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Projection and Contraction Method for the Valuation of American Options

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Abstract. An efficient numerical method is proposed for the valuation of American options via the Black-Scholes variational inequality. A far field boundary condition is employed to truncate the unbounded domain problem to produce the bounded domain problem with the associated variational inequality, to which our finite element method is applied. We prove that the matrix involved in the finite element method is symmetric and positive definite, and solve the discretized variational inequality by the projection and contraction method. Numerical experiments are conducted that demonstrate the superior performance of our method, in comparison with earlier methods.

AMS subject classifications: 35A35, 90A09, 65K10, 65M12, 65M60

Key words: American option, Black-Scholes variational inequality, finite element method, projection and contraction method.

1. Introduction

An option is a contract that permits but does not require the holder to either buy (a "call option") or sell (a "put option") a certain amount of an underlying asset at a fixed price within a fixed period of time. Options are called European or American, according to their exercise prerogative. Any European option can only be exercised on the maturity date, so it is easy to get a closed-form solution for the exercise price at any time. On the other hand, an American option may be exercised not only on its expiry date but also at any time beforehand, so there is no such closed-form solution, which makes the American option pricing problem a challenging task.

There has been extensive analytical and numerical work done on the American option pricing problem, due to its complexity and importance. The analytical research has continued since the 1970s, but the results have often been unsatisfactory. Early closed-form solutions depend upon the optimal exercise boundary that is unknown in practice [3–5,13, 16,24], and a more recent exact solution in the form of a Taylor series expansion is a very beautiful theoretical result [29]. Numerical methods for American options have attracted

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increasing interest, and are mainly of two types — viz. the Monte Carlo method [6,19,22] and the partial differential equation (PDE) method [1,9,10,14,20,23,27]. The Monte Carlo method has a high computational cost due to its slow convergence, and in this article we pursue the famous Black-Scholes PDE approach, which is widely regarded as one of most effective [7,11,15].

Numerical methods developed and extensively studied in recent decades include lattice tree methods, finite difference methods, finite element methods and spectral methods. In their seminal contribution, Cox *et al.* [8] introduced the binomial method to price American options, and its convergence was proven by Amin & Khanna [2]. The binomial method is essentially a difference method, and inspired a variety of finite difference schemes for American option pricing [9,20,26]. Refs. [14,15] discuss relevant convergence analysis. To improve the solution precision, finite element and spectral methods have received more attention recently [1,10,23,27]). One may refer to Ref. [18] and the references therein for a survey.

There are two main challenges in the numerical evaluation of American option prices:

- the solution domain of the option price is unbounded, so the truncation technique is a key issue; and
- the variational inequality under the Black-Scholes approach renders a complicated nonlinear problem, so an efficient numerical method to solve the problem quickly and accurately is extremely important in practice.

To meet the first challenge for American call options, Holmes & Yang [10] introduced a far field boundary condition, and we follow this idea to deal with put options in this article. The second challenge is our main concern here. We first discretize the American option pricing problem by a finite element method [21,28], and then solve the resulting system using the projection and contraction method [12]. Numerical experiments show that our proposed method is much faster than earlier methods within the same accuracy.

In Section 2, we summarise the linear complementary problem and corresponding variational inequality form for an American put option in the Black-Scholes model. The far field boundary condition is recalled and employed to truncate the unbounded domain. In Section 3, a finite element method is applied to the truncated variational inequality problem, and we prove that the matrix in the associated discretisation is symmetric and positive definite. The projection and contraction algorithm adopted to solve the resulting nonlinear system is also discussed in this section. In Section 4, numerical simulations are presented to compare the performance of our method against earlier methods, and our concluding remarks are in Section 5.

2. Pricing Problem on a Bounded Domain

The Black-Scholes model for an American put option that we adopt is summarised here. In particular, we represent the corresponding linear complementary problem on a bounded domain obtained via a far field boundary condition and its variational inequality form, which we will then proceed to solve using a finite element method.