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Distributed Feedback Control of the Benjamin-Bona-Mahony-Burgers Equation by a Reduced-Order Model

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Abstract. A reduced-order model for distributed feedback control of the Benjamin-Bona-Mahony-Burgers (BBMB) equation is discussed. To retain more information in our model, we first calculate the functional gain in the full-order case, and then invoke the proper orthogonal decomposition (POD) method to design a low-order controller and thereby reduce the order of the model. Numerical experiments demonstrate that a solution of the reduced-order model performs well in comparison with a solution for the full-order description.

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1. Introduction

Standard discretisation schemes (finite element, finite difference, spectral element, finite volume, etc.) may require quite high degree for accurate simulation of fluid flows, and can be expensive with respect to both storage and computing time. Reduced-order models for the simulation of nonlinear complex systems and optimal or feedback control has therefore received more attention recently. This approach involves projecting the dynamical system onto subspaces consisting of basis elements that reflect characteristics of the expected solution, in contrast to the traditional numerical methods — e.g. the elements of the subspaces in the finite element method are uncorrelated to the physical properties of the system described.

The proper orthogonal decomposition (POD) method has also received considerable attention in recent years, as a tool to analyse complex physical systems. This method begins with a set of snapshots generated by either evaluating the computed solution in transient

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problems at several instants of time or by evaluating the computed solution for several values of the parameters appearing in the problem description, or a combination of both. In this article, the snapshots are obtained by a finite element method, from which the POD basis is constructed — viz. the left singular vectors corresponding to the most dominant singular values of the matrix where the columns are the snapshot vectors. This basis is then used to determine an approximate solution for different values of the system parameters, usually by a projection procedure. POD-based model reduction has been applied with some success to several problems [2, 6, 7, 9, 12, 13, 15, 18, 19, 22–25, 27–30, 32, 33].

We propose and test a reduced-order model for a distributed feedback control problem involving the Benjamin-Bona-Mahony-Burgers (BBMB) equation, which describes the propagation of small amplitude long waves in nonlinear dispersive media [5] — viz.

$$\begin{cases} y_t - y_{xxt} - \alpha y_{xx} + \beta y_x + y y_x = f(x,t) \ \text{ in } \Omega \times [0,T]\,, \\ y(0,t) = y(L,t) = 0 \ \text{ on } [0,T]\,, \\ y(x,0) = y_0(x) \ \text{ in } \Omega\,, \end{cases} \tag{1.1}$$

where $\Omega = [0, L]$, $\alpha > 0$ and β are constants and $f(x,t)$ is a given forcing term. The physical dispersion in the BBMB equation is the same as in the Benjamin-Bona-Mahony (BBM) equation and the dissipation is the same as in the Burgers equation, providing an alternative to the Korteweg-de Vries-Burgers (KdVB) equation [17]. Stabilisation of the boundary feedback control for the BBM, KdVB and Burgers equations has been investigated in Refs. [3, 4, 16, 18]. In the case of the KdV equation, the feedback controller is locally and globally controllable and stabilisable. When it is small, the solution of the generalised regularised long wave-Burgers (GRLWB) equation decays like the solution of the corresponding linear equation [8]. Since the BBMB equation is an important case of the GRLWB equation, we can design the controller and control the system (1.1) using a linear-quadratic regulator method. Here the quadratic B-spline finite element method is adopted to convert the BBMB equation into a finite set of nonlinear ordinary differential equations, in designing the full-order control law. Using the reduced-order basis obtained by the POD method, we then design the low-order controller.

In Section 2, we describe the B-spline finite element approximation of solution of the BBMB equation. In Section 3, we briefly review POD-based reduced-order bases, and in Section 4 we discuss our numerical scheme for the distributed feedback control problem. Some numerical results are given in Section 5, followed by a brief Conclusion in Section 6.

2. Finite Element Approximation

2.1. Formulation

Finite element methods (FEM) have often been applied to solve various linear and nonlinear partial differential equations (PDE). Standard Lagrangian finite element basis functions provide only simple C^0-continuity, and therefore cannot be used for the spatial discretisation of higher-order (e.g. third-order or fourth-order) differential equations. On the other hand, B-spline basis functions of various degrees can at least achieve C^1-continuity globally, and so are often used in the numerical solution of PDE — e.g. the quadratic B-spline Galerkin method [1,20,21]. Spectral methods can also be used for discretising the BBMB equation, but we prefer a lower-order method.

Let us now consider the BBMB equation subject to boundary conditions and an initial condition as above. We use a variational formulation to define a finite element method to approximate the system (1.1) as follows: find $y \in L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{cases} \int_\Omega y_t v dx + \int_\Omega y_{xt} v' dx + \alpha \int_\Omega y_x v' dx + \beta \int_\Omega y_x v dx + \int_\Omega y y_x v dx \\ \qquad = \int_\Omega f v dx \qquad \forall\, v \in H_0^1(\Omega)\,, \\ y(x,0) = y_0(x) \quad \text{in}\, \Omega\,, \end{cases} \tag{2.1}$$

where $H_0^1 = \{y \in H^1(\Omega) : y|_\Omega = 0\}$ and $H^1(\Omega) = \{v \in L^2(\Omega) : \partial v/\partial x \in L^2(\Omega)\}$. A typical finite element approximation of (2.1) is as follows: first choose conforming finite element subspaces $V^h \subset H^1(\Omega)$ and then define $V_0^h = V^h \cap H_0^1(\Omega)$. One then seeks $y^h(t,\cdot) \in V_0^h$ such that

$$\begin{cases} \int_\Omega y_t^h v^h dx + \int_\Omega y_{xt}^h (v^h)' dx + \alpha \int_\Omega y_x^h (v^h)' dx + \beta \int_\Omega y_x^h v^h dx + \int_\Omega y^h y_x^h v^h dx \\ \qquad = \int_\Omega f v^h dx \qquad \forall\, v^h \in V_0^h(\Omega)\,, \\ y^h(x,0) = y_0^h(x) \quad \text{in}\, \Omega\,, \end{cases} \tag{2.2}$$

where $y_0^h(x) \in V_0^h$ is an approximation (e.g. a projection) of $y_0(x)$.

The interval $\Omega = [0,L]$ is divided into N finite elements of equal length h by knots x_i such that $0 = x_0 < x_1 < \cdots < x_N = L$. The set of splines $\{\eta_{-1}, \eta_0, \cdots, \eta_N\}$ form a basis for functions defined on Ω. Quadratic B-splines with the required properties are denoted and defined by [31]:

$$\eta_i(x) = \frac{1}{h^2} \begin{cases} (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2 + 3(x_i - x)^2)\,, & [x_{i-1}, x_i]\,, \\ (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2, & [x_i, x_{i+1}]\,, \\ (x_{i+2}-x)^2, & [x_{i+1}, x_{i+2}]\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where $h = x_{i+1} - x_i,\ i = -1, 0, \cdots, N$. The quadratic spline and its first derivative vanish outside the interval $[x_{i-1}, x_{i+2}]$, and the spline function value and first derivative at the knots are given by

$$\begin{cases} \eta_i(x_{i-1}) = \eta_i(x_{i+2}) = 0\,, \quad \eta_i(x_i) = \eta_i(x_{i+1}) = 1\,; \\ \eta_i^{'}(x_{i-1}) = \eta_i^{'}(x_{i+2}) = 0\,, \quad \eta_i^{'}(x_i) = \eta_i^{'}(x_{i+1}) = 1\,. \end{cases} \tag{2.3}$$

Thus an approximate solution can be written in terms of the quadratic spline functions as

$$y^h(x,t) = \sum_{i=-1}^{N} a_i(t)\eta_i(x)\,, \tag{2.4}$$

where the $a_i(t)$ are as yet undetermined coefficients.

Each spline covers three intervals such that three splines $\eta_{i-1}(x)$, $\eta_i(x)$, $\eta_{i+1}(x)$ cover each finite element $[x_i, x_{i+1}]$. All other splines are zero in this region. Using Eq. (2.4) and the spline function properties in Eq. (2.3), we can express the nodal values of function $y^h(x,t)$ and its derivative at the knot x_i and fixed time $\tilde{t}$ in terms of the coefficients $a_i(\tilde{t})$:

$$y^h(x_i,\tilde{t}) = a_{i-1}(\tilde{t}) + a_i(\tilde{t})\,, \qquad \frac{\partial y^h(x,\tilde{t})}{\partial x}\Big|_{x=x_i} = \frac{2}{h}\big(a_i(\tilde{t}) - a_{i-1}(\tilde{t})\big)\,. \tag{2.5}$$

For an homogeneous boundary condition, from Eq. (2.5) $a_{-1}(t) = -a_0(t)$ and $a_N(t) = -a_{N-1}(t)$, whence

$$y^h(x,t) = \sum_{i=0}^{N-1} a_i(t)\xi_i(x)\,, \tag{2.6}$$

where $\xi_0(x) = (\eta_0(x) - \eta_{-1}(x))$, $\xi_i(x) = \eta_i(x)\ \forall\ i = 1,2,\cdots,N-2$, $\xi_{N-1}(x) = \eta_{N-1}(x) - \eta_N(x)$. Consequently, the N unknowns $a_i(t)$ $(i = 0,1,\cdots,N-1)$ can be determined for every instant t.

In the Galerkin method, the chosen weighted function in (2.2) is $v_i^h(x) = \xi_i(x)$ for $i = 0,1,\cdots,N-1$, so on substituting Eq. (2.6) into (2.2) we obtain

$$\begin{cases} \displaystyle\sum_{i=0}^{N-1}\Big(\int_\Omega \xi_i\xi_j dx\Big)\frac{da_i(t)}{dt} + \sum_{i=0}^{N-1}\Big(\int_\Omega \xi_i'\xi_j'\, dx\Big)\frac{da_i(t)}{dt} \\ \displaystyle\qquad + \alpha\sum_{i=0}^{N-1}\Big(\int_\Omega \xi_i'\xi_j'\, dx\Big)a_i(t) + \beta\sum_{i=0}^{N-1}\Big(\int_\Omega \xi_i'\xi_j\, dx\Big)a_i(t) \\ \displaystyle\qquad + \sum_{i=0}^{N-1}\sum_{k=0}^{N-1}\Big(\int_\Omega \xi_i\xi_k'\xi_j\, dx\Big)a_i(t)a_k(t) = \int_\Omega f\xi_j\, dx\,, \\ \displaystyle\sum_{i=0}^{N-1}\Big(\int_\Omega \xi_i\xi_j\, dx\Big)a_i(0) = \int_\Omega y_0(x)\xi_j\, dx\,,\ \ j = 0,1,\cdots,N-1\,. \end{cases} \tag{2.7}$$

Assuming $m_{ij} = (\xi_i,\xi_j)$, $s_{ij} = (\xi_i',\xi_j')$, $d_{ij} = (\xi_i',\xi_j)$, $n_{ijk} = (\xi_i\xi_k',\xi_j)$, $f_j = (f,\xi_j)$, $y_0^j = (y_0,\xi_j)$, mass matrix $\mathbb{M} = (m_{ij})$, stiff matrix $\mathbb{S} = (s_{ij})$, $\mathbb{D} = (d_{ij})$, $\mathbb{N} = (n_{ijk})$, $\vec{f} = (f_0,f_1,\cdots,f_{N-1})^T$, $\vec{y}_0 = (y_0^0, y_0^1,\cdots,y_0^{N-1})$, $\vec{a}_0 = (a_0(0),a_1(0),\cdots,a_{N-1})^T$ and $\vec{a}(t) = (a_0(t),a_1(t),\cdots,a_{N-1}(t))^T$, we can then write the system (2.7) in the matrix form

$$\begin{cases} (\mathbb{M}+\mathbb{S})\dfrac{d\vec{a}}{dt} + (\alpha\mathbb{S}+\beta\mathbb{D})\vec{a} + (\vec{a})^T\mathbb{N}\vec{a} = \vec{f}\,, \\ \mathbb{M}\vec{a}_0 = \vec{y}_0\,, \end{cases} \tag{2.8}$$

a system of N nonlinear ordinary differential equations in N unknowns subject to the initial condition. Since $(\mathbb{M}+\mathbb{S})$ and $\mathbb{M}$ are invertible matrices, the system (2.8) can be written as standard first order nonlinear ordinary differential equations subject to initial conditions — viz.

$$\frac{d\vec{a}}{dt} = (\mathbb{M}+\mathbb{S})^{-1}\Big(\vec{f} - ((\alpha\mathbb{S}+\beta\mathbb{D})\vec{a} + (\vec{a})^T\mathbb{N}\vec{a})\Big), \quad \vec{a}_0 = \mathbb{M}^{-1}\vec{y}_0\,, \tag{2.9}$$

and for simplicity we take $\vec{y}_0 = \mathbb{M}^{-1}\vec{y}_0$. The terms in Eq. (2.9) on the right-hand sides are continuously differentiable, so the system (2.9) has a unique solution — and a zero equilibrium solution when the forcing term $f(x,t)$ tends to zero and the time tends to infinity. Thus on taking the equilibrium solution as the starting point, we may obtain the numerical solution of the system (1.1) by the Newton method used to generate the snapshots — i.e. the M snapshot vectors

$$\vec{a}_m = \big[a_0(t_m)\; a_1(t_m)\; \cdots\; a_{N-1}(t_m)\big]^{\mathrm{T}}, \quad m = 1,\cdots,M$$

are determined by evaluating the approximate solution of the system (2.9) at M equally spaced time values t_m, from $t_1 = 0$ to $t_M = T$.

Remark 2.1. For convenience, by "nodal value" we mean the solution of the differential equation at the knot, and "coefficient" refers to any coefficient appearing in Eqs. (2.4) and (2.5).

Remark 2.2. The property (2.5) will be used in the control solutions, so the nodal values of the full-order and reduced-order control solutions at a knot x_i equal the sum of the coefficients at the knots x_{i-1} and x_i (at other than the boundary points).

2.2. The convergence test for the quadratic B-spline method

In this subsection, we present a numerical test for the error estimate and convergence. Let us consider

$$\begin{cases} y_t - y_{xxt} - \alpha y_{xx} + \beta y_x + y y_x \\ \quad = \exp(-t)\left[\cos x - \sin x + \dfrac{1}{2}\exp(-t)\sin(2x)\right] \text{ in } \Omega\times[0,T]\,, \\ y(0,t) = y(\pi,t) = 0 \;\text{ on } [0,T]\,, \\ y(x,0) = \sin x \;\text{ in } \Omega\,, \end{cases} \tag{2.10}$$

for which the exact solution is $u(x,t) = \exp(-t)\sin x$. Table 1 shows the estimate of error $||e||_{\infty,h} = \max_{i,n}|y^h(x_i,t_n) - y(x_i,t_n)|$ for various spatial step sizes h. The time step can be $\Delta t = h$ for a second-order scheme in time, such as the Crank-Nicholson method, or $\Delta t = h^2$ for a first-order scheme. The quadratic B-spline method provides a second-order scheme for the space discretisation — cf. Refs. [1, 20, 21] and references therein for more information on B-spline FEM.

The convergence rate for the finite difference solution of the BBMB equation in the L^∞-norm was found to be about second-order, and error estimates showed that the actual error is lower than for the quadratic B-spline FEM [26].

Table 1: Convergence rate in maxmum norm.

$1/h$	$\|\|e\|\|_{\infty,h} = \max_{i,n} \|y^h(x_i,t_n) - y(x_i,t_n)\|$	convergence rate
10	1.817280262880139e-03	
20	4.743102628801399e-04	3.83
40	1.212538700144372e-04	3.91
80	3.066562894760150e-05	3.95
160	7.685621290125688e-06	3.99

3. Reduced-Order Bases via POD

Let us now briefly describe the derivation of reduced-order bases using the POD method. Given a discrete set of snapshot vectors $\mathbb{Y} = \{\vec{a}_m\}_{m=1}^M$ belonging to $\Re^N$ where $M < N$, we form the $N \times M$ snapshot matrix with columns the snapshot vectors $\vec{a}_m$ — viz.

$$\mathbb{Y} = (\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_M)\,.$$

Let

$$\mathbb{U}^T \mathbb{Y} \mathbb{V} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbb{U}$ and $\mathbb{V}$ are $N \times N$ and $M \times M$ orthogonal matrices respectively, and where $\sum = \mathrm{diag}(\sigma_1, \cdots, \sigma_{\tilde{M}})$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\tilde{M}}$ is the singular value decomposition of $\mathbb{Y}$. Here $\tilde{M}$ is the rank of $\mathbb{Y}$ — i.e. the dimension of the snapshot set $\mathbb{Y}$, which is less than M whenever the snapshot set is linearly dependent. It is well known [14] that if

$$\mathbb{U} = (\vec{\phi}_1 \ \ \vec{\phi}_2 \ \ \cdots \ \ \vec{\phi}_N) \quad \text{and} \quad \mathbb{V} = (\vec{\psi}_1 \ \ \vec{\psi}_2 \ \ \cdots \ \ \vec{\psi}_M),$$

then

$$\mathbb{Y}\vec{\psi}_i = \sigma_i \vec{\phi}_i \quad \text{and} \quad \mathbb{Y}^{\mathrm{T}}\vec{\phi}_i = \sigma_i \vec{\psi}_i \quad \text{for } i = 1, \cdots, \tilde{M}\,,$$

and hence

$$\mathbb{Y}^{\mathrm{T}}\mathbb{Y}\vec{\psi}_i = \sigma_i \vec{\phi}_i \quad \text{and} \quad \mathbb{Y}\mathbb{Y}^{\mathrm{T}}\vec{\phi}_i = \sigma_i \vec{\psi}_i \quad \text{for } i = 1, \cdots, \tilde{M}$$

such that σ_i^2, $i = 1, \cdots, \tilde{M}$ are the nonzero eigenvalues of $\mathbb{Y}^{\mathrm{T}}\mathbb{Y}$ (and also $\mathbb{Y}\mathbb{Y}^{\mathrm{T}}$) arranged in non-decreasing order. The matrix $\mathbb{C} = \mathbb{Y}^{\mathrm{T}}\mathbb{Y}$ is the correlation matrix for the set of snapshot vectors $\mathbb{Y} = \{\vec{a}_m\}_{m=1}^M$ — i.e. we have $\mathbb{C}_{mn} = \vec{a}_m^{\mathrm{T}}\vec{a}_n$.

In a reduced-order model, given a set of snapshots $\mathbb{Y} = \{\vec{a}_m\}_{m=1}^M$ belonging to $\Re^N$, the POD reduced-basis of dimension $K \leq M < N$ is the set of vectors $\{\vec{\phi}_k\}_{k=1}^K$ also belonging to $\Re^N$ that consists of the first K left singular vectors of the snapshot matrix $\mathbb{Y}$. Thus one can determine the POD basis by computing the (partial) singular value decomposition of the $N \times M$ matrix $\mathbb{Y}$. Alternatively, one can compute the (partial) eigensystem $\{\sigma_k^2, \vec{\psi}_i\}_{i=1}^K$ of the $M \times M$ correlation matrix $\mathbb{C} = \mathbb{Y}^{\mathrm{T}}\mathbb{Y}$, and then set $\vec{\phi}_k = \mathbb{Y}\vec{\psi}_k$, $k = 1, \cdots, K$. The K-dimensional POD basis has the obvious property of orthonormality.

For reduced-order models, the snapshot vectors are coefficient vectors in the expansion of the finite element approximate solution of the PDE at different instants in time, so to each snapshot vector there corresponds a finite element function. One can thus define a POD basis with respect to functions instead of vectors — i.e. start with a snapshot set consisting of finite element functions. According to the optimal properties of POD basis and some related reasoning, we can therefore rewrite the correlation matrix as $\mathbb{C} = \mathbb{Y}^{\mathrm{T}}\mathbb{M}\mathbb{Y}$ where $\mathbb{M}$ is a mass matrix [9], so we can again use singular value decomposition to determine the POD basis function. To this end, consider the mass matrix $\mathbb{M} = \mathbb{W}^{\mathrm{T}}\mathbb{W}$, where the $N \times N$ matrix $\mathbb{W}$ could be chosen to be a symmetric, positive definite square root of $\mathbb{M}$ (i.e. $\mathbb{W} = \mathbb{M}^{1/2}$), or alternatively a Cholesky factor (i.e. $\mathbb{W}^{\mathrm{T}} = \mathbb{L}$). We may then let $\bar{\mathbb{W}} = \mathbb{M}^{1/2}\mathbb{Y}$ such that $\mathbb{C} = \bar{\mathbb{W}}^{\mathrm{T}}\bar{\mathbb{W}}$, and therefore the POD bases are the first K left singular vectors of $\bar{\mathbb{W}}$.

Remark 3.1. In the computation here, the snapshots differ from the numerical solutions, as is usually noted in the reduced-order PDE model — i.e. rather than “nodal values", the snapshots are coefficients mentioned in Remark (2.1).

4. Feedback Control Design

The control problem is as follows.

Find an optimal control $u^*(t)$, which minimises the cost functional

$$J(u) = \int_0^\infty ||y(t,\cdot)||^2_{L^2(\Omega)} + |u(t)|^2 dt$$

subject to the constraint equations

$$\begin{cases} y_t - y_{xxt} = \alpha y_{xx} - \beta y_x - y y_x + f(x,t) \ \text{ in } \Omega \times [0,T]\,, \\ y(0,t) = y(L,t) = 0 \ \text{ on } [0,T]\,, \\ y(x,0) = y_0(x) \ \text{ in } \Omega\,. \end{cases} \tag{4.1}$$

We will assume the forcing term $f(x,t)$ has the special form $b(x)u(t)$, where $u(t)$ is the control input and $b(x)$ is a given function that distributes the control over the domain.

4.1. Linear quadratic regulator design

Provided the nonlinear term in the BBMB equation is small, a suboptimal feedback control u^* can be obtained from well-known linear quadratic regulator theory [11,22,30]. A full state feedback control involves finding an optimal control $u^* \in L^2([0,T), L^2(\Omega))$ by minimising the cost functional

$$J(u) = \int_0^\infty (\mathscr{Q} y(t,\cdot), y(t,\cdot))_{L^2(\Omega)} + \big(\mathscr{R} u(t), u(t))\big) dt$$

subject to the constraint equations

$$\dot{y}(t) = \mathscr{A} y(t) + \mathscr{B} u(t)\,, \quad y(0) = y_0 \quad \text{for } \ t > 0\,,$$

where $\mathscr{Q} : L^2(\Omega) \to L^2(\Omega)$ is a non-negative definite self-adjoint weighting operator for the state and $\mathscr{R} : L^2(\Omega) \to L^2(\Omega)$ is a positive definite weighting operator for the control. The optimal control $u^*(t)$ can be found as

$$u^*(t) = -\frac{1}{2}\mathscr{R}^{-1}\mathscr{B}^T \Pi y(t) = -\mathscr{K} y(t) \,,$$

where $\mathscr{K}$ is called the feedback operator and Π is the symmetric positive definite solution of the algebraic Riccati equation

$$\Pi\mathscr{A} + \mathscr{A}^T \Pi - \Pi \mathscr{B}\mathscr{R}^{-1}\mathscr{B}^T \Pi + \mathscr{Q} = 0 \,. \tag{4.2}$$

4.2. Linear feedback controllers with state estimate feedback

The design controller method adopted here is similar to the simple classical linear quadratic regulator (LQR) scheme, which assumes that the full state is “feed back" into the system by the control, but the properties (2.5) are considered in the whole process. Knowledge of the full state is not possible for many complicated physical systems, so a compensator design provides a state estimate based on state measurements used in the feedback control law. Thus it is not assumed we have knowledge of the full state, and instead assume a state measurement of the form

$$z(t) = \mathscr{C} y(t) \,, \tag{4.3}$$

where $\mathscr{C} \in \mathscr{L}(L^2(\Omega), \mathfrak{R}^m)$. We can apply the theory and results to show that a stabilising compensator-based controller can be applied to the system [10]. The observer design is mainly needed to provide the feedback control law with estimated state variables, and the control law and observer are combined into a complete system called the compensator.

This technique requires a limited measurement of the state as a condition for a system of the assumed form

$$\dot{y}(t) = \mathscr{A} y(t) + \mathscr{G}(y(t)) + \mathscr{B}u(t) \,, \quad y(0) = y_0 \,, \tag{4.4}$$

where $y(t)$ is in a state space $L^2(\Omega)$ and $u(t)$ is in a control space U. From Eq, (4.3), a state estimate $\tilde{y}(t)$ is computed by solving the observer equation

$$\dot{\tilde{y}}(t) = \mathscr{A} \tilde{y}(t) + \mathscr{G}(\tilde{y}(t)) + \mathscr{B}u(t) + \mathscr{L}\big[z(t) - \mathscr{C}\tilde{y}(t)\big] \,, \quad \tilde{y}(0) = \tilde{y}_0 \,. \tag{4.5}$$

The feedback control law is

$$u(t) = -\mathscr{K}\tilde{y}(t) \,, \tag{4.6}$$

where $\mathscr{K}$ is called the feedback operator. As usual, the operator $\mathscr{K}$ and estimator gain operator $\mathscr{L}$ are determined by the linear quadratic regulator (LQR) and Kalman estimator (LQE), respectively. From the above, we already know that

$$\mathscr{K} = \mathscr{R}^{-1}\mathscr{B}^T \Pi \,. \tag{4.7}$$

Next, $\mathscr{P}$ is found as the non-negative definite solution of

$$\mathscr{A}\mathscr{P} + \mathscr{P}\mathscr{A}^T - \mathscr{P}\mathscr{C}^T\mathscr{C}\mathscr{P} + \bar{\mathscr{Q}} = 0\,,$$

where $\bar{\mathscr{Q}}$ is a non-negative definite weighting operator. If the solution $\mathscr{P}$ exists, we can define

$$\mathscr{L} = \mathscr{P}\mathscr{C}^T. \tag{4.8}$$

From Eqs. (4.3)-(4.8), we obtain the closed loop compensator

$$\begin{cases} \dot{y}(t) = \mathscr{A}y(t) - \mathscr{B}\mathscr{K}\tilde{y}(t) + \mathscr{G}(y(t))\,, \\ \dot{\tilde{y}}(t) = \mathscr{L}\mathscr{C}y(t) + (\mathscr{A} - \mathscr{L}\mathscr{C} - \mathscr{B}\mathscr{K})\tilde{y}(t) + \mathscr{G}(\tilde{y}(t))\,, \\ y(0) = y_0\,,\ \ \tilde{y}(0) = \tilde{y}_0\,. \end{cases} \tag{4.9}$$

4.3. Reduced-order compensators

Implementation of the controller involves some numerical discretisation scheme. For example, a finite element method provides finite-dimensional approximations of Eqs. (4.3) and (4.4) of order N, where order refers to the degree of freedom of the finite element:

$$\begin{cases} \dot{y}^N(t) = \mathbb{A}^N y^N(t) + \mathbb{G}^N(y^N(t)) + \mathbb{B}^N u^N(t)\,, \\ y^N(0) = y_0^N, \\ z^N(t) = \mathbb{C}^N y^N(t)\,, \end{cases}$$

with $\mathbb{A}^N = (\mathbb{M}+\mathbb{S})^{-1}(\alpha\mathbb{S}+\beta\mathbb{D})$, the $\mathbb{B}^N$ constructed by integration of the product of $b(x)$ and the test function $\xi(x)$, and $\mathbb{G}^N(y) = (\mathbb{M}+\mathbb{S})^{-1}y^T\mathbb{N}y$. In a full-order compensator design, the order N approximations are used to compute $\mathbb{K}^N$ and $\mathbb{L}^N$. The respective finite-dimensional approximations from the compensator equation (4.5) and control law (4.6) are

$$\begin{cases} \dot{\tilde{y}}^N(t) =\mathbb{A}^N\tilde{y}^N(t) + \mathbb{G}^N(\tilde{y}^N(t)) + \mathbb{B}^N u^N(t) + \mathbb{L}^N\left[z^N(t) - \mathbb{C}^N\tilde{y}^N(t)\right]\,, \\ \tilde{y}^N(0) =\tilde{y}_0^N\,, \\ u^N(t) = -\mathbb{K}^N\tilde{y}^N(t)\,; \end{cases}$$

and the approximation to the closed-loop compensator system (henceforth referred to as full-order) is

$$\begin{cases} \dot{y}^N(t) =\mathbb{A}^N y^N(t) - \mathbb{B}^N\mathbb{K}^N\tilde{y}^N(t) + \mathbb{G}^N(y^N(t))\,, \\ \dot{\tilde{y}}^N(t) =\mathbb{L}^N\mathbb{C}^N y^N(t) + \mathbb{A}^N\tilde{y}^N(t) - \mathbb{L}^N\mathbb{C}^N\tilde{y}^N(t) - \mathbb{B}^N\mathbb{K}^N\tilde{y}^N(t) + \mathbb{G}^N(\tilde{y}^N(t))\,, \\ y^N(0) =y_0^N, \quad \tilde{y}^N(0) = \tilde{y}_0^N\,. \end{cases} \tag{4.10}$$

However, as real-time control via the full-order compensator may not be possible for the many physical problems that require large discretised systems for an adequate approximation, a reduced-order compensator is used. A “reduce-then-design” approach has the potential drawback that important physics or other information in the model can be lost before the controller is obtained [19], so we adopt a "design-then-reduce" approach — i.e. a controller is designed based on the high-order model, and then reduced:

$$\begin{cases} \dot{\tilde{y}}^K(t) = \mathbb{A}^K \tilde{y}^K(t) + \mathbb{G}^K(\tilde{y}^K(t)) + \mathbb{B}^K u^K(t) + \mathbb{L}^K\left[z^K(t) - \mathbb{C}^K \tilde{y}^K(t)\right], \\ \tilde{y}^K(0) = \tilde{y}_0^K, \end{cases} \tag{4.11}$$

$$u^K(t) = -\mathbb{K}^K \tilde{y}^K(t), \tag{4.12}$$

$$\begin{cases} \dot{y}^K(t) = \mathbb{A}^K y^K(t) + \mathbb{G}^K(y^K(t)) + \mathbb{B}^K u^K(t), \\ y^K(0) = y_0^K. \end{cases} \tag{4.13}$$

The suggested control law (4.12) is substituted into Eqs. (4.11) and (4.13), producing

$$\begin{cases} \dot{y}^K(t) = \mathbb{A}^K y^K(t) - \mathbb{B}^K \mathbb{K}^K \tilde{y}^K(t) + \mathbb{G}^K(y^K(t)), \\ \dot{\tilde{y}}^K(t) = \mathbb{L}^K \mathbb{C}^K y^K(t) + \mathbb{A}^K \tilde{y}^K(t) - \mathbb{L}^K \mathbb{C}^K \tilde{y}^K(t) - \mathbb{B}^K \mathbb{K}^K \tilde{y}^K(t) + \mathbb{G}^K(\tilde{y}^K(t)), \\ y^K(0) = y_0^K, \quad \tilde{y}^K(0) = \tilde{y}_0^K. \end{cases} \tag{4.14}$$

Reduced bases are formed using the POD process, as described in Section 3. The reduced bases are used to compute the compensator equation, feedback control law and model problem in Eqs. (4.11)-(4.12). Then the reduced systems given by Eqs. (4.14) are compared with the full-order compensator system in Eqs. (4.10), and the Backward Euler method is applied to solve the systems (4.10) and (4.14) numerically.

5. Computational Experiments

Two examples were considered.

Example 5.1. Parameters $\alpha = 0.5, \beta = 1, T = 5$ and initial condition $y_0(x) = \exp(-x)\sin(\pi x)$.

Example 5.2. Parameters $\alpha = 0.0001$, $\beta = 10$, $T = 10$, and initial condition $y_0(x) = 4x(1-x)$.

The spatial interval was taken to be $[0,1]$, full-order dimension $N = 64$, reduced-order dimension $K = 8$. and the spatial step size and time step size $h = 1/N$ and $1/100$, respectively. The control input operator was $\mathscr{B} = \int_0^L b(x)\xi_i(x)dx$, where $b(x) = x$ and $\xi_i(x)$ is a test function($i = 0, 1, \cdots, N-1$). The state weighting operator $\mathscr{Q}$ used in the Riccati equation calculations was $(\mathbb{M} + \mathbb{S})$; and we set the control weighting operator $\mathscr{R}(1,1) = 10^{-6}$, and the weighting operator $\bar{\mathscr{Q}}$ was also chosen as $(\mathbb{M} + \mathbb{S})$. Finally, we created the measurement operator $\mathscr{C}$, with $\mathscr{C}y(t,x) = 8\int_{3/4}^{5/6} y(t,x)dx$ for the state estimate feedback controller.

Fig. 1 presents uncontrolled solutions of the BBMB equation, and Figs. 2 and 3 depict full-order controlled solutions and reduced-order controlled solutions. In the whole control process, we find that both the full-order and reduced-order control are very stable, and

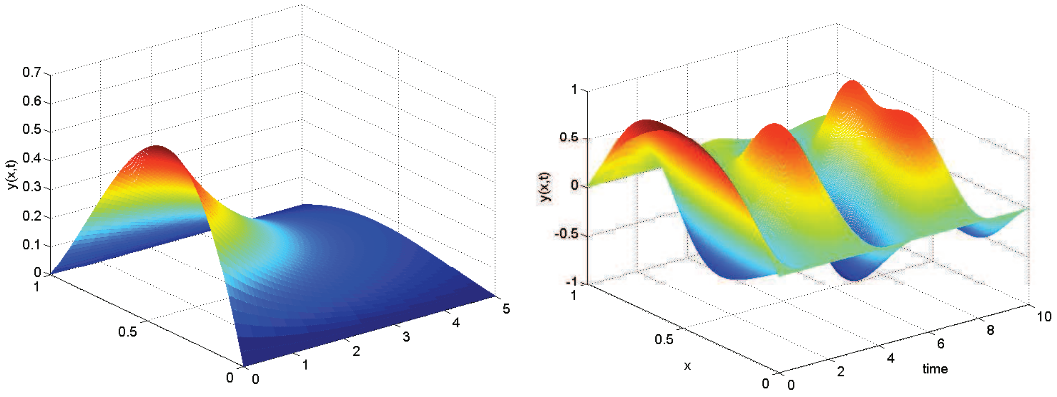

Figure 1: Uncontrolled solution of the BBMB equation for Example 5.1 (left) and Example 5.2 (right).

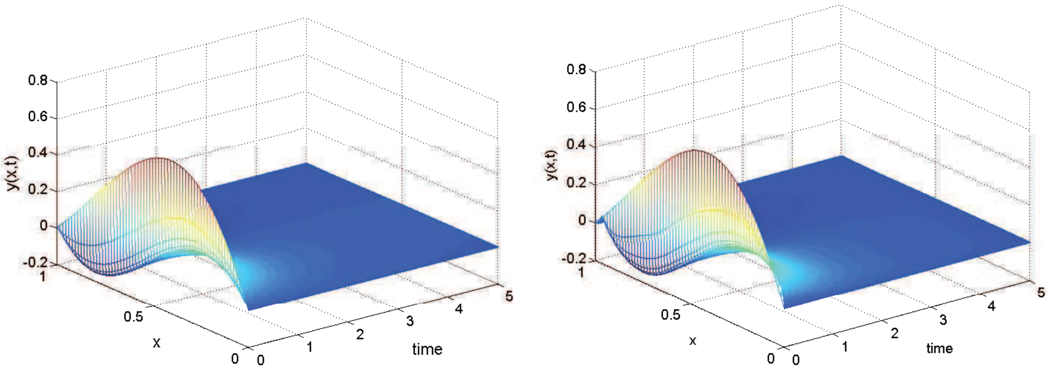

Figure 2: Full-order controlled solution (left) and reduced-order controlled solution (right) for Example 5.1.

although of two kinds (the control methods are slightly different for Example 5.2), the results obtained are very similar for both Examples.

Fig. 4 shows the L^2-norms for the solutions of the uncontrolled and controlled problem. For Example 5.1, the two kinds of control methods have almost the same effect; for Example 5.2, the full-order control is stronger than the reduced-order control until the 8th second, but after that the results are very similar. The finite element solution of the BBMB equation is smooth and slowly tends to zero when the time becomes large. For Example 5.1, the controlled solution at first rapidly reduces and then slowly approaches zero; for Example 5.2, for small coefficient α and constant β the finite element approximate solution of the BBMB equation fluctuates in a certain range, but the control solution at first rapidly reduces and then approaches zero with very small fluctuations.

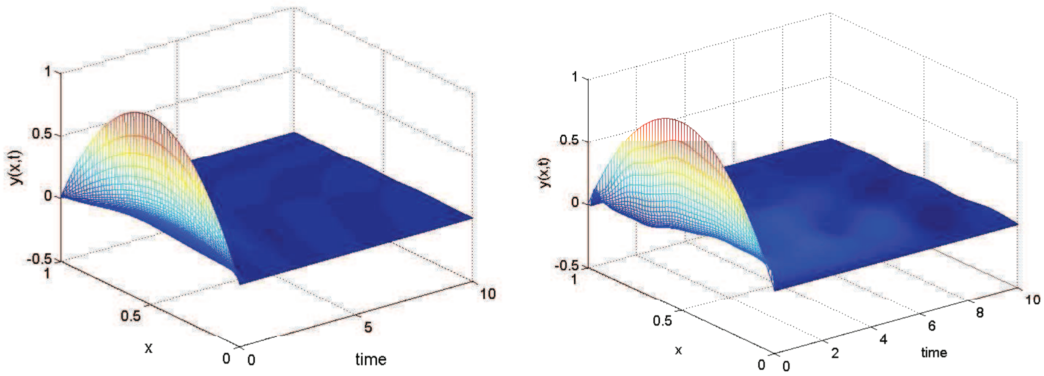

Figure 3: Full-order controlled solution (left) and reduced-order controlled solution for Example 5.2.

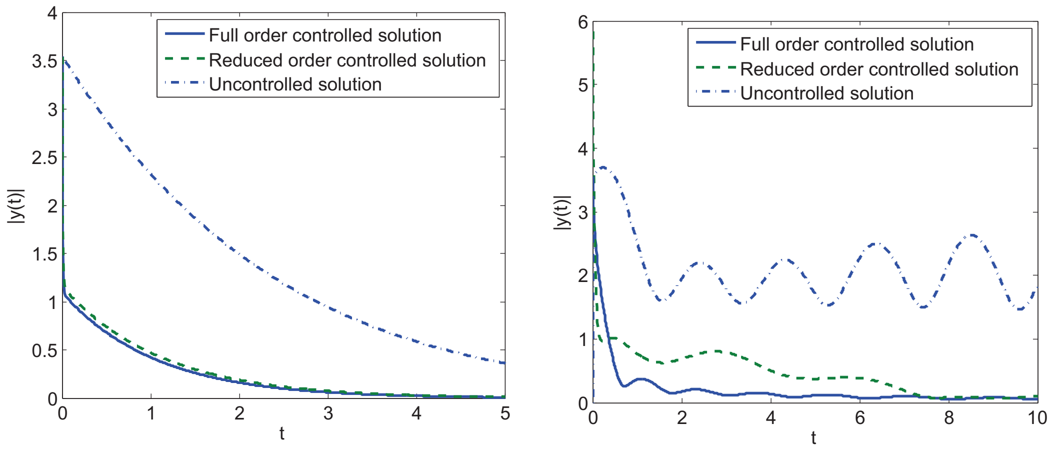

Figure 4: L^2-norms for the solutions of the uncontrolled and controlled problem vs. time for Example 5.1 (left) and Example 5.2 (right).

6. Conclusion

We have investigated a reduced order model for stabilisation of the BBMB equation through a feedback control, using a state estimate feedback. For real-time computation of the feedback control problem, we designed a reduced order model producing results that demonstrate our POD approach is feasible and efficient. In future, we intend to study a feedback control problem for a complicated system using a proper orthogonal decomposition and centroidal Veronoi tessellation (POD-CVT) reduced order model.

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References

[1] A.H.A. Ali, G.A. Gardner and L.R.T. Gardner, *A collocation solution for Burgers equation using cubic B-spline finite elements*, Comp. Meth. Appl. Mech. Engrg. **100**, 325–337 (1992).

[2] N. Aubry, W. Lian and E. Titi, *Preserving symmetries in the proper orthogonal decomposition*, SIAM J. Sci. Comp. **14**, 483-505 (1993).

[3] A. Balogh and M, Krstic, *Boundary control of the Korteweg-de Vries-Burgers equation: further results on stabilisation and well-posedness, with numerical demonstration*, IEEE Trans. Automatic Control **455**, 1739-1745 (2000).

[4] A. Balogh and M. Krstic, *Burgers equation with nonlinear boundary feedback: H^1 stability, well-posedness and simulation*, Math. Problems in Engineering **6**, 189-200 (2000).

[5] T.B. Benjamin, J.L. Bona and J.J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. Royal Soc. (London) **272** (1220), 47-78 (1972).

[6] G. Berkooz, P. Holmes and J. Lumley, *The proper orthogonal decomposition in the analysis of turbulent flows*, Ann. Rev. Fluid Mech. **25**, 539-575 (1993).

[7] G. Berkooz and E. Titi, *Galerkin projections and the proper orthogonal decomposition for equivariant equations*, Phys, Lett. A **174**, 94-102 (1993).

[8] J.L. Bona and L.H. Luo, *Asymptotic decomposition of nonlinear, dispersive wave equation with dissipation*, Physica D **152-153**, 363-383 (2001).

[9] J. Burkardt, M. Gunzburger and H.-C. Lee, *POD and CVT-based reduced-order modeling of Navier-Stokes flows*, Comp. Meth. Appl. Mech. Engrg. **196**, 337-355 (2006).

[10] J.A. Burns and S. Kang, *A control problem for Burgers' equation with bounded input/oqtput*, ICASE Report 90-45, 1990, NASA Langley research Center, Hampton, VA; Nonlinear Dynamics **2**, 235-262 (1991).

[11] C.T. Chen, *Linear System Theory and Design*, Holt, Rinehart and Winston, New York, 1984.

[12] E. Christensen, M. Brons and J. Sorensen, *Evaluation of proper orthogonal decomposition-based decomposition techniques applied to parameter-dependent nonturbulent flows*, SIAM J. Sci. Comp. **21**, 1419-1434 (2000).

[13] A. Deane, I. Kevrekidis, G. Karniadakis and S. Orszag, *Low-dimensional models for complex geometry flows: application to grooved channels and circular cylinders*, Phys. Fluids A **3**, 2337-2354 (1991).

[14] G. Golub and C. van Loan, *Matrix Computations*, Johns Hopkins University, Baltimore, 1996.

[15] M. Graham and I. Kevrekidis, *Pattern analysis and model reduction: some alternative approaches to the Karhunen-Lóeve decomposition*, Comp. Chem. Engrg. **20**, 495-506 (1996).

[16] A. Hasan, B. Foss and O.M. Aamo, *Boundary control of long waves in nonlinear dispersive systems*, in Proc. 1st Australian Control Conference, Melbourne (2011).

[17] D.J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Phil. Magazine **39**, 422-443 (1895).

[18] M. Krstic, *On global stabilisation of Burgers equaiton by boundary control*, Systems and Control. Letters. **37**, 123-141 (1999).

[19] K. Kunisch and S. Volkwein, *Control of the Burgers equation by a reduced-order approach using proper orthogonal decomposition*, J. Optim. Theory Appl. **102**, 345-371 (1999).

[20] S. Kutluay and Y. Ucar, *Numerical solutions of the coupled Burgers' equation by the Galerkin quadratic B-spline finite element method*, Math. Meth. Appl. Sci. **36**, 2403-2415 (2013).

[21] S. Kutluay and Y. Ucar, *A quadratic B-spline Galerkin approach for solving a coupled KdV equation*, Math. Model. Anal. **18**, 103-121 (2013).

[22] H.-C. Lee and G.-R. Piao, *Boundary feedback control of the Burgers equations by a reduced-order approach using centroidal Voronoi tessellations*, J. Sci. Comp. **43**, 369-387 (2010).

[23] J. Lumley, *Stochastic Tools in Turbulence*, Academic Press, New York (1971).

[24] H.V. Ly and H.T. Tran, *Modeling and control of physical processes using proper orthogonal decomposition*, Comp. Math. Appl. **33**, 223-236 (2001).

[25] S. Micu, *On the controllability of the linearised Benjamin-Bona-Mahony equation*, SIAM J. Control. Optim. **39**, 1677-1696 (2000).

[26] K. Omrani and M. Ayadi, *Finite difference discretisation of the Benjamin-Bona-Mahony-Burgers equation*, Num. Meth. Partial Differential Equations **24**, 239-248 (2008).

[27] H. Park and Y. Jang, *Control of Burgers equation by means of mode reduction*, Int. J. Eng. Sci. **38**, 785-805 (2000).

[28] H. Park and J. Lee, *Solution of an inverse heat transfer problem by means of empirical reduction of modes*, Z. Angew. Math. Phys. **51**, 17-38 (2000).

[29] H. Park and W. Lee, *An efficient method of solving the Navier-Stokes equations for flow control*, Int. J. Numer. Mech. Eng. **41**, 1133-1151 (1998).

[30] G.-R. Piao, H.-C. Lee and J.-Y. Lee, *Distributed feedback control of the Burgers equation by a reduced-order approach using weighted centroidal Voronoi tessellation*, J. KSIAM **13**, 293-305 (2009).

[31] P.M. Prenter, *Splines and Variational Methods*, Wiley, New York (1975).

[32] S. Volkwein, *Optimal control of a phase field model using the proper orthogonal decomposition*, ZAMM **81**, 83-97 (2001).

[33] S. Volkwein, *Proper orthogonal decomposition and singular value decomposition*, Spezialforschungsbereich F003 Optimierung und Kontrolle, Projektbereich Kontinuierliche Optimierung und Kontrolle, Bericht Nr. 153, Graz (1999).