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Efficient Numerical Solution of the Multi-Term Time Fractional Diffusion-Wave Equation

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Abstract. Some efficient numerical schemes are proposed to solve one-dimensional and two-dimensional multi-term time fractional diffusion-wave equation, by combining the compact difference approach for the spatial discretisation and an L1 approximation for the multi-term time Caputo fractional derivatives. The unconditional stability and global convergence of these schemes are proved rigorously, and several applications testify to their efficiency and confirm the orders of convergence.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Multi-term time fractional diffusion-wave equation, compact difference scheme, discrete energy method, convergence.

1. Introduction

Recently, fractional differential equations have been invoked in various applications. Unlike classical differential equations of integer order, where the derivatives depend only on the local behaviour of the function, fractional differential equations accumulate all of the information on the function in a weighted form. This is the so-called memory effect in physics, chemistry and other research areas — e.g. see Refs. [1–3] and references therein. In particular, the time fractional diffusion-wave equation models a wide range of important physical phenomena, including *inter alia* the propagation of mechanical waves in viscoelastic media [4], a non-Markovian diffusion process with memory [5], and charge transport in amorphous semiconductors [6].

Since analytical solutions are rare and to date restricted to simpler fractional partial differential equations, there has been increasing interest in the development of effective and easy to use numerical schemes. Yuste & Murillo [7,8] constructed difference schemes using an L1 discretisation formula for the fractional diffusion equation and an L2 discretisation formula for fractional diffusion-wave equations, respectively. The stability analysis of their

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schemes was carried out via the von Neumann method. Langlands & Henry [9] considered an implicit numerical scheme for a fractional diffusion equation, using the backward Euler approximation to discretise the first order time derivative and an L1 scheme to approximate the fractional order time derivative. Chen *et al.* [10] constructed a difference scheme based on the Grünwald-Letnikov formula. They also provided both an explicit and an implicit scheme for the two-dimensional anomalous sub-diffusion equation, using relationships between the fractional Grünwald-Letnikov and Riemann-Liouville definitions [11]. The corresponding theoretical analysis for stability and convergence was undertaken using the Fourier method, and a highly accurate algorithm was constructed exploiting Richardson extrapolation. Sun & Wu [12] derived two fully discrete difference schemes for the fractional diffusion-wave and sub-diffusion equations, and proved that the schemes are uniquely solvable, unconditionally stable, and respectively $\mathcal{O}(\tau^{3-\alpha}+h^2)$ and $\mathcal{O}(\tau^{2-\alpha}+h^2)$ convergent in the maximum norm. Recently, Zhang *et al.* [13] constructed a compact alternating direction implicit (ADI) scheme to solve two-dimensional time fractional diffusion-wave equations.

There has also been some previous work on the numerical solution of problems with multiple fractional derivatives. Diethelm & Luchko [14] gave an algorithm for solving the multi-term linear fractional differential equations based on Ref. [15], but their method may require a large amount of computational effort to calculate the associated weights. Edwards *et al.* [16] solved linear multi-term fractional differential equations through a reduction to a system of ordinary and fractional differential equations. Based on the analogue equation concept, Katsikadelis [17] presented a numerical method to solve linear multi-term fractional differential equations.

A key issue in solving fractional-order diffusion models numerically is the design of efficient algorithms for the space and time discretisation. The complexity of fractional differential equations is because the fractional derivatives are nonlocal and characterised by historic dependence and universal mutuality. Thus all previous solutions must be saved to compute the solution at the current time level, which makes the storage expensive. Due to their high spatial accuracy, compact difference methods need few grid points to produce accurate solutions. However, there appear to be very few previous studies on efficient numerical methods for problems involving multi-term fractional derivatives.

This article provides some numerical schemes to solve the one-dimensional and two-dimensional multi-term fractional differential equations of the general form (cf. [17–19])

$$P({}^C\mathscr{D}_t)u(\mathbf{X},t)=\kappa\Delta u(\mathbf{X},t)+f(\mathbf{X},t)\,,\quad \mathbf{X}\in\Omega\,,\ 0<t\leqslant T\,, \tag{1.1}$$

where κ is a positive diffusion constant. The multi-term fractional operator $P({}^C\mathscr{D}_t)$ is defined by

$$P({}^C\mathscr{D}_t)v(\mathbf{X},t)=\left({}_0^C\mathscr{D}_t^{\alpha}+\sum_{i=1}^{s}a_i\,{}_0^C\mathscr{D}_t^{\alpha_i}\right)v(\mathbf{X},t)\,,$$

where $1<\alpha_s<\cdots<\alpha_1<\alpha<2$, $a_i>0$, $i=1,2,\cdots,s$, and

$${}_0^C\mathscr{D}_t^{\alpha}v(t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{v''(s)}{(t-s)^{\alpha-1}}ds$$

defines the Caputo fractional derivative of order α.

We construct some effective and fast numerical methods for solving Eq. (1.1), and establish corresponding error estimates. To reduce the computational burden, we adopt a fourth-order compact difference method for the spatial approximation as it requires relatively few grid points to produce highly accurate solutions, and use an L1 approximation to deal with the temporal Caputo fractional derivatives. Using the discrete energy method, we prove that in the one-dimensional case our resulting compact difference scheme is unconditionally stable and globally $\mathscr{O}(\tau^{3-\alpha}+h^4)$ convergent in the maximum norm. The two-dimensional case is also discussed, and its corresponding stability and convergence results are obtained. In Section 2, we present the compact difference scheme, analyse the truncation error, and prove its stability and convergence. Our results on the two-dimensional multi-term time-fractional diffusion equation are given in Section 3. In Section 4, we discuss the results obtained from several examples to demonstrate the efficiency and accuracy of our approach, and some brief comments follow in the concluding section.

2. One-Dimensional Multi-Term Time Fractional Diffusion-Wave Equation

2.1. Derivation of the compact difference scheme

Without loss of generality, we may take $a_1 = 1$ and $\kappa = 1$ in Eq. (1.1), and so consider the following problem involving the two-term time fractional diffusion-wave equation:

$$ {}_0^C\mathscr{D}_t^{\alpha_1}u(x,t) + {}_0^C\mathscr{D}_t^{\alpha}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\ , \quad 0 < x < L\ , \quad 0 < t \leqslant T\ , \tag{2.1} $$

with initial conditions

$$ u(x,0) = \varphi_1(x)\ , \quad u_t(x,0) = \varphi_2(x)\ , \quad 0 \leqslant x \leqslant L\ , \tag{2.2} $$

and boundary conditions

$$ u(0,t) = \psi_1(t)\ , \quad u(L,t) = \psi_2(t)\ , \quad 0 < t \leqslant T\ , \tag{2.3} $$

where $1 < \alpha_1 < \alpha < 2$, $\varphi_1(x), \varphi_2(x), \psi_1(t), \psi_2(t)$ and $f(x,t)$ are known smooth functions. We first give some notations and auxiliary lemmas to be used in the construction of the compact finite difference scheme.

For the finite difference approximation, we equally sub-divide the intervals $[0,L]$ with $x_i = ih$ $(0 \leqslant i \leqslant M)$ and $[0,T]$ with $t_k = k\tau$ $(0 \leqslant k \leqslant N)$, where $h = L/M$ and $\tau = T/N$ are the respective spatial and temporal step sizes. We denote $t_{k+1/2} = (t_k + t_{k+1})/2$, and $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant M\}$, $\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant N\}$, so the computational domain $[0,L] \times [0,T]$ is covered by $\Omega_h \times \Omega_\tau$. For any grid function $v = \{v_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ defined on $\Omega_h \times \Omega_\tau$, we introduce the following notation:

$$ \delta_x v_{i-\frac{1}{2}}^k = \frac{1}{h}\left(v_i^k - v_{i-1}^k\right)\ , \qquad \delta_x^2 v_i^k = \frac{1}{h}\left(\delta_x v_{i+\frac{1}{2}}^k - \delta_x v_{i-\frac{1}{2}}^k\right)\ , $$
$$ v_i^{k+\frac{1}{2}} = \frac{1}{2}\left(v_i^{k+1} + v_i^k\right)\ , \qquad \delta_t v_i^{k+\frac{1}{2}} = \frac{1}{\tau}\left(v_i^{k+1} - v_i^k\right)\ . $$

In addition, we introduce a discrete fractional derivative operator

$$D_{\varphi}^{\alpha}u_i^{k-\frac{1}{2}}=\frac{1}{\mu_\alpha}\left[\delta_t u_i^{k-\frac{1}{2}}-\sum_{j=1}^{k-1}(b_{k-j-1}^{\alpha}-b_{k-j}^{\alpha})\delta_t u_i^{j-\frac{1}{2}}-b_{k-1}^{\alpha}\varphi(x_i)\right],\ 0\leqslant i\leqslant M,\ 1\leqslant k\leqslant N,$$

where $\mu_\alpha=\tau^{\alpha-1}\Gamma(3-\alpha)$ and $b_k^\alpha=(k+1)^{2-\alpha}-k^{2-\alpha}$; and an average operator

$$\mathscr{H}u_i=\begin{cases}\dfrac{1}{12}(u_{i-1}+10u_i+u_{i+1})=\Big(I+\dfrac{h^2}{12}\delta_x^2\Big)u_i\,, & 1\leqslant i\leqslant M-1\,,\\ u_i\,,\quad i=0,M\,,\end{cases}$$

where the I denotes identical operator. The grid function space on Ω_h is denoted by $\mathscr{V}_h=\{v\mid v=(v_0,v_1,\cdots,v_{M-1},v_M),v_0=v_M=0\}$, and for any $u,v\in\mathscr{V}_h$ we define the discrete inner product

$$(u,v)=h\sum_{i=1}^{M-1}u_iv_i$$

and adopt the L_2 norm $\|u\|=\sqrt{(u,u)}$. Further, we denote $\|\delta_x^2u\|=\sqrt{(\delta_x^2u,\delta_x^2u)}$, H^1 seminorms $|\cdot|_1,\|\cdot\|_{\mathscr{H}}$, and the maximum norm $\|\cdot\|_\infty$ as follows:

$$(\delta_xu,\delta_xv)=h\sum_{i=1}^{M}\Big(\delta_xu_{i-\frac{1}{2}}\Big)\Big(\delta_xv_{i-\frac{1}{2}}\Big),\quad |u|_1=\sqrt{(\delta_xu,\delta_xu)}\,,$$

$$\langle u,v\rangle_{\mathscr{H}}=(\delta_xu,\delta_xv)-\frac{h^2}{12}(\delta_x^2u,\delta_x^2v)\,,\quad \|u\|_{\mathscr{H}}=\sqrt{\langle u,u\rangle_{\mathscr{H}}}\,,\quad \|u\|_\infty=\max_{1\leqslant i\leqslant M-1}|u_i|\,.$$

Lemma 2.1 (cf. Refs. [21,22]). *For any grid function $u\in\mathscr{V}_h$,*

$$\|u\|_\infty\leqslant\frac{\sqrt{L}}{2}|u|_1\,.$$

Lemma 2.2 (cf. Refs. [23]). *For any grid function $u\in\mathscr{V}_h$,*

$$\frac{2}{3}|u|_1^2\leqslant\|u\|_{\mathscr{H}}^2\leqslant|u|_1^2\,.$$

Lemma 2.3 (cf. Ref. [24]). *Let $\theta(s)=(1-s)^3[5-3(1-s)^2]$. If $f(x)\in\mathscr{C}^6[x_{i-1},x_{i+1}]$ for $1\leqslant i\leqslant M-1$, then*

$$\begin{aligned}&\frac{1}{12}\big[f''(x_{i-1})+10f''(x_i)+f''(x_{i+1})\big]-\frac{1}{h^2}\left[f(x_{i-1})-2f(x_i)+f(x_{i+1})\right]\\=&\frac{h^4}{360}\int_0^1\big[f^{(6)}(x_i-sh)+f^{(6)}(x_i+sh)\big]\theta(s)ds\,.\end{aligned}$$

Lemma 2.4 (cf. Ref. [25]). *Suppose* $1<\gamma<2$, $g(t)\in\mathscr{C}^3[0,t_k]$ *and let*

$$\delta_t g^{k-\frac{1}{2}}=\frac{g(t_k)-g(t_{k-1})}{\tau},\quad 1\leqslant k\leqslant N\,,$$

$$b_k^\gamma=(k+1)^{2-\gamma}-k^{2-\gamma},\quad 0\leqslant k\leqslant N-1\,,$$

$$\begin{aligned}R^\gamma(g(t_{k-\frac{1}{2}}))\equiv&\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left[\delta_t g^{k-\frac{1}{2}}-\sum_{j=1}^{k-1}(b_{k-j-1}^\gamma-b_{k-j}^\gamma)\delta_t g^{j-\frac{1}{2}}-b_{k-1}^\gamma g'(0)\right]\\&-\frac{1}{2}\left[\frac{1}{\Gamma(2-\gamma)}\int_0^{t_k}\frac{g''(s)}{(t_k-s)^{\gamma-1}}ds+\frac{1}{\Gamma(2-\gamma)}\int_0^{t_{k-1}}\frac{g''(s)}{(t_{k-1}-s)^{\gamma-1}}ds\right].\end{aligned}$$

Then

$$|R^\gamma(g(t_{k-\frac{1}{2}}))|\leqslant\frac{1}{\Gamma(3-\gamma)}\left[\frac{2-\gamma}{12}+\frac{2^{3-\gamma}}{3-\gamma}-(1+2^{1-\gamma})+\frac{1}{6}\right]\max_{0\leqslant t\leqslant t_k}\left|g'''(t)\right|\tau^{3-\gamma}.$$

Let us now consider the grid functions

$$U_i^k=u(x_i,t_k)\,,\quad f_i^k=f(x_i,t_k)\,,\quad 0\leqslant i\leqslant M,\ \ 0\leqslant k\leqslant N\,.$$

Suppose $u(x,t)\in\mathscr{C}_{x,t}^{6,3}([0,L]\times[0,T])$. Thus from Eq. (2.1), at the point (x_i,t_k) we have

$$\begin{aligned}&\frac{1}{\Gamma(2-\alpha_1)}\int_0^{t_k}\frac{\partial^2u(x_i,s)}{\partial s^2}(t_k-s)^{1-\alpha_1}ds+\frac{1}{\Gamma(2-\alpha)}\int_0^{t_k}\frac{\partial^2u(x_i,s)}{\partial s^2}(t_k-s)^{1-\alpha}ds\\=&u_{xx}(x_i,t_k)+f(x_i,t_k)\,,\qquad 0\leqslant i\leqslant M,\ 0\leqslant k\leqslant N\,,\end{aligned}$$

whence

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{\Gamma(2-\alpha_1)}\int_0^{t_k}\frac{\partial^2u(x_i,s)}{\partial s^2}(t_k-s)^{1-\alpha_1}ds+\frac{1}{\Gamma(2-\alpha)}\int_0^{t_k}\frac{\partial^2u(x_i,s)}{\partial s^2}(t_k-s)^{1-\alpha}ds\right.\\&\left.+\frac{1}{\Gamma(2-\alpha_1)}\int_0^{t_{k-1}}\frac{\partial^2u(x_i,s)}{\partial s^2}(t_{k-1}-s)^{1-\alpha_1}ds+\frac{1}{\Gamma(2-\alpha)}\int_0^{t_{k-1}}\frac{\partial^2u(x_i,s)}{\partial s^2}(t_{k-1}-s)^{1-\alpha}ds\right]\\=&\frac{1}{2}\Big[u_{xx}(x_i,t_k)+u_{xx}(x_i,t_{k-1})\Big]+\frac{1}{2}\Big[f(x_i,t_k)+f(x_i,t_{k-1})\Big],\qquad 0\leqslant i\leqslant M,\ 1\leqslant k\leqslant N.\end{aligned}$$

Recalling $D_{\varphi_2}^{\alpha_1}$, $D_{\varphi_2}^{\alpha}$ and using Lemma 2.4,

$$\begin{aligned}D_{\varphi_2}^{\alpha_1}U_i^{k-\frac{1}{2}}+D_{\varphi_2}^{\alpha}U_i^{k-\frac{1}{2}}=&\frac{1}{2}\Big[u_{xx}(x_i,t_k)+u_{xx}(x_i,t_{k-1})\Big]+\frac{1}{2}\Big[f(x_i,t_k)+f(x_i,t_{k-1})\Big]\\&+R_i^{k-\frac{1}{2}}(u)\,,\quad 0\leqslant i\leqslant M\,,\ \ 1\leqslant k\leqslant N\,,\end{aligned}\tag{2.4}$$

where $R_i^{k-1/2}(u) = R^{\alpha_1}(u(x_i, t_{k-1/2})) + R^{\alpha}(u(x_i, t_{k-1/2}))$. Since $1 < \alpha_1 < \alpha < 2$, there exists a positive constant c_1 independent of h and τ satisfying

$$\left|R_i^{k-\frac{1}{2}}(u)\right| \leqslant c_1 \tau^{3-\alpha}, \quad 0 \leqslant i \leqslant M, \;\; 1 \leqslant k \leqslant N.$$

Applying the spatial average operator $\mathscr{H}$ on both sides of Eq. (2.4),

$$\begin{aligned}\mathscr{H} D_{\varphi_2}^{\alpha_1} U_i^{k-\frac{1}{2}} + \mathscr{H} D_{\varphi_2}^{\alpha} U_i^{k-\frac{1}{2}} = &\frac{1}{2}\Big[\mathscr{H} u_{xx}(x_i, t_k) + \mathscr{H} u_{xx}(x_i, t_{k-1})\Big] \\ &+ \frac{1}{2}\Big[\mathscr{H} f(x_i, t_k) + \mathscr{H} f(x_i, t_{k-1})\Big] + \mathscr{H} R_i^{k-\frac{1}{2}}(u), \\ &1 \leqslant i \leqslant M-1, \;\; 1 \leqslant k \leqslant N.\end{aligned}$$

From Lemma 2.3,

$$\mathscr{H} D_{\varphi_2}^{\alpha_1} U_i^{k-\frac{1}{2}} + \mathscr{H} D_{\varphi_2}^{\alpha} U_i^{k-\frac{1}{2}} = \delta_x^2 U_i^{k-\frac{1}{2}} + \mathscr{H} f_i^{k-\frac{1}{2}} + \tilde{R}_i^{k-\frac{1}{2}}, \;\; 1 \leqslant i \leqslant M-1, \;\; 1 \leqslant k \leqslant N, \tag{2.5}$$

where

$$\begin{aligned}\tilde{R}_i^{k-\frac{1}{2}} = &\mathscr{H} R_i^{k-\frac{1}{2}}(u) + \frac{h^4}{720}\int_0^1 \left[\frac{\partial^6 u}{\partial x^6}(x_i - \lambda h, t_k) + \frac{\partial^6 u}{\partial x^6}(x_i - \lambda h, t_{k-1})\right. \\ &\left. + \frac{\partial^6 u}{\partial x^6}(x_i + \lambda h, t_k) + \frac{\partial^6 u}{\partial x^6}(x_i + \lambda h, t_{k-1})\right]\theta(\lambda) d\lambda,\end{aligned}$$

and there exists a positive constant c_2 independent of h and τ satisfying

$$|\tilde{R}_i^{k-\frac{1}{2}}| \leqslant c_2(\tau^{3-\alpha} + h^4), \;\; 1 \leqslant i \leqslant M-1, \;\; 1 \leqslant k \leqslant N. \tag{2.6}$$

Further, given the boundary conditions (2.3) and the initial condition (2.2), we have

$$U_0^k = \psi_1(t_k), \qquad U_M^k = \psi_2(t_k), \qquad 1 \leqslant k \leqslant N, \tag{2.7}$$

$$U_i^0 = \varphi_1(x_i), \qquad 0 \leqslant i \leqslant M. \tag{2.8}$$

Omitting the small term $\tilde{R}_i^{k-1/2}$ in Eq. (2.5) and replacing the function U_i^k with its numerical approximation u_i^k, we obtain the following compact difference scheme L1-CD:

$$\mathscr{H} D_{\varphi_2}^{\alpha_1} u_i^{k-\frac{1}{2}} + \mathscr{H} D_{\varphi_2}^{\alpha} u_i^{k-\frac{1}{2}} = \delta_x^2 u_i^{k-\frac{1}{2}} + \mathscr{H} f_i^{k-\frac{1}{2}}, \;\; 1 \leqslant i \leqslant M-1, \;\; 1 \leqslant k \leqslant N, \tag{2.9}$$

$$u_0^k = \psi_1(t_k), \quad u_M^k = \psi_2(t_k), \;\; 1 \leqslant k \leqslant N, \tag{2.10}$$

$$u_i^0 = \varphi_1(x_i), \quad 0 \leqslant i \leqslant M. \tag{2.11}$$

At each time level, this compact difference scheme (2.9)-(2.11) is a tridiagonal system of linear algebraic equations where the coefficient matrix is strictly diagonally dominant, so it has a unique solution and can be solved via the fast tridiagonal (Thomas) algorithm — cf. Ref. [28] for details.

2.2. Stability and Convergence of the L1-CD Scheme

We now proceed to investigate the stability and convergence of the L1-CD scheme (2.9)-(2.11). Let us first introduce some lemmas that play a very important role in proving stability and convergence.

Lemma 2.5 (cf. Refs. [10,20]). *If* $1<\gamma<2$, $b_k^{\gamma}=(k+1)^{2-\gamma}-k^{2-\gamma}$, $k=0,1,2,\cdots$, *then*

$$1=b_0^{\gamma}>b_1^{\gamma}>b_2^{\gamma}>\cdots>b_k^{\gamma}>\cdots\rightarrow 0\,,$$
$$(2-\gamma)(k+1)^{1-\gamma}<b_k^{\gamma}<(2-\gamma)k^{1-\gamma}\,.$$

Lemma 2.6. *For any grid function* $\{u_i^k \mid 0\leqslant i\leqslant M,\ 0\leqslant k\leqslant N\}$ *and* $u_0^k=u_M^k=0$,

$$-\Big(\delta_x^2u^{k-\frac12},\mathscr{H}\delta_tu^{k-\frac12}\Big)=\frac{1}{2\tau}\Big(\|u^k\|_{\mathscr{H}}^2-\|u^{k-1}\|_{\mathscr{H}}^2\Big),\quad 1\leqslant k\leqslant N\,.$$

Proof. Applying the identity $\mathscr{H}u_i=(I+\frac{h^2}{12}\delta_x^2)u_i$, we obtain

$$\begin{aligned}-\Big(\delta_x^2u^{k-\frac12},\mathscr{H}\delta_tu^{k-\frac12}\Big)&=-\Big(\delta_x^2u^{k-\frac12},(I+\frac{h^2}{12}\delta_x^2)\delta_tu^{k-\frac12}\Big)\\&=-\Big(\delta_x^2u^{k-\frac12},\delta_tu^{k-\frac12}\Big)-\frac{h^2}{12}\Big(\delta_x^2u^{k-\frac12},\delta_t\delta_x^2u^{k-\frac12}\Big).\end{aligned}\tag{2.12}$$

Invoking the discrete Green formula, we rewrite the first term on the right-hand side of (2.12) as

$$\begin{aligned}-\Big(\delta_x^2u^{k-\frac12},\delta_tu^{k-\frac12}\Big)&=\Big(\delta_xu^{k-\frac12},\delta_t\delta_xu^{k-\frac12}\Big)\\&=\frac{1}{2\tau}\Big[\Big(\delta_xu^k,\delta_xu^k\Big)-\Big(\delta_xu^{k-1},\delta_xu^{k-1}\Big)\Big].\end{aligned}\tag{2.13}$$

The second term on the right-hand side of (2.12) may be rewritten

$$-\frac{h^2}{12}\Big(\delta_x^2u^{k-\frac12},\delta_t\delta_x^2u^{k-\frac12}\Big)=-\frac{h^2}{24\tau}\Big[\Big(\delta_x^2u^k,\delta_x^2u^k\Big)-\Big(\delta_x^2u^{k-1},\delta_x^2u^{k-1}\Big)\Big].\tag{2.14}$$

Consequently, substituting Eqs. (2.13) and (2.14) into Eq. (2.12) and noting the definition of $\|\cdot\|_{\mathscr{H}}$, we arrive at

$$-\Big(\delta_x^2u^{k-\frac12},\mathscr{H}\delta_tu^{k-\frac12}\Big)=\frac{1}{2\tau}\Big(\|u^k\|_{\mathscr{H}}^2-\|u^{k-1}\|_{\mathscr{H}}^2\Big).$$ □

In proceeding to determine the stability of the L1-CD difference scheme (2.9)-(2.11) with respect to the initial values $\varphi_1(x)$, $\varphi_2(x)$ and the forcing term f, we now prove the following *a priori* estimate.

Theorem 2.1. *Let $\{v_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ be the solution of the difference system*

$$\mathscr{H} D_{\varphi_2}^{\alpha_1} v_i^{k-\frac{1}{2}} + \mathscr{H} D_{\varphi_2}^{\alpha} v_i^{k-\frac{1}{2}} = \delta_x^2 v_i^{k-\frac{1}{2}} + g_i^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N, \tag{2.15}$$

$$v_0^k = 0, \quad v_M^k = 0, \quad 1 \leqslant k \leqslant N, \tag{2.16}$$

$$v_i^0 = v^0(x_i), \quad 0 \leqslant i \leqslant M, \tag{2.17}$$

where $v^0(x_0) = v^0(x_M) = 0$. Then for $1 \leqslant k \leqslant N$,

$$\|v^k\|_{\mathscr{H}}^2 \leqslant \|v^0\|_{\mathscr{H}}^2 + \left(\frac{t_k^{2-\alpha_1}}{\Gamma(3-\alpha_1)} + \frac{t_k^{2-\alpha}}{\Gamma(3-\alpha)} \right) \|\mathscr{H}\varphi_2\|^2 + \Gamma(2-\alpha) t_k^{\alpha-1} \tau \sum_{j=1}^{k} \|g^{j-\frac{1}{2}}\|^2 .$$

Proof. Multiplying Eq. (2.15) by $h\mathscr{H}\delta_t v_i^{k-\frac{1}{2}}$ and summing over i from 1 to $M-1$,

$$\begin{aligned}&\left(\mathscr{H} D_{\varphi_2}^{\alpha_1} v^{k-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right) + \left(\mathscr{H} D_{\varphi_2}^{\alpha} v^{k-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\ =&\left(\delta_x^2 v^{k-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right) + \left(g^{k-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right).\end{aligned} \tag{2.18}$$

From the definitions of $D_{\varphi_2}^{\alpha_1}$ and $D_{\varphi_2}^{\alpha}$,

$$\begin{aligned}&\left(\mathscr{H} D_{\varphi_2}^{\alpha_1} v^{k-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\ =&\frac{1}{\mu_{\alpha_1}} \left\| \mathscr{H}\delta_t v^{k-\frac{1}{2}} \right\|^2 - \frac{1}{\mu_{\alpha_1}} \sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_1} - b_{k-j}^{\alpha_1}\right) \left(\mathscr{H}\delta_t v^{j-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\ &- \frac{b_{k-1}^{\alpha_1}}{\mu_{\alpha_1}} \left(\mathscr{H}\varphi_2, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right),\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\left(\mathscr{H} D_{\varphi_2}^{\alpha} v^{k-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\ =&\frac{1}{\mu_{\alpha}} \left\| \mathscr{H}\delta_t v^{k-\frac{1}{2}} \right\|^2 - \frac{1}{\mu_{\alpha}} \sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha} - b_{k-j}^{\alpha}\right) \left(\mathscr{H}\delta_t v^{j-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\ &- \frac{b_{k-1}^{\alpha}}{\mu_{\alpha}} \left(\mathscr{H}\varphi_2, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right).\end{aligned} \tag{2.20}$$

Invoking Lemma 2.6, we obtain the following estimate:

$$-\left(\delta_x^2 v^{k-\frac{1}{2}}, \mathscr{H}\delta_t v^{k-\frac{1}{2}}\right) = \frac{1}{2\tau} \left(\|v^k\|_{\mathscr{H}}^2 - \|v^{k-1}\|_{\mathscr{H}}^2\right). \tag{2.21}$$

Substituting Eqs. (2.19)-(2.21) into Eq. (2.18), from Lemma 2.5 and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\left(\frac{1}{\mu_{\alpha_1}}+\frac{1}{\mu_\alpha}\right)\left\|\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right\|^2+\frac{1}{2\tau}\left(\|v^k\|_{\mathscr{H}}^2-\|v^{k-1}\|_{\mathscr{H}}^2\right)\\
=&\frac{1}{\mu_{\alpha_1}}\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha_1}-b_{k-j}^{\alpha_1}\right)\left(\mathscr{H}\delta_t v^{j-\frac{1}{2}},\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)+\frac{b_{k-1}^{\alpha_1}}{\mu_{\alpha_1}}\left(\mathscr{H}\varphi_2,\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\
&+\frac{1}{\mu_\alpha}\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha}-b_{k-j}^{\alpha}\right)\left(\mathscr{H}\delta_t v^{j-\frac{1}{2}},\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\
&+\frac{b_{k-1}^{\alpha}}{\mu_\alpha}\left(\mathscr{H}\varphi_2,\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)+\left(g^{k-\frac{1}{2}},\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right)\\
\leqslant&\frac{1}{2\mu_{\alpha_1}}\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha_1}-b_{k-j}^{\alpha_1}\right)\left\|\mathscr{H}\delta_t v^{j-\frac{1}{2}}\right\|^2+\frac{1-b_{k-1}^{\alpha_1}}{2\mu_{\alpha_1}}\left\|\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right\|^2\\
&+\frac{b_{k-1}^{\alpha_1}}{2\mu_{\alpha_1}}\|\mathscr{H}\varphi_2\|^2+\frac{b_{k-1}^{\alpha_1}}{2\mu_{\alpha_1}}\left\|\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right\|^2+\frac{1}{2\mu_\alpha}\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha}-b_{k-j}^{\alpha}\right)\left\|\mathscr{H}\delta_t v^{j-\frac{1}{2}}\right\|^2\\
&+\frac{1-b_{k-1}^{\alpha}}{2\mu_\alpha}\left\|\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right\|^2+\frac{b_{k-1}^{\alpha}}{2\mu_\alpha}\|\mathscr{H}\varphi_2\|^2+\frac{b_{k-1}^{\alpha}}{2\mu_\alpha}\left\|\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right\|^2+\left|(g^{k-\frac{1}{2}},\mathscr{H}\delta_t v^{k-\frac{1}{2}})\right|\ ,
\end{aligned}$$

or

$$\begin{aligned}
&\left(\frac{1}{\mu_{\alpha_1}}+\frac{1}{\mu_\alpha}\right)\left\|\mathscr{H}\delta_t v^{k-\frac{1}{2}}\right\|^2+\frac{1}{\tau}\left(\|v^k\|_{\mathscr{H}}^2-\|v^{k-1}\|_{\mathscr{H}}^2\right)\\
\leqslant&\frac{1}{\mu_{\alpha_1}}\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha_1}-b_{k-j}^{\alpha_1}\right)\left\|\mathscr{H}\delta_t v^{j-\frac{1}{2}}\right\|^2+\frac{b_{k-1}^{\alpha_1}}{\mu_{\alpha_1}}\|\mathscr{H}\varphi_2\|^2\\
&+\frac{1}{\mu_\alpha}\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha}-b_{k-j}^{\alpha}\right)\left\|\mathscr{H}\delta_t v^{j-\frac{1}{2}}\right\|^2+\frac{b_{k-1}^{\alpha}}{\mu_\alpha}\|\mathscr{H}\varphi_2\|^2+2\left|(g^{k-\frac{1}{2}},\mathscr{H}\delta_t v^{k-\frac{1}{2}})\right|\ ,\\
&\qquad 1\leqslant k\leqslant N\ . \qquad (2.22)
\end{aligned}$$

Let

$$F^0=\|v^0\|_{\mathscr{H}}^2$$

and

$$F^k=\|v^k\|_{\mathscr{H}}^2+\tau\sum_{j=1}^{k}\left(\frac{b_{k-j}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{k-j}^{\alpha}}{\mu_\alpha}\right)\left\|\mathscr{H}\delta_t v^{j-\frac{1}{2}}\right\|^2,\quad 1\leqslant k\leqslant N.$$

Then on multiplying (2.22) by τ and using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
F^k \leqslant & F^{k-1}+\tau\left(\frac{b_{k-1}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{k-1}^{\alpha}}{\mu_{\alpha}}\right)\|\mathscr{H}\varphi_2\|^2+2\tau\left|(g^{k-\frac{1}{2}},\mathscr{H}\delta_t v^{k-\frac{1}{2}})\right| \\
\leqslant & F^0+\tau\sum_{j=1}^{k}\left(\frac{b_{j-1}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{j-1}^{\alpha}}{\mu_{\alpha}}\right)\|\mathscr{H}\varphi_2\|^2+2\tau\sum_{j=1}^{k}\left|(g^{j-\frac{1}{2}},\mathscr{H}\delta_t v^{j-\frac{1}{2}})\right| \\
\leqslant & F^0+\tau\sum_{j=1}^{k}\left(\frac{b_{j-1}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{j-1}^{\alpha}}{\mu_{\alpha}}\right)\|\mathscr{H}\varphi_2\|^2+\tau\sum_{j=1}^{k}\left[\frac{1}{\frac{b_{k-j}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{k-j}^{\alpha}}{\mu_{\alpha}}}\left\|g^{j-\frac{1}{2}}\right\|^2\right. \\
& \left.+\left(\frac{b_{k-j}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{k-j}^{\alpha}}{\mu_{\alpha}}\right)\left\|\mathscr{H}\delta_t v^{j-\frac{1}{2}}\right\|^2\right], \qquad 1\leqslant k\leqslant N,
\end{aligned}$$

or

$$\|v^k\|_{\mathscr{H}}^2 \leqslant \|v^0\|_{\mathscr{H}}^2+\tau\sum_{j=1}^{k}\left(\frac{b_{j-1}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{j-1}^{\alpha}}{\mu_{\alpha}}\right)\|\mathscr{H}\varphi_2\|^2+\tau\sum_{j=1}^{k}\frac{1}{\frac{b_{k-j}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{k-j}^{\alpha}}{\mu_{\alpha}}}\left\|g^{j-\frac{1}{2}}\right\|^2, \quad 1\leqslant k\leqslant N. \tag{2.23}$$

Let us first estimate the second term on the right-hand side of Eq. (2.23). Noting that $\sum_{j=1}^{k} b_{j-1}^{\gamma}=k^{2-\gamma},\ \gamma=\alpha_1,\alpha$, we obtain

$$\tau\sum_{j=1}^{k}\left(\frac{b_{j-1}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{j-1}^{\alpha}}{\mu_{\alpha}}\right)\|\mathscr{H}\varphi_2\|^2=\left(\frac{t_k^{2-\alpha_1}}{\Gamma(3-\alpha_1)}+\frac{t_k^{2-\alpha}}{\Gamma(3-\alpha)}\right)\|\mathscr{H}\varphi_2\|^2. \tag{2.24}$$

Next, let us estimate the third term on the right-hand side of Eq. (2.23). Noting that $b_{k-j}^{\gamma}\geqslant(2-\gamma)(k-j+1)^{1-\gamma}\geqslant(2-\gamma)k^{1-\gamma},\gamma=\alpha_1,\alpha$ when $1\leqslant j\leqslant k$, we arrive at

$$\frac{b_{k-j}^{\gamma}}{\mu_{\gamma}}=\frac{b_{k-j}^{\gamma}}{\Gamma(3-\gamma)\tau^{\gamma-1}}\geqslant\frac{(2-\gamma)k^{1-\gamma}}{\Gamma(3-\gamma)\tau^{\gamma-1}}=\frac{t_k^{1-\gamma}}{\Gamma(2-\gamma)}. \tag{2.25}$$

Moreover, invoking Eq. (2.25) we find that

$$\tau\sum_{j=1}^{k}\frac{1}{\frac{b_{k-j}^{\alpha_1}}{\mu_{\alpha_1}}+\frac{b_{k-j}^{\alpha}}{\mu_{\alpha}}}\left\|g^{j-\frac{1}{2}}\right\|^2\leqslant\tau\sum_{j=1}^{k}\frac{\mu_{\alpha}}{b_{k-j}^{\alpha}}\left\|g^{j-\frac{1}{2}}\right\|^2\leqslant\Gamma(2-\alpha)t_k^{\alpha-1}\tau\sum_{j=1}^{k}\left\|g^{j-\frac{1}{2}}\right\|^2. \tag{2.26}$$

The desired result follows on inserting Eqs. (2.24) and (2.26) into Eq. (2.23). □

We conclude the following stability statement from Theorem 2.1.

Theorem 2.2. *The L1-CD scheme* (2.9)-(2.11) *is unconditionally stable with respect to the initial values $\varphi_1(x)$ and $\varphi_2(x)$ and the inhomogeneous term f.*

We now consider the convergence of the L1-CD scheme (2.9)-(2.11). Let

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\ .$$

Subtracting Eqs. (2.9)-(2.11) from Eqs. (2.5), (2.7) and (2.8) respectively, we get the error equations

$$\begin{aligned}
&\mathscr{H} D_0^{\alpha_1} e_i^{k-\frac{1}{2}} + \mathscr{H} D_0^{\alpha} e_i^{k-\frac{1}{2}} = \delta_x^2 e_i^{k-\frac{1}{2}} + \tilde{R}_i^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N\ ,\\
&e_0^k = 0, \quad e_M^k = 0, \quad 1 \leqslant k \leqslant N\ ,\\
&e_i^0 = 0, \quad 0 \leqslant i \leqslant M\ .
\end{aligned}$$

Invoking Eq. (2.6) and Theorem 2.1, we obtain

$$\begin{aligned}
\|e^k\|_{\mathscr{H}}^2 &\leqslant \Gamma(2-\alpha) t_k^{\alpha-1} \tau \sum_{j=1}^{k} \left\| \tilde{R}^{j-\frac{1}{2}} \right\|^2 \leqslant\ \Gamma(2-\alpha) t_k^{\alpha-1} \tau \sum_{j=1}^{k} L c_2^2 (\tau^{3-\alpha} + h^4)^2\\
&\leqslant c_2^2 L \Gamma(2-\alpha) T^{\alpha} (\tau^{3-\alpha} + h^4)^2\ ,
\end{aligned}$$

and applying Lemmas 2.1 and 2.2 we have the following theorem.

Theorem 2.3. *Assume that* $u(x,t) \in \mathscr{C}_{x,t}^{6,3}([0,L]\times[0,T])$ *is the solution of the problem* (2.1)-(2.3)*, and* $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ *is solution of the L1-CD scheme* (2.9)-(2.11)*, respectively. Then*

$$\|e^k\|_{\infty} \leqslant \frac{c_2 L}{4} \sqrt{6\Gamma(2-\alpha) T^{\alpha}} (\tau^{3-\alpha} + h^4)\ , \quad 0 \leqslant k \leqslant N\ .$$

3. Two-Dimensional Multi-Term Time Fractional Diffusion-Wave Equation

3.1. Derivation of the compact ADI difference scheme

Let us now consider the numerical solution of the following problem involving the two-dimensional multi-term time fractional diffusion-wave equation:

$$\begin{aligned}
&{}_0^C\mathscr{D}_t^{\alpha_1} u(x,y,t) + {}_0^C\mathscr{D}_t^{\alpha} u(x,y,t) = \Delta u(x,y,t) + f(x,y,t)\ , (x,y) \in \Omega,\ 0 < t \leqslant T\ , && (3.1)\\
&u(x,y,0) = \varphi_1(x,y)\ , \quad u_t(x,y,0) = \varphi_2(x,y)\ , \qquad (x,y) \in \bar{\Omega} = \Omega \cup \partial\Omega\ , && (3.2)\\
&u(x,y,t) = \psi(x,y,t)\ , \qquad (x,y) \in \partial\Omega\ ,\ 0 < t \leqslant T\ , && (3.3)
\end{aligned}$$

where $\Omega = (0,L_1)\times(0,L_2)$, $1 < \alpha_1 < \alpha < 2$ and $\varphi_1(x,y), \varphi_2(x,y), \psi(x,y,t)$ and $f(x,y,t)$ are known smooth functions.

Set $x_i = ih_1$ and $y_j = jh_2$ with $h_1 = L_1/M_1$ and $h_2 = L_2/M_2$, where M_1 and M_2 are positive integers. Define $\Omega_{h_1} = \{x_i |\ 0 \leq i \leq M_1\}$ and $\Omega_{h_2} = \{y_j |\ 0 \leq j \leq M_2\}$,

$\bar{\Omega}_h = \Omega_{h_1} \times \Omega_{h_2}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$. The definitions of $\tau,\ t_k$ are the same as in Section 2. For any grid function $v = \{v_{ij} \mid 0 \le i \le M_1, 0 \le j \le M_2\}$, we denote

$$\delta_x v_{i-\frac{1}{2},j} = \frac{1}{h_1}\left(v_{ij} - v_{i-1,j}\right), \qquad \delta_x^2 v_{ij} = \frac{1}{h_1}\left(\delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j}\right),$$

$$\delta_y\delta_x v_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{1}{h_2}\left(\delta_x v_{i-\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j-1}\right), \qquad \delta_y\delta_x^2 v_{i,j-\frac{1}{2}} = \frac{1}{h_2}\left(\delta_x^2 v_{ij} - \delta_x^2 v_{i,j-1}\right).$$

Similarly, the notations $\delta_y v_{i,j-1/2}$, $\delta_y^2 v_{ij}$, $\delta_x\delta_y^2 v_{i-1/2,j}$, $\delta_x^2\delta_y^2 v_{ij}$ can be defined. The discrete Laplace operator is denoted by $\Delta_h v_{ij} = \delta_x^2 v_{ij} + \delta_y^2 v_{ij}$, and the spatial average operators are defined as

$$\mathscr{H}_x v_{ij} = \begin{cases} \dfrac{1}{12}\left(v_{i-1,j} + 10v_{ij} + v_{i+1,j}\right) = \left(I + \dfrac{h_1^2}{12}\delta_x^2\right)v_{ij}, & 1 \le i \le M_1 - 1,\ 0 \le j \le M_2, \\ v_{ij}, & i = 0 \quad \text{or} \quad M_1,\ 0 \le j \le M_2, \end{cases}$$

$$\mathscr{H}_y v_{ij} = \begin{cases} \dfrac{1}{12}\left(v_{i,j-1} + 10v_{ij} + v_{i,j+1}\right) = \left(I + \dfrac{h_2^2}{12}\delta_y^2\right)v_{ij}, & 1 \le j \le M_2 - 1,\ 0 \le i \le M_1, \\ v_{ij}, & j = 0 \quad \text{or} \quad M_2,\ 0 \le i \le M_1. \end{cases}$$

We introduce the space of grid functions on $\bar{\Omega}_h$:

$$V_h^* = \left\{ v \mid v = \{v_{ij} \mid (x_i, y_j) \in \bar{\Omega}_h\} \text{ and } v_{ij} = 0 \text{ if } (x_i, y_j) \in \partial\Omega_h \right\}.$$

For any grid functions $u, v \in V_h^*$, we define the discrete inner product

$$(u, v) = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} u_{ij} v_{ij}$$

and denote $\|v\| = \sqrt{(v,v)}$. Similarly, we define $\|\delta_x^2 v\|$, $\|\delta_y^2 v\|$ and $\|\delta_x^2\delta_y^2 v\|$; and denote

$$\|\delta_x v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2-1} \left|\delta_x v_{i-\frac{1}{2},j}\right|^2},$$

$$\|\delta_x\delta_y v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \left|\delta_x\delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}\right|^2},$$

$$\|\delta_y\delta_x^2 v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} \left|\delta_y\delta_x^2 v_{i,j-\frac{1}{2}}\right|^2},$$

and the semi-norms $\|\delta_y v\|$, $\|\delta_x\delta_y^2 v\|$ can be defined similarly. The discrete H^1 semi-norm and H^1 norm are respectively

$$\|\nabla_h v\| = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}, \qquad \|v\|_{H^1} = \sqrt{\|v\|^2 + \|\nabla_h v\|^2}.$$

Finally, for any grid function $v \in V_h^*$, we denote

$$\|\delta_x v\|_A = \sqrt{\|\delta_x v\|^2 - \frac{h_1^2}{12}\|\delta_x^2 v\|^2}\,,$$
$$\|\delta_y v\|_B = \sqrt{\|\delta_y v\|^2 - \frac{h_2^2}{12}\|\delta_y^2 v\|^2}\,,$$

and

$$\|v\|_{h^*} = \sqrt{\|\mathscr{H}_y \delta_x v\|_A^2 + \|\mathscr{H}_x \delta_y v\|_B^2}\,.$$

We next construct our compact alternating direction implicit (ADI) difference scheme to solve the problem (3.1)-(3.3).

Suppose that $u(x,y,t) \in \mathscr{C}_{x,y,t}^{6,6,3}(\Omega \times [0,T])$. On considering Eq. (3.1) at the point $(x_i, y_j, t_{n-1/2})$ and then using Lemma 2.4, we have

$$\begin{aligned}
&D_{\varphi_2}^{\alpha_1} U_{ij}^{n-\frac{1}{2}} + D_{\varphi_2}^{\alpha} U_{ij}^{n-\frac{1}{2}} \\
=&\frac{1}{2}\Big[u_{xx}(x_i,y_j,t_n) + u_{xx}(x_i,y_j,t_{n-1})\Big] + \frac{1}{2}\Big[u_{yy}(x_i,y_j,t_n) + u_{yy}(x_i,y_j,t_{n-1})\Big] \\
&+ \frac{1}{2}\Big[f(x_i,y_j,t_n) + f(x_i,y_j,t_{n-1})\Big] + R^{\alpha_1}\Big(u(x_i,y_j,t_{n-\frac{1}{2}})\Big) + R^{\alpha}\Big(u(x_i,y_j,t_{n-\frac{1}{2}})\Big)\,, \\
&\qquad\qquad (x_i,y_j) \in \overline{\Omega}_h\,,\ 1 \leqslant n \leqslant N\,. \qquad (3.4)
\end{aligned}$$

Applying the spatial average operator $\mathscr{H}_x\mathscr{H}_y$ on both sides of Eq. (3.4), we obtain

$$\begin{aligned}
&\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha_1} U_{ij}^{n-\frac{1}{2}} + \mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha} U_{ij}^{n-\frac{1}{2}} \\
=&\frac{\mathscr{H}_x\mathscr{H}_y}{2}\Big[u_{xx}(x_i,y_j,t_n) + u_{xx}(x_i,y_j,t_{n-1})\Big] + \frac{\mathscr{H}_x\mathscr{H}_y}{2}\Big[u_{yy}(x_i,y_j,t_n) + u_{yy}(x_i,y_j,t_{n-1})\Big] \\
&+ \frac{\mathscr{H}_x\mathscr{H}_y}{2}\Big[f(x_i,y_j,t_n) + f(x_i,y_j,t_{n-1})\Big] + \mathscr{H}_x\mathscr{H}_y R^{\alpha_1}\Big(u(x_i,y_j,t_{n-\frac{1}{2}})\Big) \\
&+ \mathscr{H}_x\mathscr{H}_y R^{\alpha}\Big(u(x_i,y_j,t_{n-\frac{1}{2}})\Big)\,, \qquad (x_i,y_j) \in \Omega_h\,,\ 1 \leqslant n \leqslant N\,.
\end{aligned}$$

From Lemma 2.3,

$$\begin{aligned}
&\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha_1} U_{ij}^{n-\frac{1}{2}} + \mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha} U_{ij}^{n-\frac{1}{2}} \\
=&(\mathscr{H}_y \delta_x^2 + \mathscr{H}_x \delta_y^2) U_{ij}^{n-\frac{1}{2}} + \mathscr{H}_x\mathscr{H}_y f_{ij}^{n-\frac{1}{2}} + \bar{R}_{ij}^{n-\frac{1}{2}}\,, \quad (x_i,y_j) \in \Omega_h\,,\ 1 \leqslant n \leqslant N\,, \qquad (3.5)
\end{aligned}$$

where

$$\begin{aligned}\bar{R}_{ij}^{n-\frac{1}{2}} =&\mathscr{H}_x\mathscr{H}_y R^{\alpha_1}\left(u(x_i,y_j,t_{n-\frac{1}{2}})\right)+\mathscr{H}_x\mathscr{H}_y R^{\alpha}\left(u(x_i,y_j,t_{n-\frac{1}{2}})\right)\\&+\frac{h_1^4}{720}\mathscr{H}_y\int_0^1\left[\frac{\partial^6 u}{\partial x^6}(x_i-sh,y_j,t_n)+\frac{\partial^6 u}{\partial x^6}(x_i-sh,y_j,t_{n-1})\right.\\&\left.+\frac{\partial^6 u}{\partial x^6}(x_i+sh,y_j,t_n)+\frac{\partial^6 u}{\partial x^6}(x_i+sh,y_j,t_{n-1})\right]\theta(s)ds\\&+\frac{h_2^4}{720}\mathscr{H}_x\int_0^1\left[\frac{\partial^6 u}{\partial y^6}(x_i,y_j-sh,t_n)+\frac{\partial^6 u}{\partial y^6}(x_i,y_j-sh,t_{n-1})\right.\\&\left.+\frac{\partial^6 u}{\partial y^6}(x_i,y_j+sh,t_n)+\frac{\partial^6 u}{\partial y^6}(x_i,y_j+sh,t_{n-1})\right]\theta(s)ds\;.\end{aligned}$$

Adding a small term $\frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-1/2}$ into Eq. (3.5), we have

$$\begin{aligned}&\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha_1}U_{ij}^{n-\frac{1}{2}}+\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha}U_{ij}^{n-\frac{1}{2}}+\frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}}\\&=\left(\mathscr{H}_y\delta_x^2+\mathscr{H}_x\delta_y^2\right)U_{ij}^{n-\frac{1}{2}}+\mathscr{H}_x\mathscr{H}_y f_{ij}^{n-\frac{1}{2}}+R_{ij}^{n-\frac{1}{2}}\;,\quad (x_i,y_j)\in\Omega_h\;,\;1\leqslant n\leqslant N\;,\end{aligned}\tag{3.6}$$

where

$$R_{ij}^{n-1/2}=\bar{R}_{ij}^{n-1/2}+\frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}}\;.$$

Since $\frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-1/2}=\mathscr{O}(\tau^{1+\alpha})$, there exists a positive constant c_3 independent of h_1, h_2 and τ satisfying

$$\left|R_{ij}^{n-\frac{1}{2}}\right|\leqslant c_3(\tau^{3-\alpha}+h_1^4+h_2^4)\;,\quad (x_i,y_j)\in\Omega_h\;,\;1\leqslant n\leqslant N\;.\tag{3.7}$$

In addition, given the initial and boundary conditions (3.2) and (3.3) we find that

$$U_{ij}^0=\varphi_1(x_i,y_j)\;,\qquad (x_i,y_j)\in\bar{\Omega}_h\;,\tag{3.8}$$

$$U_{ij}^n=\psi(x_i,y_j,t_n)\;,\quad (x_i,y_j)\in\partial\Omega_h\;,\;1\leqslant n\leqslant N\;.\tag{3.9}$$

Omitting the small term $R_{ij}^{n-1/2}$ in Eq. (3.6), and replacing the function U_{ij}^n with its numerical approximation u_{ij}^n, and noting Eqs. (3.8) and (3.9), we obtain the following compact difference scheme L1-CADI:

$$\begin{aligned}&\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha_1}u_{ij}^{n-\frac{1}{2}}+\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha}u_{ij}^{n-\frac{1}{2}}+\frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}}\\&=\left(\mathscr{H}_y\delta_x^2+\mathscr{H}_x\delta_y^2\right)u_{ij}^{n-\frac{1}{2}}+\mathscr{H}_x\mathscr{H}_y f_{ij}^{n-\frac{1}{2}}\;,\quad (x_i,y_j)\in\Omega_h\;,\;1\leqslant n\leqslant N\;,\end{aligned}\tag{3.10}$$

$$u_{ij}^0=\varphi_1(x_i,y_j)\;,\quad (x_i,y_j)\in\bar{\Omega}_h\;,\tag{3.11}$$

$$u_{ij}^n=\psi(x_i,y_j,t_n)\;,\quad (x_i,y_j)\in\partial\Omega_h\;,\;1\leqslant n\leqslant N\;.\tag{3.12}$$

We determine $\{u_{ij}^n\}$ by solving two sets of independent one-dimensional problems as follows. Introducing the intermediate variables

$$u_{ij}^* = \left[\mathscr{H}_y - \frac{\mu_{\alpha_1}\mu_\alpha \tau}{2(\mu_{\alpha_1}+\mu_\alpha)}\delta_y^2\right]u_{ij}^n\,, \quad 0 \leqslant i \leqslant M_1\,,\ 1 \leqslant j \leqslant M_2 - 1\,.$$

To compute $\{u_{ij}^*\}$ for fixed $j \in \{1,2,\cdots,M_2-1\}$, we solve

$$\begin{cases} \left[\mathscr{H}_x - \dfrac{\mu_{\alpha_1}\mu_\alpha \tau}{2(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\right]u_{ij}^* = \left[\mathscr{H}_x + \dfrac{\mu_{\alpha_1}\mu_\alpha \tau}{2(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\right]\left[\mathscr{H}_y + \dfrac{\mu_{\alpha_1}\mu_\alpha \tau}{2(\mu_{\alpha_1}+\mu_\alpha)}\delta_y^2\right]u_{ij}^{n-1} \\ \qquad + \dfrac{\mu_\alpha \tau}{\mu_{\alpha_1}+\mu_\alpha}\displaystyle\sum_{l=1}^{n-1}(b_{n-l-1}^{\alpha_1} + b_{n-l}^{\alpha_1})\mathscr{H}_x\mathscr{H}_y\delta_t u_{ij}^{l-\frac{1}{2}} \\ \qquad + \dfrac{\mu_{\alpha_1} \tau}{\mu_{\alpha_1}+\mu_\alpha}\displaystyle\sum_{l=1}^{n-1}(b_{n-l-1}^{\alpha} - b_{n-l}^{\alpha})\mathscr{H}_x\mathscr{H}_y\delta_t u_{ij}^{l-\frac{1}{2}} + \dfrac{\mu_\alpha \tau b_{n-1}^{\alpha_1}}{\mu_{\alpha_1}+\mu_\alpha}\mathscr{H}_x\mathscr{H}_y(\varphi_2)_{ij} \\ \qquad + \dfrac{\mu_{\alpha_1} \tau b_{n-1}^{\alpha}}{\mu_{\alpha_1}+\mu_\alpha}\mathscr{H}_x\mathscr{H}_y(\varphi_2)_{ij} + \dfrac{\mu_\alpha\mu_{\alpha_1}\tau}{\mu_{\alpha_1}+\mu_\alpha}\mathscr{H}_x\mathscr{H}_y f_{ij}^{n-\frac{1}{2}}\,, \quad 1 \le i \le M_1 - 1\,, \\ u_{0j}^* = \left[\mathscr{H}_y - \dfrac{\mu_{\alpha_1}\mu_\alpha \tau}{2(\mu_{\alpha_1}+\mu_\alpha)}\delta_y^2\right]u_{0j}^n\,, \quad u_{M_1 j}^* = \left[\mathscr{H}_y - \dfrac{\mu_{\alpha_1}\mu_\alpha \tau}{2(\mu_{\alpha_1}+\mu_\alpha)}\delta_y^2\right]u_{M_1 j}^n\,. \end{cases}$$

Once $\{u_{ij}^*\}$ is available, we solve the following system for fixed $i \in \{1,2,\cdots,M_1-1\}$ to obtain the solution $\{u_{ij}^n\}$:

$$\begin{cases} \left[\mathscr{H}_y - \dfrac{\mu_{\alpha_1}\mu_\alpha \tau}{2(\mu_{\alpha_1}+\mu_\alpha)}\delta_y^2\right]u_{ij}^n = u_{ij}^*\,, \quad 1 \le j \le M_2 - 1\,, \\ u_{i0}^n = \psi(x_i, y_0, t_n)\,, \quad u_{iM_2}^n = \psi(x_i, y_{M_2}, t_n)\,. \end{cases}$$

3.2. Stability and convergence of the L1-CADI scheme

To analyse the stability and convergence of the L1-CADI scheme (3.10)-(3.12), we first introduce some lemmas.

Lemma 3.1 (cf. Ref. [13]). *For any grid function* $\{v_{ij}^n \mid (x_i, y_j) \in \bar{\Omega}_h,\ 0 \leqslant n \leqslant N\}$ *and* $v_{ij}^n = 0$ *on* $\partial\Omega_h$,

$$\Big((\mathscr{H}_y\delta_x^2 + \mathscr{H}_x\delta_y^2)v^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\Big) = -\frac{1}{2\tau}\Big(\|v^n\|_{h^*}^2 - \|v^{n-1}\|_{h^*}^2\Big).$$

Lemma 3.2 (cf. Ref. [13]). *For any grid function* $v \in V_h^*$,

$$\big(\delta_x^2\delta_y^2 v, \mathscr{H}_x\mathscr{H}_y v\big) \ge \frac{1}{3}\|\delta_x\delta_y v\|^2\,.$$

Lemma 3.3 (cf. Ref. [13]). *For any grid function $v \in V_h^*$,*

$$\frac{4}{3}\sqrt{\frac{(L_1^2+L_2^2)}{6(L_1^2+L_2^2)+L_1^2L_2^2}}\|v\|_{H^1} \le \|v\|_{h^*} \le \frac{4}{3}\|v\|_{H^1}\,.$$

In order to show the stability of the L1-CADI scheme (3.10)-(3.12) to the initial values $\varphi_1(x,y)$, $\varphi_2(x,y)$ and the forcing term f, we prove the following *a priori* estimate.

Theorem 3.1. *Let $\{v_{ij}^n \mid (x_i,y_j)\in\bar{\Omega}_h,\ 0\leqslant n\leqslant N\}$ be the solution of the difference system*

$$\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha_1} v_{ij}^{n-\frac{1}{2}} + \mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha} v_{ij}^{n-\frac{1}{2}} + \frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\delta_y^2\delta_t v_{ij}^{n-\frac{1}{2}}$$
$$= (\mathscr{H}_y\delta_x^2 + \mathscr{H}_x\delta_y^2)v_{ij}^{n-\frac{1}{2}} + g_{ij}^{n-\frac{1}{2}},\ \ (x_i,y_j)\in\Omega_h\,,\ 1\leqslant n\leqslant N\,, \tag{3.13}$$
$$v_{ij}^0 = v^0(x_i,y_j)\,,\ \ (x_i,y_j)\in\bar{\Omega}_h\,, \tag{3.14}$$
$$v_{ij}^n = 0\,,\ \ (x_i,y_j)\in\partial\Omega_h\,,\ 1\leqslant n\leqslant N\,, \tag{3.15}$$

where $v^0(x_i,y_j)\equiv 0$ on $(x_i,y_j)\in\partial\Omega_h$. Then for $1\leqslant n\leqslant N$,

$$\|v^n\|_{h^*}^2 \leqslant \|v^0\|_{h^*}^2 + \left(\frac{t_n^{2-\alpha_1}}{\Gamma(3-\alpha_1)} + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)}\right)\left\|\mathscr{H}_x\mathscr{H}_y\varphi_2\right\|^2 + \Gamma(2-\alpha)t_n^{\alpha-1}\tau\sum_{l=1}^{n}\left\|g^{l-\frac{1}{2}}\right\|^2\,.$$

Proof. Taking the inner product (3.13) with $\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}$, we have the following equality:

$$\left(\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha_1} v^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right) + \left(\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha} v^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)$$
$$+\frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\left(\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)$$
$$=\left((\mathscr{H}_y\delta_x^2+\mathscr{H}_x\delta_y^2)v^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right) + \left(g^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)\,. \tag{3.16}$$

Recalling $D_{\varphi_2}^{\alpha_1}$ and $D_{\varphi_2}^{\alpha}$, we have

$$\left(\mathscr{H}_x\mathscr{H}_y D_{\varphi_2}^{\alpha_1} v^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)$$
$$=\frac{1}{\mu_{\alpha_1}}\left\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right\|^2 - \frac{1}{\mu_{\alpha_1}}\sum_{l=1}^{n-1}\left(b_{n-l-1}^{\alpha_1} - b_{n-l}^{\alpha_1}\right)\left(\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)$$
$$-\frac{b_{n-1}^{\alpha_1}}{\mu_{\alpha_1}}\left(\mathscr{H}_x\mathscr{H}_y\varphi_2, \mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)\,, \tag{3.17}$$

$$\begin{aligned}&\left(\mathscr{H}_x\mathscr{H}_y D^{\alpha}_{\varphi_2} v^{n-\frac{1}{2}}, \mathscr{H}_x\mathscr{H}_y \delta_t v^{n-\frac{1}{2}}\right)\\ =&\frac{1}{\mu_\alpha}\left\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right\|^2-\frac{1}{\mu_\alpha}\sum_{l=1}^{n-1}\left(b^{\alpha}_{n-l-1}-b^{\alpha}_{n-l}\right)\left(\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)\\ &-\frac{b^{\alpha}_{n-1}}{\mu_\alpha}\left(\mathscr{H}_x\mathscr{H}_y\varphi_2,\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right). \end{aligned}\tag{3.18}$$

Invoking Lemmas 3.1 and 3.2, we obtain the following estimates:

$$\left((\mathscr{H}_y\delta_x^2+\mathscr{H}_x\delta_y^2)v^{n-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)=-\frac{1}{2\tau}\left(\|v^n\|^2_{h^*}-\|v^{n-1}\|^2_{h^*}\right),\tag{3.19}$$

$$\left(\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right)\geq\frac{1}{3}\left\|\delta_x\delta_y\delta_t v^{n-\frac{1}{2}}\right\|^2.\tag{3.20}$$

On substituting Eqs. (3.17)-(3.20) into Eq. (3.16), from the Cauchy-Schwarz inequality we deduce that

$$\begin{aligned}&\left(\frac{1}{\mu_{\alpha_1}}+\frac{1}{\mu_\alpha}\right)\left\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right\|^2+\frac{1}{2\tau}\left(\|v^n\|^2_{h^*}-\|v^{n-1}\|^2_{h^*}\right)+\frac{1}{3}\left\|\delta_x\delta_y\delta_t v^{n-\frac{1}{2}}\right\|^2\\ \leq&\frac{1}{\mu_{\alpha_1}}\sum_{l=1}^{n-1}(b^{\alpha_1}_{n-l-1}-b^{\alpha_1}_{n-l})(\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}})+\frac{b^{\alpha_1}_{n-1}}{\mu_{\alpha_1}}(\mathscr{H}_x\mathscr{H}_y\varphi_2,\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}})\\ &+\frac{1}{\mu_\alpha}\sum_{l=1}^{n-1}(b^{\alpha}_{n-l-1}-b^{\alpha}_{n-l})(\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}})+\frac{b^{\alpha}_{n-1}}{\mu_\alpha}(\mathscr{H}_x\mathscr{H}_y\varphi_2,\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}})\\ &+(g^{n-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}})\\ \leq&\frac{1}{2\mu_{\alpha_1}}\sum_{l=1}^{n-1}(b^{\alpha_1}_{n-l-1}-b^{\alpha_1}_{n-l})(\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}}\|^2+\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\|^2)+\frac{b^{\alpha_1}_{n-1}}{2\mu_{\alpha_1}}\|\mathscr{H}_x\mathscr{H}_y\varphi_2\|^2\\ &+\frac{b^{\alpha_1}_{n-1}}{2\mu_{\alpha_1}}\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\|^2+\frac{1}{2\mu_\alpha}\sum_{l=1}^{n-1}(b^{\alpha}_{n-l-1}-b^{\alpha}_{n-l})(\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}}\|^2+\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\|^2)\\ &+\frac{b^{\alpha}_{n-1}}{2\mu_\alpha}\|\mathscr{H}_x\mathscr{H}_y\varphi_2\|^2+\frac{b^{\alpha}_{n-1}}{2\mu_\alpha}\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\|^2+|(g^{n-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}})|,\end{aligned}$$

implying that

$$\begin{aligned}&\left(\frac{1}{\mu_{\alpha_1}}+\frac{1}{\mu_\alpha}\right)\left\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}}\right\|^2+\frac{1}{\tau}\left(\|v^n\|^2_{h^*}-\|v^{n-1}\|^2_{h^*}\right)\\ \leq&\frac{1}{\mu_{\alpha_1}}\sum_{l=1}^{n-1}\left(b^{\alpha_1}_{n-l-1}-b^{\alpha_1}_{n-l}\right)\left\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}}\right\|^2+\frac{b^{\alpha_1}_{n-1}}{\mu_{\alpha_1}}\|\mathscr{H}_x\mathscr{H}_y\varphi_2\|^2\\ &+\frac{1}{\mu_\alpha}\sum_{l=1}^{n-1}\left(b^{\alpha}_{n-l-1}-b^{\alpha}_{n-l}\right)\left\|\mathscr{H}_x\mathscr{H}_y\delta_t v^{l-\frac{1}{2}}\right\|^2+\frac{b^{\alpha}_{n-1}}{\mu_\alpha}\|\mathscr{H}_x\mathscr{H}_y\varphi_2\|^2\\ &+2\left|(g^{n-\frac{1}{2}},\mathscr{H}_x\mathscr{H}_y\delta_t v^{n-\frac{1}{2}})\right|.\end{aligned}\tag{3.21}$$

On denoting

$$G^n = \|v^n\|_{h^*}^2 + \tau \sum_{l=1}^{n} \left(\frac{b_{n-l}^{\alpha_1}}{\mu_{\alpha_1}} + \frac{b_{n-l}^{\alpha}}{\mu_{\alpha}} \right) \left\| \mathscr{H}_x \mathscr{H}_y \delta_t v^{l-\frac{1}{2}} \right\|^2 , \ 1 \le n \le N \quad \text{and} \quad G^0 = \|v^0\|_{h^*}^2 ,$$

Eq. (3.21) can be written as

$$G^n \leqslant G^{n-1} + \tau \left(\frac{b_{n-1}^{\alpha_1}}{\mu_{\alpha_1}} + \frac{b_{n-1}^{\alpha}}{\mu_{\alpha}} \right) \left\| \mathscr{H}_x \mathscr{H}_y \varphi_2 \right\|^2 + 2\tau \left| (g^{n-\frac{1}{2}}, \mathscr{H}_x \mathscr{H}_y \delta_t v^{n-\frac{1}{2}}) \right| , \quad 1 \leqslant n \leqslant N .$$

Recalling the analytical method and tools in the proof of Theorem 2.1 or given in Refs. [13, 26], we similarly obtain the desired result. ☐

The following stability statement follows immediately from Theorem 3.1.

Theorem 3.2. *The L1-CADI scheme* (3.10)-(3.12) *is unconditionally stable with respect to the initial values $\varphi_1(x,y)$ and $\varphi_2(x,y)$ and the inhomogeneous term f.*

Let us now consider the convergence of the L1-CADI scheme (3.10)-(3.12). On setting

$$e_{ij}^n = U_{ij}^n - u_{ij}^n , \quad (x_i, y_j) \in \bar{\Omega}_h , \quad 0 \leqslant n \leqslant N ,$$

and subtracting Eqs. (3.10)-(3.12) from Eqs. (3.6), (3.8) and (3.9) respectively, we get the error equations

$$\begin{aligned} & \mathscr{H}_x \mathscr{H}_y D_0^{\alpha_1} e_{ij}^{n-\frac{1}{2}} + \mathscr{H}_x \mathscr{H}_y D_0^{\alpha} e_{ij}^{n-\frac{1}{2}} + \frac{\mu_{\alpha_1} \mu_\alpha \tau^2}{4(\mu_{\alpha_1} + \mu_\alpha)} \delta_x^2 \delta_y^2 \delta_t e_{ij}^{n-\frac{1}{2}} \\ & = (\mathscr{H}_y \delta_x^2 + \mathscr{H}_x \delta_y^2) e_{ij}^{n-\frac{1}{2}} + R_{ij}^{n-\frac{1}{2}} , \quad (x_i, y_j) \in \Omega_h , \ 1 \leqslant n \leqslant N , \\ & e_{ij}^0 = 0 , \quad (x_i, y_j) \in \bar{\Omega}_h , \\ & e_{ij}^n = 0 , \quad (x_i, y_j) \in \partial \Omega_h , \ 1 \leqslant n \leqslant N . \end{aligned}$$

With the help of Eq. (3.7), it then follows from Theorem 3.1 that

$$\|e^n\|_{h^*}^2 \leqslant \Gamma(2-\alpha) t_n^{\alpha-1} \tau \sum_{l=1}^{n} \|R^{l-\frac{1}{2}}\|^2 \leqslant c_3^2 L_1 L_2 \Gamma(2-\alpha) T^\alpha (\tau^{3-\alpha} + h_1^4 + h_2^4)^2 ,$$

and applying Lemma 3.3 yields the following convergence result.

Theorem 3.3. *Assume that $u(x,y,t) \in \mathscr{C}_{x,y,t}^{6,6,3}(\Omega \times [0,T])$ is the solution of* (3.1)-(3.3) *and $\{u_{ij}^n \mid (x_i,y_j) \in \bar{\Omega}_h,\ 0 \leqslant n \leqslant N\}$ is the solution of the L1-CADI scheme* (3.10)-(3.12). *Then*

$$\|e^n\|_{H^1} \leqslant \frac{3c_3}{4} \sqrt{\left(6 + \frac{L_1^2 L_2^2}{L_1^2 + L_2^2}\right) L_1 L_2 \Gamma(2-\alpha) T^\alpha} (\tau^{3-\alpha} + h_1^4 + h_2^4) , \quad 0 \leqslant n \leqslant N .$$

4. Numerical Experiments

We now present some calculations using the numerical methods previously discussed, where the errors involved are measured by comparing the numerical solutions with the exact solutions.

Example 4.1. Let $L = \pi, T = 1$. In order to test the convergence rate of the proposed methods, we refer to the exact solution of the problem (2.1)–(2.3) — viz.

$$u(x,t) = t^{1+\alpha_1+\alpha}\sin x \ .$$

It can readily be checked that the corresponding source term $f(x,t)$ and the respective initial and boundary conditions are

$$f(x,t) = \left(\frac{\Gamma(2+\alpha_1+\alpha)}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{\Gamma(2+\alpha_1+\alpha)}{\Gamma(2+\alpha_1)}t^{1+\alpha_1} + t^{1+\alpha_1+\alpha}\right)\sin x \ ,$$

and

$$\varphi_1(x) = 0 \ , \quad \varphi_2(x) = 0 \ , \quad \psi_1(t) = 0 \ , \quad \psi_2(t) = 0 \ .$$

We compute the maximum norm errors of the numerical solution

$$e_\infty(h,\tau) = \max_{0\le k\le N} \|U^k - u^k\|_\infty \ ,$$

and characterise the temporal convergence order and the spatial convergence order as

$$\text{Order1} = \log_2\left(\frac{e_\infty(h,2\tau)}{e_\infty(h,\tau)}\right) , \qquad \text{Order2} = \log_2\left(\frac{e_\infty(2h,\tau)}{e_\infty(h,\tau)}\right) .$$

In order to show the effectiveness of the L1-CD scheme, we construct the corresponding Crank-Nicolson scheme L1-CND, which is also computationally efficient when the storage is inexpensive:

$$\begin{aligned}
&D^{\alpha_1}_{\varphi_2} u_i^{k-\frac{1}{2}} + D^{\alpha}_{\varphi_2} u_i^{k-\frac{1}{2}} = \delta_x^2 u_i^{k-\frac{1}{2}} + f_i^{k-\frac{1}{2}} \ , \quad 1 \leqslant i \leqslant M-1, \ \ 1 \leqslant k \leqslant N \ ,\\
&u_0^k = \psi_1(t_k), \quad u_M^k = \psi_2(t_k) \ , \quad 1 \leqslant k \leqslant N \ ,\\
&u_i^0 = \varphi_1(x_i) \ , \quad 0 \leqslant i \leqslant M \ ,
\end{aligned}$$

with the truncation error $\mathscr{O}(\tau^{3-\alpha} + h^2)$. We compare the the numerical results from the L1-CD scheme with the results obtained from the L1-CND scheme.

The first computational investigation concerns the temporal errors and convergence orders. In order to find the temporal convergence order, the space step h should be chosen sufficiently small to prevent large spatial discretisation error. The computational results of the L1-CD and the L1-CND schemes with $h = \pi/20$ and $h = \pi/100$ are presented in Table 1, respectively. It is observed that both schemes generate $3-\alpha$ temporal convergence order, consistent with our theoretical analysis.

Table 1: Numerical convergence of the L1-CND and L1-CD schemes in the temporal direction for Example 4.1.

α_1, α	τ	L1-CND scheme ($h=\pi/100$) $e_\infty(h,\tau)$	Order1	L1-CD scheme ($h=\pi/20$) $e_\infty(h,\tau)$	Order1
	1/10	1.387e-1	*	1.387e-1	*
$\alpha_1=1.15$	1/20	6.485e-2	1.096	6.485e-2	1.096
$\alpha=1.95$	1/40	3.081e-2	1.074	3.080e-2	1.074
	1/80	1.475e-2	1.062	1.475e-2	1.063
	1/160	7.093e-3	1.056	7.090e-3	1.057
	1/10	5.793e-2	*	5.793e-2	*
$\alpha_1=1.35$	1/20	2.180e-2	1.410	2.179e-2	1.410
$\alpha=1.65$	1/40	8.266e-3	1.399	8.262e-3	1.399
	1/80	3.152e-3	1.391	3.149e-3	1.392
	1/160	1.208e-3	1.383	1.205e-3	1.386

Table 2: Numerical convergence of the L1-CND and L1-CD schemes in the spatial direction for Example 4.1.

α_1, α	L1-CND scheme ($\tau=1/20000$) h	$e_\infty(h,\tau)$	Order2	L1-CD scheme ($\tau=1/200000$) h	$e_\infty(h,\tau)$	Order2
	$\pi/4$	3.870e-3	*	$\pi/2$	2.096e-3	*
$\alpha_1=1.1$	$\pi/8$	9.803e-4	1.981	$\pi/4$	1.243e-4	4.076
$\alpha=1.3$	$\pi/16$	2.459e-4	1.995	$\pi/8$	7.638e-6	4.025
	$\pi/32$	6.155e-5	1.998	$\pi/16$	4.756e-7	4.005
	$\pi/4$	2.526e-3	*	$\pi/2$	1.369e-3	*
$\alpha_1=1.3$	$\pi/8$	6.407e-4	1.979	$\pi/4$	8.124e-5	4.075
$\alpha=1.5$	$\pi/16$	1.610e-4	1.993	$\pi/8$	4.999e-6	4.022
	$\pi/32$	4.050e-5	1.991	$\pi/16$	3.195e-7	3.968

Secondly, we test the spatial errors and convergence orders of the two schemes by letting h vary and fixing the time step size τ sufficiently small to avoid temporal error. Table 2 shows the maximum norm errors and spatial convergence orders of the L1-CD scheme and the L1-CND scheme with different α_1,α. As predicted in our theoretical estimates, the L1-CD scheme attains fourth-order spatial accuracy while the L1-CND scheme has second-order spatial accuracy.

Next, in order to quantify some features of the computational efficiencies of the L1-CD scheme more precisely, we consider the CPU time for both schemes. As mentioned before, since fractional derivatives are non-local operators, they require a large memory storage capacity if low-order finite difference methods are employed for the spatial approximation. For the L1-CND scheme, the optimal step sizes satisfy $\tau^{3-\alpha}\approx h^2$, or $N\approx[M^{\frac{2}{3-\alpha}}]$. For the L1-CD scheme, the optimal step sizes satisfy $\tau^{3-\alpha}\approx h^4$, or $N\approx[M^{\frac{4}{3-\alpha}}]$. From Table 3, it is clear that the two schemes provide almost the same accuracy for the same temporal grid size, but the L1-CD scheme needs fewer spatial grid points and less CPU time. Thus the L1-CD scheme reduces both the storage requirement and the necessary CPU time successfully.

Example 4.2. Let $T=1$, $\Omega=(0,\pi)\times(0,\pi)$. We consider the exact solution of the problem

Table 3: The maximum norm error and CPU time of the L1-CND and L1-CD schemes for Example 4.1.

		L1-CND scheme			L1-CD scheme		
α_1, α	N	M	$e_\infty(h,\tau)$	CPU time(s)	M	$e_\infty(h,\tau)$	CPU time(s)
	44	25	1.130e-3	0.0581	5	1.029e-3	0.0308
	225	100	6.655e-5	1.2490	10	6.337e-5	0.3184
$\alpha_1=1.1,\ \alpha=1.3$	585	225	1.274e-5	8.2231	15	1.205e-5	1.4025
	1151	400	3.962e-6	34.0598	20	3.763e-6	4.2526
	1947	625	1.601e-6	110.9587	25	1.517e-6	10.6970
	141	25	1.816e-3	0.1999	5	1.702e-3	0.1114
	1194	100	1.121e-4	8.9086	10	1.102e-4	2.1454
$\alpha_1=1.1,\ \alpha=1.7$	4157	225	2.209e-5	124.5583	15	2.160e-5	19.5761
	10073	400	6.984e-6	1012.6392	20	6.867e-6	110.1357
	20015	625	2.859e-6	5602.6990	25	2.806e-6	480.6011

(3.1)–(3.3) as follows:

$$u(x,y,t)=t^{3+\alpha_1+\alpha}\sin x\sin y\ .$$

It is again not difficult to obtain the corresponding forcing term $f(x,y,t)$, and the initial and boundary conditions $\varphi_1(x,y),\varphi_2(x,y)$ and $\psi(x,y,t)$.

In order to test the convergence rate of the proposed methods, we use the same spacing h in each direction ($h_1=h_2=h$), and compute the maximum norm errors of the numerical solution

$$E_\infty(h,\tau)=\max_{\substack{(x_i,y_j)\in\bar{\Omega}_h\\ 0\le n\le N}}\left|u(x_i,y_j,t_n)-u_{ij}^n\right|\ ,$$

via

$$\text{Order3}=\log_2\left(\frac{E_\infty(h,2\tau)}{E_\infty(h,\tau)}\right),\qquad \text{Order4}=\log_2\left(\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\right).$$

As for Example 4.1, we give the following ADI scheme (L1-ADI) to compare with the L1-CADI scheme:

$$\begin{aligned}
&D_{\varphi_2}^{\alpha_1}u_{ij}^{n-\frac12}+D_{\varphi_2}^{\alpha}u_{ij}^{n-\frac12}+\frac{\mu_{\alpha_1}\mu_\alpha\tau^2}{4(\mu_{\alpha_1}+\mu_\alpha)}\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac12}=\Delta_h u_{ij}^{n-\frac12}+f_{ij}^{n-\frac12},\ (x_i,y_j)\in\Omega_h\,,1\leqslant n\leqslant N,\\
&u_{ij}^0=\varphi_1(x_i,y_j),\qquad (x_i,y_j)\in\bar{\Omega}_h\ ,\\
&u_{ij}^n=\psi(x_i,y_j,t_n),\quad (x_i,y_j)\in\partial\Omega_h\ ,\ 1\leqslant n\leqslant N\,,
\end{aligned}$$

where the truncation error of the L1-ADI scheme is $\mathcal{O}(\tau^{3-\alpha}+h_1^2+h_2^2)$.

As before, the numerical accuracy in time is first verified. For fixed space step sizes $h=\pi/20$ and $h=\pi/200$ respectively, the results we obtain on varying the temporal step size τ are displayed in Table 4. We conclude that there is again $(3-\alpha)$-order convergence in time.

Secondly, we test the spatial errors and convergence orders of the two schemes, by letting h vary and fixing the time step τ sufficiently small to avoid significant temporal

Table 4: Numerical convergence of the L1-ADI and L1-CADI schemes in the temporal direction for Example 4.2.

α_1, α	τ	L1-ADI scheme ($h=\pi/200$) $E_\infty(h,\tau)$	Order3	L1-CADI scheme ($h=\pi/20$) $E_\infty(h,\tau)$	Order3
$\alpha_1=1.2$, $\alpha=1.9$	1/10	2.079e-1	*	2.079e-1	*
	1/20	9.184e-2	1.179	9.184e-2	1.179
	1/40	4.168e-2	1.140	4.168e-2	1.140
	1/80	1.917e-2	1.121	1.917e-2	1.121
	1/160	8.874e-3	1.111	8.874e-3	1.111
$\alpha_1=1.1$, $\alpha=1.3$	1/10	3.408e-2	*	3.409e-2	*
	1/20	1.019e-2	1.742	1.019e-2	1.742
	1/40	3.046e-3	1.742	3.048e-3	1.741
	1/80	9.120e-4	1.740	9.137e-4	1.738
	1/160	2.737e-4	1.737	2.754e-4	1.730

Table 5: Numerical convergence of the the L1-ADI and L1-CADI schemes in the spatial direction for Example 4.2.

α_1, α	L1-ADI scheme ($\tau=1/20000$) h	$E_\infty(h,\tau)$	Order4	L1-CADI scheme ($\tau=1/200000$) h	$E_\infty(h,\tau)$	Order4
$\alpha_1=1.35$ $\alpha=1.65$	$\pi/4$	7.468e-3	*	$\pi/2$	1.072e-3	*
	$\pi/8$	1.979e-3	1.916	$\pi/4$	6.370e-5	4.073
	$\pi/16$	5.032e-4	1.976	$\pi/8$	3.981e-6	4.000
	$\pi/32$	1.275e-4	1.981	$\pi/16$	3.153e-7	3.658
$\alpha_1=1.45$ $\alpha=1.55$	$\pi/4$	7.689e-3	*	$\pi/2$	1.104e-3	*
	$\pi/8$	2.037e-3	1.917	$\pi/4$	6.552e-5	4.074
	$\pi/16$	5.170e-4	1.978	$\pi/8$	4.046e-6	4.018
	$\pi/32$	1.302e-4	1.989	$\pi/16$	2.722e-7	3.893

error. Table 5 gives the maximum norm errors and spatial convergence orders for the two schemes. As predicted by our theoretical estimates, the L1-CADI scheme attains fourth-order spatial accuracy whereas the L1-ADI scheme has second-order spatial accuracy. In Table 6, we display some CPU time results for the L1-CADI and L1-ADI schemes. It is clear that the two schemes generate almost the same accuracy for the same temporal grid size, while the L1-CADI scheme needs fewer spatial grid points and less CPU time, and therefore requires less storage and CPU time.

We compute the problem for a longer time by fixing $T=10$ and $M_1=M_2=4,5,\cdots,14$, and still choosing the optimal step sizes $\tau^{3-\alpha}\approx h^2$ for the L1-ADI scheme and $\tau^{3-\alpha}\approx h^4$ for the L1-CADI scheme, respectively. Fig. 1 shows the maximum error and CPU time of both schemes for $t=1,2,\cdots,10$ when $\alpha_1=1.1$, $\alpha=1.2$, and also the efficiency of the L1-CADI scheme.

Example 4.3. Let $L=1$, $T=1$. Consider the following two-term time fractional diffusion wave equation

$$ {}_0^C\mathscr{D}_t^{\alpha_1}u(x,t)+{}_0^C\mathscr{D}_t^{\alpha}u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{4.1}$$

Table 6: The maximum norm error and CPU time of the L1-ADI and L1-CADI schemes for Example 4.2.

$\alpha_1,\quad \alpha$	N	L1-ADI scheme $M_1=M_2$	$E_\infty(h,\tau)$	CPU time(s)	L1-CADI scheme $M_1=M_2$	$E_\infty(h,\tau)$	CPU time(s)
$\alpha_1=1.1,\ \alpha=1.3$	44	25	1.756e-7	0.9955	5	1.594e-7	0.2609
	97	49	2.506e-9	8.6312	7	2.384e-9	1.3806
	176	81	1.015e-10	46.4399	9	9.844e-11	5.3702
	282	121	8.013e-12	181.5510	11	7.852e-12	16.7878
$\alpha_1=1.5,\ \alpha=1.7$	141	25	4.127e-11	3.3463	5	3.748e-11	1.0444
	398	49	6.739e-14	44.0832	7	6.412e-14	10.4380
	863	81	5.607e-16	370.4846	9	5.440e-16	68.4346
	1601	121	1.224e-17	2865.1578	11	1.199e-17	330.9768

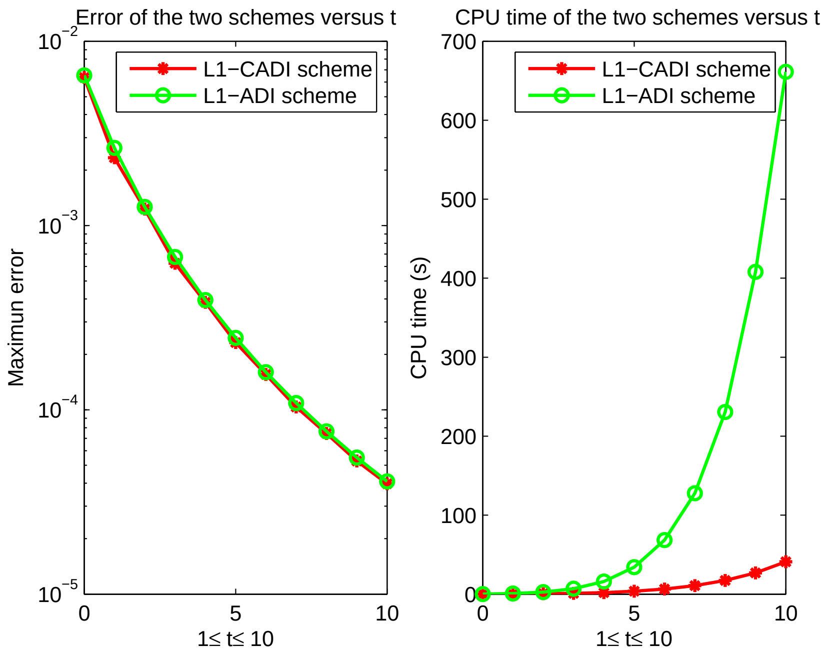

Figure 1: Error and CPU time of the L1-ADI and L1-CADI schemes.

subject to zero boundary conditions and the following initial conditions

$$u(x,0)=\sin(\pi x)\,,\quad u_t(x,0)=0\,,\quad 0\le x\le 1\,. \tag{4.2}$$

Denote

$$F_\infty(\tau)=\max_{0\le i\le M}\left|u_i^N(h,\tau)-u_i^{2N}\Big(h,\frac{\tau}{2}\Big)\right| \text{ for sufficiently small fixed } h\,,$$

and

$$G_\infty(h)=\max_{0\le i\le M}\left|u_i^N(h,\tau)-u_{2i}^N\Big(\frac{h}{2}\tau\Big)\right| \quad \text{for sufficiently small fixed } \tau.$$

We have tested the algorithms L1-CD and L1-CND for (4.1)-(4.2). The results at time $T=1$ with $h=1/10000$ are reported in Fig. 2. We show the errors in the L_∞-norm as

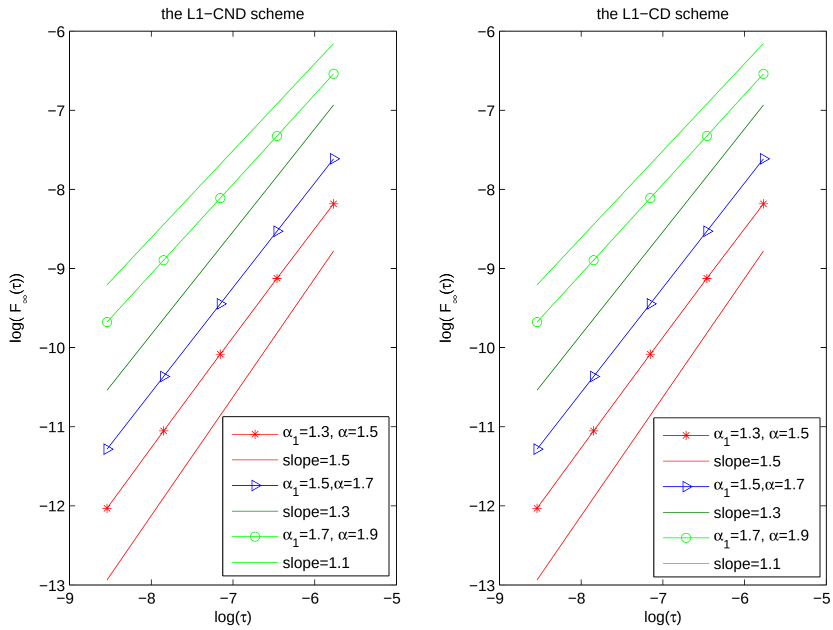

Figure 2: Convergence orders in temporal direction of the L1-CND and L1-CD schemes for Example 4.3.

function of N for different α_1 and α. The slopes are 1.5, 1.3 and 1.1 respectively, in good agreement with the theoretical result of $3-\alpha$.

The errors obtained by the L1-CD and L1-CND schemes at time $T = 1$ with $\tau = 1/100000$ are shown in Fig. 3. It is clear that the L1-CD scheme attains fourth-order spatial accuracy while the L1-CND scheme has second-order spatial accuracy, as theoretically predicted.

Example 4.4. Consider the following two-term time fractional equation in the 2D domain $\Omega=(0,1)\times(0,1)$

$$\begin{aligned}{}_0^C\mathscr{D}_t^{\alpha_1}u(x,y,t)+{}_0^C\mathscr{D}_t^{\alpha}u(x,y,t)=&\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}+e^{x+y}\Big[\frac{\Gamma(3+\alpha+\alpha_1)}{\Gamma(3+\alpha)}t^{2+\alpha}\\&+\frac{\Gamma(3+\alpha+\alpha_1)}{\Gamma(3+\alpha_1)}t^{2+\alpha_1}-2t^{2+\alpha+\alpha_1}\Big],\end{aligned}\tag{4.3}$$

subject to zero initial conditions and the boundary condition

$$u(x,y,t)|_{\partial\Omega}=e^{x+y}t^{2+\alpha+\alpha_1}|_{\partial\Omega}.\tag{4.4}$$

The exact solution of the problem (4.3)–(4.4) as follows:

$$u(x,y,t)=e^{x+y}t^{2+\alpha+\alpha_1}.$$

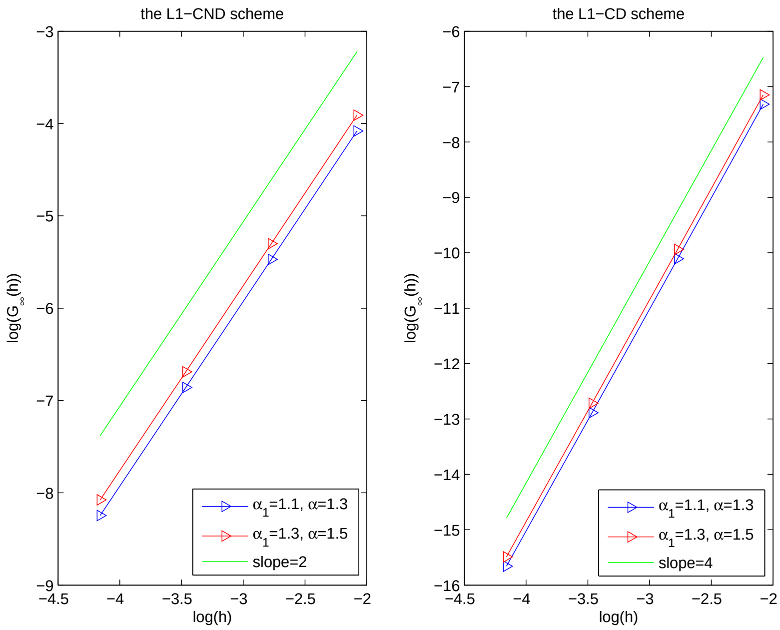

Figure 3: Convergence orders in spatial direction of the L1-CND and L1-CD schemes for Example 4.3.

With the fixed spatial step size h, the problem (4.3)-(4.4) is first solved numerically using the L1-CADI and L1-ADI schemes, respectively. The computational results are reported in Table 7, from which the $(3-\alpha)$th-order convergence in time of both schemes is evident.

Secondly, we examine the numerical accuracy of the L1-CADI and L1-ADI schemes in space. The computational results in Table 8 show that the spatial convergence order of the L1-CADI scheme is fourth-order, while the L1-ADI scheme has second-order spatial accuracy.

As for Example 4.1, we choose the optimal step sizes $\tau^{3-\alpha} \approx h^2$ for the L1-ADI scheme and $\tau^{3-\alpha} \approx h^4$ for the L1-CADI scheme, respectively. In Table 9, we display some resulting CPU times for the L1-CADI and L1-ADI schemes. It is clear that the two schemes generate almost the same accuracy for the same temporal grid size, but the L1-CADI scheme again needs fewer spatial grid points and less CPU time, so requires less storage.

5. Conclusions

Some effective and fast numerical methods have been constructed for multi-term time fractional diffusion-wave equations. Our approach is based on the L1 approximation for the Caputo fractional derivative in the temporal direction, and a compact difference method for the spatial approximation with fourth-order accuracy that reduces storage requirements. Using some novel techniques, we rigorously proved the unique solvability, unconditional stability and global convergence for both the one-dimensional and two-dimensional cases. Numerical examples verify the effectiveness of these compact difference schemes. The

Table 7: Numerical convergence of the L1-ADI and L1-CADI schemes in the temporal direction for Example 4.4.

$\alpha_1,\quad\alpha$	τ	L1-ADI scheme ($h=1/200$) $E_\infty(h,\tau)$	Order3	L1-CADI scheme ($h=1/20$) $E_\infty(h,\tau)$	Order3
$\alpha_1=1.2,\ \alpha=1.9$	1/10	3.591e-1	$*$	3.592e-1	$*$
	1/20	1.678e-1	1.098	1.678e-1	1.098
	1/40	7.809e-2	1.104	7.810e-2	1.104
	1/80	3.632e-2	1.104	3.633e-2	1.104
	1/160	1.690e-2	1.104	1.690e-2	1.104
$\alpha_1=1.1,\ \alpha=1.3$	1/10	2.919e-2	$*$	2.923e-2	$*$
	1/20	8.773e-3	1.734	8.785e-3	1.734
	1/40	2.626e-3	1.740	2.630e-3	1.740
	1/80	7.865e-4	1.740	7.880e-4	1.739
	1/160	2.359e-4	1.737	2.368e-4	1.735

Table 8: Numerical convergence of the the L1-ADI and L1-CADI schemes in the spatial direction for Example 4.4.

$\alpha_1,\ \alpha$	L1-ADI scheme ($\tau=1/20000$) h	$E_\infty(h,\tau)$	Order4	L1-CADI scheme ($\tau=1/200000$) h	$E_\infty(h,\tau)$	Order4
$\alpha_1=1.1$ $\alpha=1.15$	1/4	1.228e-3	$*$	1/4	1.857e-3	$*$
	1/8	3.470e-4	1.823	1/8	1.174e-4	3.984
	1/16	8.805e-5	1.978	1/16	7.327e-6	4.002
	1/32	2.210e-5	1.994	1/32	4.548e-7	4.010

Table 9: The maximum norm error and CPU time of the L1-ADI and L1-CADI schemes for Example 4.4.

$\alpha_1,\quad\alpha$	N	L1-ADI scheme $M_1=M_2$	$E_\infty(h,\tau)$	CPU time(s)	L1-CADI scheme $M_1=M_2$	$E_\infty(h,\tau)$	CPU time(s)
$\alpha_1=1.1,\ \alpha=1.3$	26	16	6.598e-5	0.3599	4	6.951e-5	0.1621
	68	36	1.254e-6	4.9273	6	1.334e-6	0.9599
	133	64	7.458e-8	32.6547	8	8.031e-8	4.3982
	225	100	7.984e-9	157.0927	10	8.684e-9	15.5447
$\alpha_1=1.5,\ \alpha=1.7$	71	16	9.129e-7	0.9941	4	9.053e-7	0.4062
	248	36	2.706e-9	21.0838	6	2.725e-9	5.6526
	601	64	4.124e-11	257.7156	8	4.259e-11	47.1732
	1194	100	1.576e-12	1924.6880	10	1.663e-12	270.1644

methods and techniques discussed could be extended to other kinds of multi-term time-space fractional differential equations, and to equations with a nonlinear source term.

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