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Perturbation Bound for the Eigenvalues of a Singular Diagonalizable Matrix

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Abstract. In this short note, we present a sharp upper bound for the perturbation of eigenvalues of a singular diagonalizable matrix given by Stanley C. Eisenstat [3].

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1. Introduction

For $A \in \mathbb{C}^{n\times n}$, the smallest nonnegative integer k satisfying the rank equation,

$$rank(A^k) = rank(A^{k+1})$$

is called the index of the matrix A [1,9]. If $X \in \mathbb{C}^{n\times n}$ is the unique solution of the three matrix equations

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA\,,$$

we call X the Drazin inverse A^D. If $index(A) = 1$, then the Drazin inverse is reduced to the group inverse denoted by $A^\sharp$ [1,9].

Let us now recall the classical Bauer-Fike theorem of 1960 and its version from 1999.

Theorem 1.1. (Bauer-Fike Theorem [2,4]) *Let A be diagonalizable — i.e. $A = X\Lambda X^{-1}$, where the diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, λ_i is the eigenvalue of A. Let E be the perturbation of A and μ the eigenvalue of $A+E$. Then*

$$\min_i \left|\lambda_i - \mu\right| \le \kappa_2(X)\,\|E\|_2\,. \tag{1.1}$$

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If A is invertible, then

$$\min_i \left|\frac{\lambda_i-\mu}{\lambda_i}\right| \le \kappa_2(X)\left\|A^{-1}E\right\|_2, \tag{1.2}$$

where $\kappa_2(X)=\|X^{-1}\|_2\|X\|_2$ *is the condition number of X with respect to the* 2*-norm.*

Wei *et al.* [7, 8] explored how to extend the classical Bauer-Fike theorem to include the singular case, with the help of the group inverse. Later, Eisenstat [3] gave a different version as follows:

Theorem 1.2. *Suppose that A is singular diagonalizable — i.e.* $A=X\begin{pmatrix}\Lambda_1 & \\ & \mathbf{0}\end{pmatrix}X^{-1}$*, where* $\Lambda_1=diag(\lambda_1,\lambda_2,\cdots,\lambda_r)$*,* λ_i $(i=1,2,\cdots,r)$ *is the nonzero eigenvalue of A. Let E be the perturbation of A, and* μ *the eigenvalue of* $A+E$*. If* $|\mu|>\kappa_2(X)\|E\|_2$*, then*

$$\min_i \left|\frac{\lambda_i-\mu}{\lambda_i}\right| \le \sqrt{1+\alpha^2}\kappa_2(X)\left\|A^{\sharp}E\right\|_2, \tag{1.3}$$

where $\alpha=\kappa_2(X)\|E\|_2/\sqrt{|\mu|^2-(\kappa_2(X)\|E\|_2)^2}$.

2. Main Results

In this section, we present our main result that improves the upper bound of Ref. [3].

Theorem 2.1. *Assume that A is singular diagonalizable and E is the perturbation of A, and* μ *is the eigenvalue of* $A+E$*. If* $|\mu|>\|X^{-1}(I-AA^{\sharp})EX\|_2$*. Then*

$$\min_i \left|\frac{\lambda_i-\mu}{\lambda_i}\right| \le \sqrt{1+\beta^2}\left\|X^{-1}A^{\sharp}EX\right\|_2, \tag{2.1}$$

where $\beta=\|X^{-1}(I-AA^{\sharp})EX\|_2/\sqrt{|\mu|^2-\|X^{-1}(I-AA^{\sharp})EX\|_2^2}$.

Proof. Let $A=X\begin{pmatrix}\Lambda_1 & \\ & \mathbf{0}\end{pmatrix}X^{-1}$, where $\Lambda_1=diag(\lambda_1,\lambda_2,\cdots,\lambda_r)$ is a nonsingular diagonal matrix. Let x be an eigenvector of $A+E$ associated with μ, and denote

$$X^{-1}EX=\begin{pmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{pmatrix} \quad\text{and}\quad X^{-1}x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

Since $\mu x=(A+E)x$,

$$\mu\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\mu X^{-1}x=X^{-1}(A+E)XX^{-1}x=\begin{pmatrix}E_{11}+\Lambda_1 & E_{12}\\ E_{21} & E_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix},$$

so that

$$\mu x_2 = \begin{pmatrix} \mathbf{0} & I \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

After a little algebra, we have

$$A^{\sharp} = X \begin{pmatrix} \Lambda_1^{-1} & \\ & \mathbf{0} \end{pmatrix} X^{-1}, \quad AA^{\sharp} = X \begin{pmatrix} I & \\ & \mathbf{0} \end{pmatrix} X^{-1}, \quad I - AA^{\sharp} = X \begin{pmatrix} \mathbf{0} & \\ & I \end{pmatrix} X^{-1}$$

and

$$X^{-1}(I - AA^{\sharp})EX = X^{-1}(I - AA^{\sharp})XX^{-1}EX = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ E_{21} & E_{22} \end{pmatrix},$$

so

$$\mu x_2 = \begin{pmatrix} \mathbf{0} & I \end{pmatrix} X^{-1}(I - AA^{\sharp})EX \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

On taking the 2-norm of both sides we have

$$\begin{aligned} |\mu| \|x_2\|_2 &\le \left\| \begin{pmatrix} E_{21} & E_{22} \end{pmatrix} \right\|_2 \sqrt{\|x_1\|_2^2 + \|x_2\|_2^2} \\ &= \left\| X^{-1}(I - AA^{\sharp})EX \right\|_2 \sqrt{\|x_1\|_2^2 + \|x_2\|_2^2} \end{aligned}$$

— i.e. $\|x_2\|_2^2 \le \beta^2 \|x_1\|_2^2$. It is easy to verify that

$$\begin{aligned} \left\| X^{-1}(I - AA^{\sharp})EX \right\|_2 &= \left\| \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ E_{21} & E_{22} \end{pmatrix} \right\|_2 \le \left\| \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \right\|_2 \\ &= \left\| X^{-1}EX \right\|_2 \le \kappa_2(X) \|E\|_2. \end{aligned}$$

Since

$$\begin{aligned} \alpha &= \frac{\kappa_2(X)\|E\|_2}{\sqrt{|\,\mu\,|^2 - \left(\kappa_2(X)\|E\|_2\right)^2}} = \frac{1}{\sqrt{|\,\mu\,|^2 \,/\, \left(\kappa_2(X)\|E\|_2\right)^2 - 1}}, \\ \beta &= \frac{\left\| X^{-1}(I - AA^{\sharp})EX \right\|_2}{\sqrt{|\,\mu\,|^2 - \left\| X^{-1}(I - AA^{\sharp})EX \right\|_2^2}} = \frac{1}{\sqrt{|\,\mu\,|^2 \,/\, \left(\left\| X^{-1}(I - AA^{\sharp})EX \right\|_2 \right)^2 - 1}}, \end{aligned}$$

it is obvious that $\beta \le \alpha$. On the other hand, we have

$$\begin{pmatrix} I - \mu\Lambda_1^{-1} & \\ & \mathbf{0} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = X^{-1}A^{\sharp}(A - \mu I)XX^{-1}x = -X^{-1}A^{\sharp}Ex$$

and

$$
\begin{aligned}
(I-\mu\Lambda_1^{-1})x &= -\begin{pmatrix} I & \mathbf{0} \end{pmatrix} X^{-1}A^{\sharp}EX \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} I & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Lambda^{-1} & \\ & \mathbf{0} \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} I & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Lambda_1^{-1}E_{11} & \Lambda_1^{-1}E_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{aligned}
$$

Taking the 2-norm of both sides and noting that $\|x_2\|_2 \le \beta\|x_1\|_2$, we therefore obtain

$$
\begin{aligned}
\min_{\lambda_i\neq 0}\left|\frac{\lambda_i-\mu}{\lambda_i}\right|\|x_1\|_2 &\le \left\|\begin{pmatrix} I & \mathbf{0} \end{pmatrix}\begin{pmatrix} \Lambda_1^{-1}E_{11} & \Lambda_1^{-1}E_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right\|_2 \sqrt{\|x_1\|_2^2+\|x_2\|_2^2} \\
&= \left\|\begin{pmatrix} \Lambda_1^{-1}E_{11} & \Lambda_1^{-1}E_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right\|_2 \sqrt{\|x_1\|_2^2+\|x_2\|_2^2} \\
&\le \sqrt{1+\beta^2}\left\|\begin{pmatrix} \Lambda_1^{-1}E_{11} & \Lambda_1^{-1}E_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right\|_2 \|x_1\|_2 \\
&\le \sqrt{1+\beta^2}\left\|X^{-1}A^{\sharp}EX\right\|_2 \|x_1\|_2 ,
\end{aligned}
$$

which completes the proof. □

Remark 2.1. If $|\mu| > \kappa_2(X)\|(I-AA^{\sharp})E\|_2$, then we take

$$
\beta = \frac{\kappa_2(X)\|(I-AA^{\sharp})E\|_2}{\sqrt{|\,\mu\,|^2-(\kappa_2(X)\|(I-AA^{\sharp})E\|_2)^2}}
$$

so that

$$
\min_i\left|\frac{\lambda_i-\mu}{\lambda_i}\right| \le \sqrt{1+\beta^2}\kappa_2(X)\left\|A^{\sharp}E\right\|_2 .
$$

3. Examples

We now discuss two examples illustrating the improvement over the bound in Ref. [3].

Consider the matrix $A\in\mathbb{R}^{3\times 3}$ given by

$$
A = \begin{pmatrix} -0.25 & 0.5\times 10^{10} & 1.25\times 10^{10} \\ -0.5\times 10^{-10} & 1 & 1.5 \\ -1.25\times 10^{-10} & 1.5 & 1.75 \end{pmatrix}
$$

with the three eigenvalues

$$
\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 0 \ \text{ such that } A = X\,diag(1,2,0)X^{-1},
$$

where

$$X = \begin{pmatrix} 3 & 2 & 1 \\ 2\times 10^{-10} & 2\times 10^{-10} & 2\times 10^{-10} \\ 1\times 10^{-10} & 2\times 10^{-10} & -1\times 10^{-10} \end{pmatrix},$$

$$X^{-1} = \begin{pmatrix} 0.75 & -0.5\times 10^{10} & -0.25\times 10^{10} \\ -0.5 & 0.5\times 10^{10} & 0.5\times 10^{10} \\ -0.25 & 0.5\times 10^{10} & -0.25\times 10^{10} \end{pmatrix}.$$

We choose the perturbation matrix E such that $|E| \le 10^{-6} \times |A|$, where $|E|$ is the absolute matrix of

$$E = 10^{-16} \times X \begin{pmatrix} 10^5 & -1\times 10^{10} & 0 \\ 1 & 10^5 & 0 \\ 0 & 0 & 1 \end{pmatrix} X^{-1},$$

We compute

$$X^{-1}AA^{\sharp}EX = 10^{-16} \times \begin{pmatrix} 10^5 & -1\times 10^{10} & 0 \\ 1 & 10^5 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^{-1}(I - AA^{\sharp})EX = 10^{-16} \times diag(0,0,1) \quad \text{and} \quad \Lambda_1^{-1} = diag(1,0.5)\,.$$

The matrix $A+E$ has the three eigenvalues

$$\mu_1 = 1 + 0.9976 \times 10^{-11}, \quad \mu_2 = 2 + 1.0006 \times 10^{-11}, \quad \mu_3 = 0.6067 \times 10^{-17}.$$

Let us now compare the two assumptions in Refs. [3,8], respectively — viz.

$$\kappa_2(X)\|E\|_2 = 7.4246 \times 10^{14} \gg \mu_i\,, \quad (i = 1,2)$$

and

$$\left\|X^{-1}(I - AA^{\sharp})EX\right\|_2 = 1.0000 \times 10^{-16} \ll \mu_i\,, \quad (i = 1,2)\,.$$

It is easy to see that our assumption is weaker than that of Ref. [3] so we cannot apply Theorem 1.2, but our bound holds — i.e.

$$\sqrt{1+\beta^2}\,\left\|X^{-1}A^{\sharp}EX\right\|_2 = 1.0000 \times 10^{-6}.$$

Let us now consider another matrix $A \in \mathbb{R}^{3\times 3}$ given by

$$A = \begin{pmatrix} 1.75 & 0.5\times 10^{-5} & 2.75\times 10^{-5} \\ -0.5\times 10^{5} & 2 & 3.5 \\ -2.75\times 10^{5} & 3.5 & 4.25 \end{pmatrix}$$

with the three eigenvalues

$$\lambda_1 = 3\,,\ \lambda_2 = 5\,,\ \lambda_3 = 0 \ \text{ such that } \ A = X diag(3,5,0) X^{-1}\,,$$

where

$$X = \begin{pmatrix} 3 & 2 & 1 \\ 2\times 10^5 & 2\times 10^5 & 2\times 10^5 \\ 10^5 & 2\times 10^5 & -1\times 10^5 \end{pmatrix},$$

$$X^{-1} = \begin{pmatrix} 0.75 & -0.5\times 10^{-5} & -0.25\times 10^{-5} \\ -0.5 & 0.5\times 10^{-5} & 0.5\times 10^{-5} \\ -0.25 & 0.5\times 10^{-5} & -0.25\times 10^{-5} \end{pmatrix}.$$

We select the perturbation matrix E satisfying $|E| \le 10^{-10} \times |A|$ — viz.

$$E = 10^{-11} \times X \begin{pmatrix} 10^{-5} & -1\times 10^{-10} & 0 \\ 1 & 10^{-5} & 0 \\ 0 & 0 & 1 \end{pmatrix} X^{-1}.$$

Then

$$X^{-1}AA^{\sharp}EX = \begin{pmatrix} 10^{-16} & -1\times 10^{-21} & 0 \\ 1\times 10^{-11} & 10^{-16} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$X^{-1}(I - AA^{\sharp})EX = 10^{-11} \times diag(0,0,1), \quad \Lambda_1^{-1} = diag(0.3333, 0.2000),$$

and $A + E$ has the three eigenvalues

$$\mu_1 = 3 + 6.217248937900877\times 10^{-15}, \quad \mu_2 = 5 - 7.105427357601002\times 10^{-15},$$
$$\mu_3 = 1\times 10^{-11}.$$

Now we can compare with the relative error bounds of Refs. [3,8], with

$$\kappa_2(X)\|E\|_2 = 0.70544766163927 < \mu_i, \quad (i = 1,2),$$

and

$$\left\|X^{-1}(I - AA^{\sharp})EX\right\|_2 = 1\times 10^{-11} \ll \mu_i, \quad (i = 1,2).$$

The bound in Ref. [3] is

$$\sqrt{1+\alpha^2}\kappa_2(X)\left\|A^{\sharp}E\right\|_2 = 0.15277727491342,$$

whereas our new bound is

$$\sqrt{1+\beta^2}\left\|X^{-1}A^{\sharp}EX\right\|_2 = 2.000000000377778\times 10^{-12}.$$

The relative error bounds for λ_1 and λ_2 are

$$\left|\frac{\lambda_1 - \mu_1}{\lambda_1}\right| = 1.998401444325282\times 10^{-15},$$
$$\left|\frac{\lambda_2 - \mu_2}{\lambda_2}\right| = 1.443289932012704\times 10^{-15}.$$

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