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Projection and Contraction Method for the Valuation of American Options

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Abstract. An efficient numerical method is proposed for the valuation of American options via the Black-Scholes variational inequality. A far field boundary condition is employed to truncate the unbounded domain problem to produce the bounded domain problem with the associated variational inequality, to which our finite element method is applied. We prove that the matrix involved in the finite element method is symmetric and positive definite, and solve the discretized variational inequality by the projection and contraction method. Numerical experiments are conducted that demonstrate the superior performance of our method, in comparison with earlier methods.

AMS subject classifications: 35A35, 90A09, 65K10, 65M12, 65M60

Key words: American option, Black-Scholes variational inequality, finite element method, projection and contraction method.

1. Introduction

An option is a contract that permits but does not require the holder to either buy (a "call option") or sell (a "put option") a certain amount of an underlying asset at a fixed price within a fixed period of time. Options are called European or American, according to their exercise prerogative. Any European option can only be exercised on the maturity date, so it is easy to get a closed-form solution for the exercise price at any time. On the other hand, an American option may be exercised not only on its expiry date but also at any time beforehand, so there is no such closed-form solution, which makes the American option pricing problem a challenging task.

There has been extensive analytical and numerical work done on the American option pricing problem, due to its complexity and importance. The analytical research has continued since the 1970s, but the results have often been unsatisfactory. Early closed-form solutions depend upon the optimal exercise boundary that is unknown in practice [3–5, 13, 16, 24], and a more recent exact solution in the form of a Taylor series expansion is a very beautiful theoretical result [29]. Numerical methods for American options have attracted

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increasing interest, and are mainly of two types — viz. the Monte Carlo method [6,19,22] and the partial differential equation (PDE) method [1,9,10,14,20,23,27]. The Monte Carlo method has a high computational cost due to its slow convergence, and in this article we pursue the famous Black-Scholes PDE approach, which is widely regarded as one of most effective [7,11,15].

Numerical methods developed and extensively studied in recent decades include lattice tree methods, finite difference methods, finite element methods and spectral methods. In their seminal contribution, Cox *et al.* [8] introduced the binomial method to price American options, and its convergence was proven by Amin & Khanna [2]. The binomial method is essentially a difference method, and inspired a variety of finite difference schemes for American option pricing [9,20,26]. Refs. [14,15] discuss relevant convergence analysis. To improve the solution precision, finite element and spectral methods have received more attention recently [1,10,23,27]). One may refer to Ref. [18] and the references therein for a survey.

There are two main challenges in the numerical evaluation of American option prices:

- the solution domain of the option price is unbounded, so the truncation technique is a key issue; and
- the variational inequality under the Black-Scholes approach renders a complicated nonlinear problem, so an efficient numerical method to solve the problem quickly and accurately is extremely important in practice.

To meet the first challenge for American call options, Holmes & Yang [10] introduced a far field boundary condition, and we follow this idea to deal with put options in this article. The second challenge is our main concern here. We first discretize the American option pricing problem by a finite element method [21,28], and then solve the resulting system using the projection and contraction method [12]. Numerical experiments show that our proposed method is much faster than earlier methods within the same accuracy.

In Section 2, we summarise the linear complementary problem and corresponding variational inequality form for an American put option in the Black-Scholes model. The far field boundary condition is recalled and employed to truncate the unbounded domain. In Section 3, a finite element method is applied to the truncated variational inequality problem, and we prove that the matrix in the associated discretisation is symmetric and positive definite. The projection and contraction algorithm adopted to solve the resulting nonlinear system is also discussed in this section. In Section 4, numerical simulations are presented to compare the performance of our method against earlier methods, and our concluding remarks are in Section 5.

2. Pricing Problem on a Bounded Domain

The Black-Scholes model for an American put option that we adopt is summarised here. In particular, we represent the corresponding linear complementary problem on a bounded domain obtained via a far field boundary condition and its variational inequality form, which we will then proceed to solve using a finite element method.

2.1. Black-Scholes model

For simplicity, we consider only an American put option, since American call options can be obtained by put-call parity [1,15]. Let S, t, σ, r, q, K and T denote the price for some underlying asset of a certain amount, time, volatility, interest rate, dividend rate, exercise price and date of expiration, respectively. Then the option value $P = P(S,t)$ satisfies the following free boundary problem [11]:

$$
\begin{aligned}
&P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} + (r-q)SP_S - rP = 0\,, && B(t) < S < +\infty\,, \quad 0 \le t < T\,,\\
&P(S,T) = G(S)\,, && B(T) \le S < +\infty\,,\\
&P(B(t),t) = G(B(t))\,, && 0 \le t < T\,,\\
&P_S(B(t),t) = -1\,, && 0 \le t < T\,,\\
&\lim_{S\to+\infty} P(S,t) = 0\,, && 0 \le t < T\,,
\end{aligned}
\tag{2.1}
$$

where the subscripts S and t denote the respective differentiation, $G(S) = \max\{K - S, 0\}$, and $B(t)$ is the unknown optimal exercise boundary that satisfies [15,18]

$$KX \le B(t) \le B(T)\,.$$

Here KX is the optimal exercise boundary of a permanent American put option, where

$$X = \frac{-r+q+\frac{1}{2}\sigma^2 - \sqrt{(-r+q+\frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{-r+q-\frac{1}{2}\sigma^2 - \sqrt{(-r+q+\frac{1}{2}\sigma^2)^2 + 2r\sigma^2}} \quad \text{and} \quad B(T) = K\min(r/q, 1)\,.$$

2.2. Linear complementary problem

The Black-Scholes model implies that the put option price $P(S,t)$ satisfies (2.1) when $S > B(t)$, and of course the American put option price is $P(S,t) = G(S)$ when $S \le B(t)$. The price $P(S,t)$ of the American put option can therefore be characterised mathematically by the following backward differential linear complementary problem [25]:

$$
\begin{aligned}
&\left(P_t + \frac{\sigma^2}{2} S^2 P_{SS} + (r-q)SP_S - rP\right)(P - G(S)) = 0\,, && 0 \le S < +\infty\,, \quad 0 \le t < T\,,\\
&P(S,T) = G(S)\,, && 0 \le S < +\infty\,,\\
&P(0,t) = G(0)\,, && 0 \le t < T\,,\\
&\lim_{S\to+\infty} P(S,t) = 0\,, && 0 \le t < T\,,
\end{aligned}
\tag{2.2}
$$

subject to the constraints

$$P_t + \frac{\sigma^2}{2} S^2 P_{SS} + (r-q)SP_S - rP \le 0\,, \quad P \ge G(S)\,. \tag{2.3}$$

Using standard variable transforms [9, 18]

$$P(S,t)=Ke^{-\alpha y-\beta\tau}\phi(y,\tau)\,,\qquad T-t=\tau/\frac{\sigma^2}{2},\qquad S=Ke^{y}\,, \tag{2.4}$$

where $\alpha=(r-q)/\sigma^2-1/2$ and $\beta=\alpha^2+2r/\sigma^2$, we can rewrite (2.2) as a forward differential linear complementary problem with constant coefficients — viz.

$$\begin{aligned}
&(\phi_\tau-\phi_{xx})(\phi-g)=0\,, && -\infty<x<+\infty\,,\quad 0<\tau\le T^*\,,\\
&\phi(x,0)=g(x,0)\,, && -\infty<x<+\infty\,,\\
&\lim_{x\to-\infty}\phi(x,\tau)=g(x,\tau)\,, && 0<\tau\le T^*\,,\\
&\lim_{x\to+\infty}\phi(x,\tau)=0\,, && 0<\tau\le T^*\,;
\end{aligned} \tag{2.5}$$

and the function G, the expiration date T and the constraints (2.3) render

$$\begin{aligned}
&g(x,\tau)=e^{\alpha x+\beta\tau}\max\{1-e^x,0\}\,,\qquad T^*=\frac{\sigma^2}{2}T\,,\\
&\phi_\tau-\phi_{xx}\ge 0\,,\qquad \phi-g\ge 0\,.
\end{aligned} \tag{2.6}$$

We now introduce the far field estimate, which plays a pivotal role in solving the option pricing problem on some truncated domain. We can adapt a result used by Holmes & Yang [10] for American call options, to deal with American put options considered here. Other methods to truncate the unbounded domain are discussed in Refs. [1, 9, 17, 27].

Lemma 2.1 ([30]). *For a given positive number $\varepsilon\in(0,1)$, we have*

$$P(S,t)\le\varepsilon\,,\quad \forall\, S\ge Ke^{L_1}\,,\ \ 0\le t<T\,,$$

where

$$L_1=-2.5T^*\alpha+0.5\sqrt{25(T^*)^2\alpha^2-20T^*\log(\varepsilon/\sqrt{5}K)}\,.$$

From Lemma 2.1, the right boundary in (2.5) can be truncated at $x=L_1$. For the left boundary, the boundary condition can be satisfied at $x=-L_2\le\log X$, since $P(S,t)=G(S)$ when $S\le B(t)$. For convenience, we let $L=\max\{L_1,L_2\}$ and $\Omega=[-L,L]$, such that the approximating bounded domain problem corresponding to (2.5)-(2.6) is formulated as

$$\begin{aligned}
&(u_\tau-u_{xx})(u-g)=0\,, && -L\le x\le L\,,\quad 0<\tau\le T^*\,,\\
&u(x,0)=g(x,0)\,, && -L\le x\le L\,,\\
&u(-L,\tau)=g(-L,\tau)\,, && 0<\tau\le T^*\,,\\
&u(L,\tau)=0\,, && 0<\tau\le T^*\,,
\end{aligned} \tag{2.7}$$

with constraints

$$u_\tau-u_{xx}\ge 0\,,\quad u-g\ge 0\,. \tag{2.8}$$

2.3. Variational inequality

Let us now summarise the variational inequality form corresponding to (2.7)-(2.8), which we will proceed to solve by our finite element method. We define

$$H^1_\tau(\Omega) := \left\{ v \in H^1(\Omega);\ v \geq g(x,\tau)\,,\ v(-L) = g(-L,\tau)\,,\ v(L) = 0 \right\},$$

then the variational inequality form of the problem reads as follows:

(VF:) Find $u(\tau) \in H^1_\tau(\Omega)$ such that $u(x,0) = g(x,0)$ and

$$(u_\tau, v-u) + (u_x, v_x - u_x) \geq 0 \quad \forall\ v \in H^1_\tau(\Omega) \ \ a.e.\ \ 0 < \tau \leq T^*. \tag{2.9}$$

Lemma 2.2 ([28]). *If $u \in L^2(0,T^*;H^2(\Omega))$, $u_\tau \in L^2(0,T^*;L^2(\Omega))$, then u is the solution of the linear complementary problem* (2.7)-(2.8) *if and only if u is the solution of the variational inequality problem* (2.9).

A brief sketch of a proof of Lemma 2.2 is as follows — cf. also Ref. [28].

Proof. If u is the solution of the linear complementary problem, from (2.7) an integration yields

$$(u_\tau, u-g) + (u_x, u_x - g_x) = 0\,, \qquad 0 < \tau \leq T^*. \tag{2.10}$$

Letting $v \in H^1_\tau(\Omega)$, we have $v \geq g$, and combining with the constraint condition $u_\tau - u_{xx} \geq 0$ leads to

$$(u_\tau, v-g) + (u_x, v_x - g_x) \geq 0\,, \qquad \forall\ v \in H^1_\tau(\Omega) \ \ a.e.\ \ 0 < \tau \leq T^*. \tag{2.11}$$

Subtracting Eq. (2.10) from inequality (2.11), it is easy to see that $u \in H^1_\tau(\Omega)$ satisfies the variational inequality (2.9). Conversely, if $u \in H^1_\tau(\Omega)$ is the solution of the variational inequality (2.9) and satisfies the premise, on integrating by parts we get

$$(u_\tau - u_{xx}, v-u) \geq 0\,, \qquad \forall\ v \in H^1_\tau(\Omega) \ \ a.e.\ \ 0 < \tau \leq T^*. \tag{2.12}$$

Since $v \in H^1_\tau(\Omega)$ is arbitrary, we can let $v = u + \psi \in H^1_\tau(\Omega)\ \forall\ \psi \in C_0^\infty(\Omega\ \psi \geq 0)$ such that

$$u_\tau - u_{xx} \geq 0\,,$$

so with $u \geq g$ we have

$$(u_\tau - u_{xx}, u-g) \geq 0\,. \tag{2.13}$$

Letting $v = g \in H^1_\tau(\Omega)$ in (2.12) and combining with (2.13), we have

$$(u_\tau - u_{xx}, u-g) = 0\,,$$

hence from

$$u_\tau - u_{xx} \geq 0\,, \qquad u - g \geq 0$$

we obtain

$$(u_\tau - u_{xx})(u-g) = 0$$

almost everywhere. ∎

3. Finite Element Method

We now discretize the variational inequality form (2.9) under our finite element method and establish that the associated matrix for the nonlinear system is positive definite, before presenting the projection and contraction algorithm used to obtain the numerical solution.

3.1. Notation

We define the respective temporal and spatial partitions of $[0, T^*]$ and $\Omega = [-L, L]$ as

$$J_\tau : 0 = \tau_0 < \tau_1 < \cdots < \tau_M = T \quad \text{and} \quad I_h : -L = x_0 < x_1 < \cdots < x_N = L \, .$$

For each spatial element $I_n := (x_{n-1}, x_n)$ with length $h_n := x_n - x_{n-1}$ $(n = 1, 2, \cdots, N)$, the mesh size of the partition is $h := \max_{1 \le n \le N} h_n$. Similarly, $J_m := (\tau_{m-1}, \tau_m)$, $k_m = \tau_m - \tau_{m-1}$ $(m = 1, \cdots, M)$ and $\triangle\tau = \max_{1 \le m \le M} k_m$ denote the temporal element, its associated local step size and the overall step size, respectively.

Jiang [15] mentions that a put option should be exercised if the stock price S is equal to or less than $B(t)$ at time t, but otherwise it is better to hold. The most important points in the option value function are therefore around the high contact point $B(t)$, which is the curve obtained by the transformation $S = Ke^x$. We will consider a lattice with $x_{min} = -L$, $x_{max} = L$ and symmetry about $x_{N/2} = 0$, and require an even number of intervals N. When the x_n are equidistant, the $S_n := Ke^{x_n}$ distribution is rather disadvantageous for our purposes, as there are then very few points placed around $B(t)$. To correct this, we need to distribute more knots x_n close to 0, whereas towards x_{min} and x_{max} the distances between the knots may be larger. Thus we use a graded mesh in the spatial direction [21]:

$$x_n = \text{sign}(2n - N) L \left(\frac{2n - N}{N} \right)^2 , \qquad n = 0, 1, \cdots, N \, .$$

In the temporal direction, an equidistant partition is used:

$$\triangle\tau = \frac{T^*}{M} \, , \quad \tau_m = m \triangle\tau \, , \qquad m = 0, 1, \cdots, M \, .$$

3.2. Discrete approximation

In this subsection, we consider the completely discrete form of the variational inequality (2.9) using a finite element method in the spatial direction and the backward Euler method in the temporal direction. The corresponding matrix form is presented, and we establish that the associated matrix is symmetric positive definire.

Suppose $V_h \subset H^1(\Omega)$ is a piecewise linear polynomial space, and the piecewise linear finite element space is

$$S_\tau^1(\Omega) := \left\{ v_h \in V_h(\Omega); \; v(x_n) \ge g(x_n, \tau) \; \forall \; x_n \, , \; v_h(-L) = g(-L, \tau) \, , \; v_h(L) = 0 \right\} .$$

The semi-discrete form for the problem (2.9) is then:
(**SDF**:) Find $u_h(\tau) \in S^1_\tau(\Omega)$, such that $u_h(x,0) = g_I(x,0)$ and satisfying

$$(u_{h\tau}, v_h - u_h) + (u_{hx}\ , v_{hx} - u_{hx}) \geq 0, \quad \forall\ v_h \in S^1_\tau(\Omega)\ \ a.e.\ \ 0 < \tau \leq T^*, \tag{3.1}$$

where $g_I(x,0)$ denotes the linear interpolant approximation of $g(x,0)$ belonging to $S^1_0(\Omega)$. We further discretize the semi-discrete variational inequality (3.1) about the time variable by the backward Euler method. Thus at a fixed point $\tau = \tau_m$ the completely discrete approximation of the variational inequality (2.9) is

$$(u^m_h - u^{m-1}_h, v_h - u^m_h) + \triangle\tau(u^m_{hx}\ , v_{hx} - u^m_{hx}) \geq 0\ , \quad \forall\ v_h \in S^1_{\tau_m}(\Omega)\ . \tag{3.2}$$

If a basis of $S^1_\tau(\Omega)$ is given by $\{\varphi_j(x)\}_{j=0}^{N-1}$ such that

$$\varphi_0(x) = \begin{cases} \dfrac{x - x_1}{x_0 - x_1}, & x \in [x_0, x_1)\ , \\ 0, & x \in \Omega \setminus [x_0, x_1)\ , \end{cases}$$
$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j)\ , \\ \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, & x \in [x_j, x_{j+1})\ , \\ 0\ , & x \in \Omega \setminus [x_{j-1}, x_{j+1})\ , \end{cases}$$

then at each time level $\tau = \tau_m$ the finite element approximation of the solution is

$$u^m_h = \sum_{j=1}^{N-1} u^m_j \varphi_j(x) + g(-L, \tau_m)\varphi_0(x)\ , \quad m = 0, 1, \cdots, M\ .$$

Now the corresponding matrix form of inequality (3.2) is

$$(V - U^m)^T((\triangle\tau A + B)U^m - BU^{m-1} + F^m) \geq 0 \qquad \forall\ V \geq G^m,\ \ m = 1, \cdots, M\ , \tag{3.3}$$

where

$$\begin{aligned} &V = (v_1, v_2, \cdots, v_{N-1})\ , \qquad U^m = (u^m_1, u^m_2, \cdots, u^m_{N-1})\ , \\ &A = \big((\varphi'_j, \varphi'_i)\big)_{N-1\times N-1}\ , \qquad B = \big((\varphi_j, \varphi_i)\big)_{N-1\times N-1}\ , \\ &F^m = \left(\left(\frac{h_1}{6} - \frac{\triangle\tau}{h_1}\right)g(-L, \tau_m) - \frac{h_1}{6}g(-L, \tau_{m-1}), 0, \cdots, 0\right)^T, \\ &G^m = \Big(g(x_1, \tau_m), \cdots g(x_{N-1}, \tau_m)\Big)^T. \end{aligned}$$

Writing $D = \triangle\tau A + B$ and $W^m = -BU^{m-1} + F^m$, we can simplify (3.3) as

$$(V - U^m)^T(DU^m + W^m) \geq 0\ , \qquad \forall\ V \geq G^m\ . \tag{3.4}$$

Furthermore, on defining $\overline{V}=V-G^m$, $\overline{U}^m=U^m-G^m$ and $\overline{W}^m=W^m+DG^m$ we arrive at

$$(\overline{V}-\overline{U}^m)^T(D\overline{U}^m+\overline{W}^m)\geq 0\,, \qquad \forall\,\overline{V}\geq 0\,, \tag{3.5}$$

which is our completely discretized form of the variational inequality (2.9). We now prove that the associated matrix is symmetric positive definite.

Theorem 3.1. *The matrix D is symmetric and positive definite.*

Proof. For simplicity, let us write

$$D=\begin{pmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & a_{N-2} & b_{N-2} & c_{N-2} \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} \end{pmatrix},$$

where

$$\begin{aligned} a_i &= -\frac{\triangle\tau}{h_i}+\frac{h_i}{6}\,, && i=2,\cdots,N-1\,, \\ b_i &= \triangle\tau\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)+\frac{1}{3}(h_i+h_{i+1})\,, && i=1,\cdots,N-1\,, \\ c_i &= -\frac{\triangle\tau}{h_{i+1}}+\frac{h_{i+1}}{6}\,, && i=1,\cdots,N-2\,. \end{aligned}$$

Given the structure of D, we can verify it is a irreducible symmetric matrix. Furthermore,

$$\begin{aligned} &|b_1|>|c_1|\,, \\ &|b_i|>|a_i|+|c_i|\,, \qquad i=2,\cdots,N-2\,, \\ &|b_{N-1}|>|a_{N-1}|\,, \end{aligned}$$

so D is a diagonally dominant matrix, and given that $b_i>0$ we conclude that it is a symmetric positive definite matrix. □

Remark 3.1. We also observe that D is an M-matrix provided $h^2/(\triangle\tau)$ is sufficiently small.

3.3. Projection and contraction method

The projection and contraction method used to solve the variational inequality (3.5) is tabulated on the next page. He [12] has already noted that the projection and contraction method efficiently solves inequalities such as (3.5) when D is a symmetric positive definite matrix, and he also discussed the convergence of the method.

Algorithm 3.1 Projection and Contraction Method

- For $m = 1 : M$
  - Let $k = 0$, $\beta^{(0)} = 1$, $\nu \in (0,1)$, $\mu \in (0,1)$, $\varepsilon = 10^{-6}$, $\overline{U} = \overline{U}^{m-1}$, $F_U = D\overline{U} + \overline{W}^m$, $tol = abs\big(\overline{U} - \max\big(\overline{U} - F_U, 0\big)\big)$.
  - While $(tol > \varepsilon)$
    * $\overline{U}^{(k)} = \overline{U}$, $F_U^{(k)} = F_U$, $\overline{U} = \max(\overline{U}^{(k)} - \beta^{(k)} F_U^{(k)}, 0)$, $F_U = D\overline{U} + \overline{W}^m$, $d_U = \overline{U}^{(k)} - \overline{U}$, $d_F = \beta^{(k)}(F_U^{(k)} - F_U)$, $\rho^{(k)} = \|d_F\|/\|d_U\|$;
    * While $(\rho^{(k)} > \nu)$;
      · $\beta^{(k)} = \frac{2}{3}\beta^{(k)} \min(1, 1/\rho^{(k)})$, $\overline{U} = \max(\overline{U}^{(k)} - \beta^{(k)} F_U^{(k)}, 0)$, $F_U = D\overline{U} + \overline{W}^m$, $d_U = \overline{U}^{(k)} - \overline{U}$, $d_F = \beta^{(k)}(F_U^{(k)} - F_U)$, $\rho^{(k)} = \|d_F\|/\|d_U\|$;
    * End
    * $d_{UF} = d_U - d_F$, $r_1 = (d_U, d_{UF})$, $r_2 = (d_{UF}, d_{UF})$, $\alpha^* = r_1/r_2$, $\overline{U} = \overline{U}^{(k)} - \alpha^* \gamma d_{UF}$, $F_U = D\overline{U} + \overline{W}^m$, $tol = abs\big(\overline{U} - \max(\overline{U} - F_U, 0)\big)$,
    * if $\rho^{(k)} < \mu$
      $\beta^{(k)} = \beta^{(k)} * \gamma$;
    * end;
    * $\beta^{(k+1)} = \beta^{(k)}$ and $k = k + 1$.
  - end
- end

4. Numerical Experiments

We now discuss some numerical simulations for American option pricing, to illustrate the theoretical analysis in Section 3 and to verify the efficiency of our projection and contraction finite element method (PCFEM). We consider one-year ($T = 1$) American put options, and assume that the volatility of the underlying assets is $\sigma = 0.2$ and the strike price is $K = 10$ in Eq. (2.2). For comparison, we consider three cases of interest rate r and dividend rate q.

The three cases considered are:

- Case I: $r < q$ with $r = 0.005$ and $q = 0.01$;
- Case II: $r = q$ with $r = q = 0.01$;
- Case III: $r > q$ with $r = 0.05$ and $q = 0.01$.

We chose $\varepsilon = 10^{-6}$ in Lemma 2.1, and adopted the truncated lengths $L = 1.902341$, 1.326117 and 1.276117 for the three cases $r < q$, $r = q$ and $r > q$, respectively. The so-

Table 1: l_2 error (at $t=0$) on the spatial notes defined in Subsection 3.1, and the time cost per option value computed by the CSPM, FFEM and PCFEM for three cases.

Case	Method	M	N	Error/10^{-2}	Time/s
$r<q$	CSPM	10000	43	3.171908	18.637314
	FFEM	512	AUTO	5.570208	1.591291
	PCFEM	200	20	4.116932	0.169713
$r=q$	CSPM	10000	33	1.771630	15.317074
	FFEM	512	AUTO	3.461689	1.490190
	PCFEM	200	20	3.125822	0.159173
$r>q$	CSPM	10000	15	1.091294	10.135287
	FFEM	512	AUTO	1.183515	1.165953
	PCFEM	200	20	2.564035	0.158804

lution obtained via the binomial method [15] with 40000 points in the temporal direction was adopted as the numerical approximation of the exact solution. The relevant parameters in the projection and contraction method were chosen to be $\nu = 0.9$, $\mu = 0.4$, and $\gamma = 1.5$. We compared results from our PCFEM with results from two other methods — viz. the Chebyshev spectral method (CPSM) [23] and the front-fixing finite element method (FFEM) [10].

The option values at $t = 0$ using our PCFEM for the three cases shown in Fig. 1 are denoted by "o", the dotted lines stand for the values of $(K-S)^+$, and the solid lines represent the option values at $t = 0$ computed by the binomial method with 40000 points in the temporal direction. From Fig. 1, we observe that the option value computed by our PCFEM approximates the exact solution well. We compare the CSPM, FFEM and PCFEM in Table 1, from which we conclude that our PCFEM is much faster than both the CSPM and the FFEM for the same accuracy. This confirms the that our PCFEM is more effective for option pricing, and we also note in passing that it is more easily applied in practice.

Remark 4.1. The error form used in Table 1 is the l_2 error $\sum_{i=0}^{N}(P(S_i,0)-P_h(S_i,0))^2)^{1/2}$, where P and P_h are the exact solution and numerical solutions, respectively. In options trading, many people care more about the l_∞ error and relative l_∞ error, which are respectively 10^{-3} and 10^{-4} for our PCFEM. Generally, these error values are acceptable for most stock options.

Next, we verified the convergence of our proposed method. Reviewing the procedure for solving the variational problem (2.9), we see that the error of the solution $u(x,\tau)$ at $\tau = T^*$ mainly consists of the temporal error, iterative error and spatial error. In order to show the relationship between the numbers of spatial points N and the spatial error, the other errors must be small enough. We chose $M = 10000$ in the temporal direction, and $\varepsilon = 10^{-10}$ in Algorithm 3.1. We then divided the x-domain into 1000 uniform intervals, and tested the L_2 error under this partition. Fig. 2 shows that the convergence rate of our method is $\mathcal{O}(1/N^2)$.

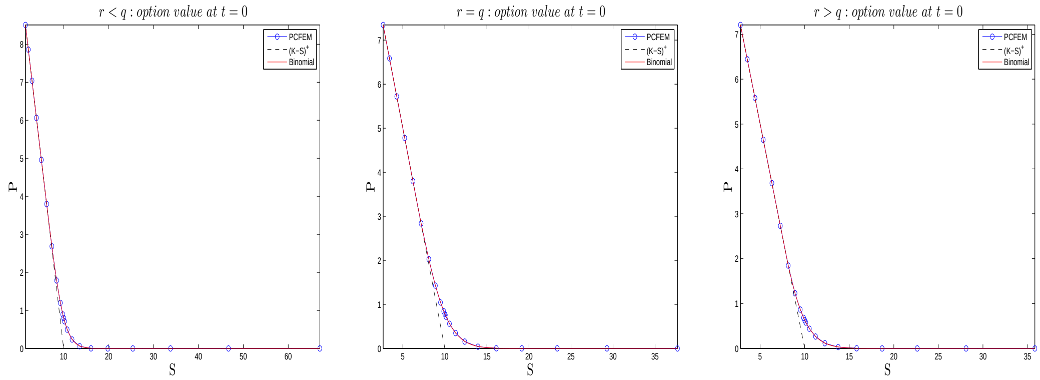

Figure 1: The option values at $t = 0$ computed by our PCFEM with $M = 200$, $N = 20$ and the binomial method with 40000 points, for the three cases. Case I: $r < q$ (left); Case II: $r = q$ (middle); Case III: $r > q$ (right).

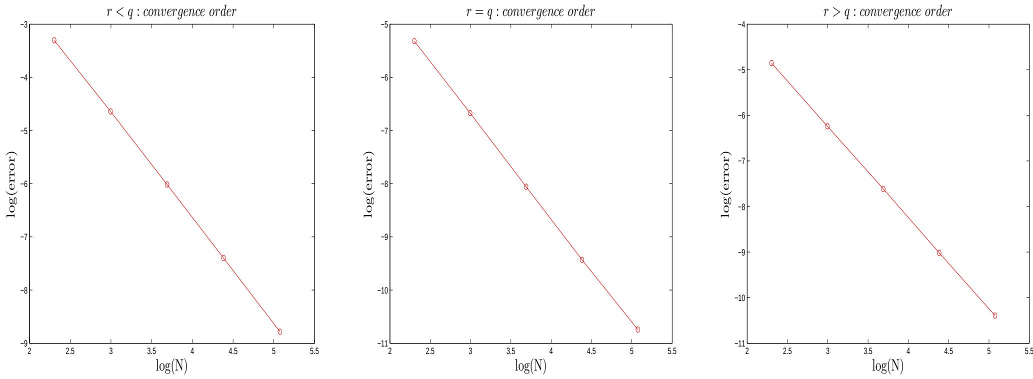

Figure 2: The relationship between the partition numbers N and the option errors on $u(x, T^*)$ for the three cases. Case I: $r < q$ (left); Case II: $r = q$ (middle); Case III: $r > q$ (right).

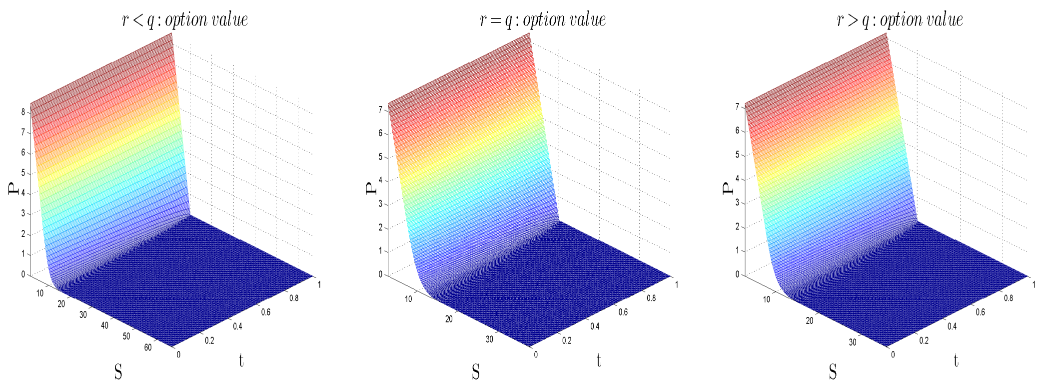

Figure 3: The option values computed by our PCFEM. Case I: $r < q$ (left); Case II: $r = q$ (middle); Case III: $r > q$ (right).

Finally, the numerical results we obtained for the American option are represented in Fig. 3, for the three cases.

5. Conclusions

We have numerically solved the Black-Scholes model via the variational inequality for the problem rendered on a bounded rectangular domain by a far field truncated technique. The valuation problem for American options is discretized by a finite element method. We have shown that the associated discrete matrix is symmetric positive definite, in the matrix form solved by our projection and contraction method. Numerical simulations verify the theoretical analysis and the efficiency of the proposed method.

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