{diag}(a_1, a_2, \cdots, a_n)$. Let $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, $\mathrm{vec}(A) = [\mathbf{a}_1^T\ \mathbf{a}_2^T \cdots \mathbf{a}_n^T]^T$, and $D_A = D_{\mathrm{vec}(A)}$.

In order to define mixed and component-wise condition numbers, the following form of component-wise distance will be useful — for any $c \in \mathbb{R}$,

$$c^{\ddagger} = \begin{cases} 1/c\,, & \text{if } c \neq 0\,, \\ 1\,, & \text{otherwise.} \end{cases}$$

Furthermore, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we define component-wise division by

$$\frac{\mathbf{a}}{\mathbf{b}} = D_{\mathbf{b}}^{\ddagger}\mathbf{a}\,, \tag{2.1}$$

where $D_{\mathbf{b}}^{\ddagger} = \mathrm{diag}(b_1^{\ddagger}, b_2^{\ddagger}, \cdots, b_n^{\ddagger})$. The component-wise distance between $\mathbf{a}$ and $\mathbf{b}$ is then defined by

$$d(\mathbf{a}, \mathbf{b}) = \left\| \frac{\mathbf{a}-\mathbf{b}}{\mathbf{b}} \right\|_\infty = \max_{1\leqslant i\leqslant n} \left\{ |b_i^{\ddagger}||a_i - b_i| \right\}. \tag{2.2}$$