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# A New GSOR Method for Generalised Saddle Point Problems

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**Abstract.** A novel generalised successive overrelaxation (GSOR) method for solving generalised saddle point problems is proposed, based on splitting the coefficient matrix. The proposed method is shown to converge under suitable restrictions on the iteration parameters, and we present some illustrative numerical results.

**AMS subject classifications**: 65F10, 65F50

**Key words**: Generalised saddle point problem, generalised successive overrelaxation method, splitting, convergence analysis.

## 1. Introduction

The generalised saddle point problem considered here has the form

$$\begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix}, \tag{1.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $C \in R^{n\times n}$ is a symmetric positive semidefinite matrix, $B \in R^{m\times n}$ is a matrix of full column rank where $m \geq n$ and the superscript $T$ denotes its transpose, and $b \in R^m$ and $q \in R^n$ are given vectors. Linear systems of the form (1.1) arise in a variety of scientific and engineering applications, including mixed or hybrid finite element approximations of second-order elliptic problems [5,42], computational fluid dynamics [19,21,22,25], least squares problems [2], inversion of geophysical data [32], stationary semiconductor device [45,47], elasticity problems and Stokes equations [5].

In recent years, many iterative methods have been introduced to solve the problem (1.1), including Uzawa-type schemes [14, 20, 23, 25, 33, 34, 51, 53], iterative projection methods [3], block and approximate Schur complement preconditioners [17,19,22,35,40, 42,43], iterative null space methods [1,26,48], splitting methods [4,7,9–13,18,29,30,36, 38,39,41,46,50], indefinite preconditioning [31,37], and preconditioning methods based

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on approximate factorisation of the coefficient matrix [6,8,28,44]. A classical approach to solve (1.1) is the successive overrelaxation (SOR) iteration method [49], which can involve relatively low computation per iterative step. However, the SOR method requires a good or optimal iteration parameter to achieve comparable rates of convergence. To address this, Bai *et al.* [15] proposed a generalised SOR method, where another parameter is introduced for solving the problem (1.1) when $C = 0$. Bai & Wang [16] then developed parameterised inexact Uzawa methods to solve large sparse generalised saddle point problems via the generalised SOR method, where another symmetric positive definite matrix is introduced. Based on the SOR-like methods, Feng & Shao [24] proposed a generalised SOR-like method by introducing uncertain parameters, and Guo *et al.* [27] considered a new splitting of the coefficient matrix in a modified SOR-like method for solving the system (1.1) with $C = 0$. Zhang & Lu [52] established a generalised symmetric SOR method based on the well-known symmetric SOR iteration method for the saddle point problem.

However, all of the SOR-like methods mentioned above need to solve a linear algebraic system each step, which is difficult and time-consuming. In this article, we propose a new generalised successive overrelaxation method for the generalised saddle point problem (1.1) based on splitting the matrix $A$. Thus instead of solving the linear system with the large coefficient matrix $A$, we only need to solve a system with a triangular matrix in each step. Moreover, we shall show the convergence of our method under suitable restrictions on the iteration parameters.

Throughout , we use the following notation: $R^{m\times n}$ and $C^{m\times n}$ are the set of $m \times n$ real and complex matrices; $R^m = R^{m\times 1}$ and $C^m = C^{m\times 1}$. $\mathrm{i} = \sqrt{-1}$ is the imaginary unit; and for $H \in R^{n\times n}$, we write $H^{-1}$, $\mathrm{rank}(H)$, $\mathscr{N}\{H\}$, $\mathscr{R}\{H\}$, $\Lambda(H)$ and $\rho(H)$ to denote the inverse, the rank, the null space, the image space, the spectrum and the spectral radius of the matrix $H$, respectively. For $x \in C^n$, $x^*$ and $\|x\|$ respectively denote the conjugate transpose and the norm of the vector $x$, and $I_l$ denotes the identity matrix of order $l$.

The organisation of this paper is as follows. In Section 2, we present our new generalised successive overrelaxation method for solving generalised saddle point problems. We discuss its convergence in Section 3, present the results of numerical experiments in Section 4 to show the effectiveness of our method, and then briefly summarise our conclusions in Section 5.

## 2. The Generalised SOR Method

In this section, we consider a new generalised successive overrelaxation method for solving the generalised saddle point problems (1.1). Firstly, we split the matrix $A$ into the following form:

$$A = D - L - L^T, \tag{2.1}$$

where $D = \mathrm{diag}\{a_{11}, a_{22}, \cdots, a_{mm}\}$ is the diagonal matrix incorporating the diagonal entries of $A$, and $-L$ is the strictly lower triangular part of the matrix $A$. Thus the coefficient matrix

$\mathscr{A}$ in (1.1) has the following expansion:

$$\mathscr{A} =: \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} = \mathscr{D} - \mathscr{L} - \mathscr{U},$$

where

$$\mathscr{D} = \begin{pmatrix} D & 0 \\ 0 & Q \end{pmatrix}, \quad \mathscr{L} = \begin{pmatrix} L & 0 \\ B^T & 0 \end{pmatrix}, \quad \mathscr{U} = \begin{pmatrix} L^T & -B \\ 0 & Q-C \end{pmatrix},$$

and $Q \in R^{n\times n}$ is a given symmetric positive definite matrix. Combining this splitting of $\mathscr{A}$ with the Gauss-Seidel method, we can derive an iterative scheme:

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = (\mathscr{D}-\mathscr{L})^{-1}\mathscr{U} \begin{pmatrix} x^k \\ y^k \end{pmatrix} + (\mathscr{D}-\mathscr{L})^{-1} \begin{pmatrix} b \\ -q \end{pmatrix}.$$

On substituting the expressions of $\mathscr{D}$, $\mathscr{L}$ and $\mathscr{U}$ into the above equality, we have

$$\begin{cases} x^{k+1} = x^k + (D-L)^{-1}(b - Ax^k - By^k)\,, \\ y^{k+1} = y^k + Q^{-1}(B^T x^{k+1} - Cy^k - q)\,. \end{cases} \tag{2.2}$$

Alternatively, by applying the successive overrelaxation (SOR) method [49] to the same splitting, we can obtain another iterative formula:

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = (\mathscr{D}-\omega\mathscr{L})^{-1}[(1-\omega)\mathscr{D} + \omega\mathscr{U}] \begin{pmatrix} x^k \\ y^k \end{pmatrix} + \omega(\mathscr{D}-\omega\mathscr{L})^{-1} \begin{pmatrix} b \\ -q \end{pmatrix}.$$

Clearly, the above equality can be reduced to

$$\begin{cases} x^{k+1} = x^k + \omega(D-\omega L)^{-1}(b - Ax^k - By^k)\,, \\ y^{k+1} = y^k + \omega Q^{-1}(B^T x^{k+1} - Cy^k - q)\,, \end{cases} \tag{2.3}$$

where $\omega$ is a nonzero real number. Inspired by the ideas in Refs. [15, 16], we provide a new generalised SOR iteration method by introducing another new relaxation parameter.

Let $\omega$ and $\tau$ be two nonzero real numbers, and $\Omega$ be a diagonal matrix in the following form:

$$\Omega = \begin{pmatrix} \omega I_m & 0 \\ 0 & \tau I_n \end{pmatrix}.$$

Then we consider the following generalised SOR iteration scheme for solving the generalised saddle point problem (1.1):

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = (\mathscr{D}-\Omega\mathscr{L})^{-1}[(I-\Omega)\mathscr{D} + \Omega\mathscr{U}] \begin{pmatrix} x^k \\ y^k \end{pmatrix} + (\mathscr{D}-\Omega\mathscr{L})^{-1}\Omega \begin{pmatrix} b \\ -q \end{pmatrix}. \tag{2.4}$$

From the definitions of $\mathscr{D}$, $\mathscr{L}$, $\mathscr{U}$ and $\Omega$, we have

$$\mathscr{D} - \Omega\mathscr{L} = \begin{pmatrix} D-\omega L & 0 \\ -\tau B^T & Q \end{pmatrix} \tag{2.5}$$

and

$$(I-\Omega)\mathscr{D}+\Omega\mathscr{U}=\begin{pmatrix} (1-\omega)D+\omega L^T & -\omega B \\ 0 & Q-\tau C \end{pmatrix}$$
$$=\begin{pmatrix} D-\omega L-\omega A & -\omega B \\ 0 & Q-\tau C \end{pmatrix}. \tag{2.6}$$

This shows that

$$(\mathscr{D}-\Omega\mathscr{L})^{-1}=\begin{pmatrix} (D-\omega L)^{-1} & 0 \\ \tau Q^{-1}B^T(D-\omega L)^{-1} & Q^{-1} \end{pmatrix},$$

whence

$$\begin{aligned}\mathscr{H}(\omega,\tau) &=: (\mathscr{D}-\Omega\mathscr{L})^{-1}[(I-\Omega)\mathscr{D}+\Omega\mathscr{U}] \\ &= \begin{pmatrix} I-\omega(D-\omega L)^{-1}A & -\omega(D-\omega L)^{-1}B \\ \tau Q^{-1}B^T[I-\omega(D-\omega L)^{-1}A] & I-\tau Q^{-1}C-\omega\tau Q^{-1}B^T(D-\omega L)^{-1}B \end{pmatrix},\end{aligned} \tag{2.7}$$

and

$$\mathscr{M}(\omega,\tau)=:(\mathscr{D}-\Omega\mathscr{L})^{-1}\Omega=\begin{pmatrix} \omega(D-\omega L)^{-1} & 0 \\ \omega\tau Q^{-1}B^T(D-\omega L)^{-1} & \tau Q^{-1} \end{pmatrix}. \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.4), it follows that

$$\begin{cases} x^{k+1}=x^k+\omega(D-\omega L)^{-1}(b-Ax^k-By^k), \\ y^{k+1}=y^k+\tau Q^{-1}(B^Tx^{k+1}-Cy^k-q). \end{cases}$$

Clearly, if $\omega=\tau$ the above iteration scheme reduces to the SOR method (2.3); and if $\omega=\tau=1$ the above iteration scheme reduces to the GS method (2.2). In summary, we can derive the following new generalised successive overrelaxation method for solving the generalised saddle point problem.

`The NSOR method.`
Let $Q\in R^{n\times n}$ be s symmetric positive definite matrix. Given initial vectors $x^0\in R^m$ and $y^0\in R^n$, and two nonzero relaxation factors $\omega$, $\tau$, for $k=0,1,2,\cdots$ compute (until satisfactory numerical convergence)

$$\begin{cases} x^{k+1}=x^k+\omega(D-\omega L)^{-1}(b-Ax^k-By^k)\,, \\ y^{k+1}=y^k+\tau Q^{-1}(B^Tx^{k+1}-Cy^k-q)\,. \end{cases} \tag{2.9}$$

It is notable that our NSOR method (2.9) can be reformulated as

$$z^{k+1}=\mathscr{H}(\omega,\tau)z^k+\mathscr{M}(\omega,\tau)f\,,$$

where $\mathscr{H}(\omega,\tau)$ and $\mathscr{M}(\omega,\tau)$ are defined in (2.7) and (2.8), and

$$z^k=\begin{pmatrix} x^k \\ y^k \end{pmatrix}, \qquad f=\begin{pmatrix} b \\ -q \end{pmatrix}.$$

Furthermore, on letting

$$\mathcal{N}(\omega,\tau) =: \mathcal{M}(\omega,\tau)^{-1} - \mathcal{A} = \begin{pmatrix} \dfrac{1}{\omega}(D-\omega L)-A & -B \\ 0 & \dfrac{1}{\tau}Q-C \end{pmatrix}$$

we see that

$$\mathcal{A} = \mathcal{M}(\omega,\tau)^{-1} - \mathcal{N}(\omega,\tau) \tag{2.10}$$

defines a splitting of $\mathcal{A}$, implying that our NSOR method (2.9) can also be induced by this matrix splitting (2.10). In addition, we know that

$$\mathcal{H}(\omega,\tau) = \mathcal{M}(\omega,\tau)\mathcal{N}(\omega,\tau)$$

is the iteration matrix of the NSOR method. Alternatively, the matrix $\mathcal{M}(\omega,\tau)^{-1}$ can be used to precondition the system (1.1).

In order to establish the convergence theorem of the NSOR method (2.9), we need to clarify the concepts of positive definite and positive semidefinite. Thus for a general (nonsymmetric) matrix $X \in R^{n\times n}$, we say that $X$ is positive definite or positive semidefinite if its symmetric part $X + X^T$ is positive definite or positive semidefinite, respectively. We have the following two lemmas.

**Lemma 2.1.** *Let $A \in R^{m\times m}$ be a symmetric positive definite matrix and $B \in R^{m\times n}$ a matrix of full column rank, with $m \geq n$. If $0 < \omega < 2$, then $D-\omega L$ is positive definite and $B^T(D-\omega L)^{-1}B$ is nonsingular.*

*Proof.* It is easy to see that

$$D-\omega L + (D-\omega L)^T = 2D - \omega L - \omega L^T = 2D - \omega(D-A) = (2-\omega)D + \omega A$$

is symmetric positive definite, which implies that $D-\omega L$ is positive definite. For the second result, on noticing that $B$ is full column rank and

$$B^T(D-\omega L)^{-1}B + B^T(D-\omega L)^{-T}B = B^T(D-\omega L)^{-1}[(D-\omega L)^T + D-\omega L](D-\omega L)^{-T}B,$$

we have that $B^T(D-\omega L)^{-1}B + B^T(D-\omega L)^{-T}B$ is symmetric positive definite. Thus $B^T(D-\omega L)^{-1}B$ is positive definite, and indeed nonsingular. □

From Lemma 2.1, the matrix $D-\omega L$ is nonsingular if $0 < \omega < 2$, then the NSOR method (2.9) is well defined — and henceforth we assume that $0 < \omega < 2$.

**Lemma 2.2** (cf. Ref. [53])**.** *Both roots of the complex quadratic equation $\lambda^2 - \mu\lambda + \eta = 0$ are less than one in modulus if and only if $|\eta| < 1$ and $|\mu - \bar{\mu}\eta| + |\eta|^2 < 1$.*

## 3. Convergence Analysis

We now examine the convergence of the NSOR method (2.9) under suitable restrictions on the two iteration parameters. The following theorem characterises some properties of the eigenpairs of the matrix $\mathscr{H}(\omega,\tau)$, where we introduce the notation:

$$\alpha = \frac{x^*Lx}{x^*Dx}, \qquad \beta = \frac{x^*Ax}{x^*Dx}, \qquad \gamma = \frac{x^*BQ^{-1}B^Tx}{x^*Dx}. \tag{3.1}$$

**Theorem 3.1.** *Let $A \in R^{m\times m}$ and $Q \in R^{n\times n}$ be symmetric positive definite matrices, $C \in R^{n\times n}$ be symmetric positive semidefinite, and $B \in R^{m\times n}$ have full column rank, with $m > n$. Assume that $\lambda$ and $z = (x^T, y^T)^T \in C^{m+n}$ with $x \in C^m$ and $y \in C^n$ are the eigenvalue and eigenvector of the iteration matrix $\mathscr{H}(\omega,\tau)$ respectively, and $\alpha$, $\beta$ and $\gamma$ as defined in* (3.1). *Then for any nonzero parameters $\omega$ and $\tau$:*

*(a) if $C = 0$, then $\lambda$ satisfies the quadratic equation*

$$(1-\omega\alpha)\lambda^2 + (2\omega\alpha + \omega\beta + \tau\omega\gamma - 2)\lambda + 1 - \omega\alpha - \omega\beta = 0\,; \tag{3.2}$$

*(b) if $C = \delta Q$ with $\delta \neq 0$ a real constant, then either $\lambda = 1 - \delta\tau$ or $\lambda$ satisfies the quadratic equation*

$$(1-\omega\alpha)\lambda^2 + [(2-\tau\delta)\omega\alpha + \omega\beta + \tau\omega\gamma + \tau\delta - 2]\lambda + (1-\tau\delta)(1-\omega\alpha - \omega\beta) = 0\,. \tag{3.3}$$

*Proof.* Combining the relation of $\mathscr{H}(\omega,\tau)z = \lambda z$ with (2.7), it follows that

$$[(I-\Omega)\mathscr{D} + \Omega\mathscr{U}]z = \lambda(\mathscr{D} - \Omega\mathscr{L})z\,,$$

which together with (2.5) and (2.6) leads to

$$\begin{cases} (1-\lambda)(D-\omega L)x - \omega Ax = \omega By\,, \\ (\lambda - 1)Qy + \tau Cy = \lambda\tau B^Tx\,. \end{cases} \tag{3.4}$$

We assert that $\lambda \neq 1$ and $x \neq 0$. Otherwise, if $\lambda = 1$ then (3.4) reduces to $-Ax = By$ and $Cy = B^Tx$ such that $x = -A^{-1}By$ and $Cy = -B^TA^{-1}By$, which implies that

$$(C + B^TA^{-1}B)y = 0\,.$$

Noticing that $A$ is symmetric positive definite, $C$ is symmetric positive semidefinite and $B$ is full column rank, we can see that $C + B^TA^{-1}B$ is a symmetric positive definite matrix. Together with the above equality, this implies that $y = 0$. Thus we have $x = -A^{-1}By = 0$, which yields $z = 0$, a contradiction to the assumption that $z$ is an eigenvector. On the other hand, if $x = 0$ then the first equality in (3.4) reduces to $By = 0$ such that $y = 0$, and again we have a contradiction.

Let us first consider the case that $C = 0$. Obviously (3.4) reduces to

$$\begin{cases} (1-\lambda)(D-\omega L)x - \omega Ax = \omega By\,, \\ (\lambda - 1)Qy = \lambda\tau B^Tx\,. \end{cases} \tag{3.5}$$

From the second equation in (3.5), we obtain

$$y=\frac{\lambda\tau}{\lambda-1}Q^{-1}B^Tx\ ,$$

and substituting this relationship into the first equality in (3.5) yields

$$(1-\lambda)(D-\omega L)x-\omega Ax=\frac{\lambda\tau\omega}{\lambda-1}BQ^{-1}B^Tx\ ,$$

which can be simplified as

$$(\lambda-1)^2(D-\omega L)x+(\lambda-1)\omega Ax+\lambda\tau\omega BQ^{-1}B^Tx=0\ .$$

Since $x\neq 0$ and $D$ is a positive definite matrix, we have $x^*Dx\neq 0$. Multiplying the above equality from the left with $x^*/(x^*Dx)$ gives

$$(\lambda-1)^2\left(1-\omega\frac{x^*Lx}{x^*Dx}\right)+(\lambda-1)\omega\frac{x^*Ax}{x^*Dx}+\lambda\tau\omega\frac{x^*BQ^{-1}B^Tx}{x^*Dx}=0\ ,$$

and hence from (3.1), we have

$$(\lambda-1)^2(1-\omega\alpha)+(\lambda-1)\omega\beta+\lambda\tau\omega\gamma=0\ .$$

On rearranging this equation, we immediately get the first result.

Next, we consider the case that $C=\delta Q$ with $\delta\neq 0$. Obviously (3.4) becomes

$$\begin{cases}(1-\lambda)(D-\omega L)x-\omega Ax=\omega By\ ,\\ (\lambda-1+\tau\delta)Qy=\lambda\tau B^Tx\ .\end{cases}\tag{3.6}$$

Firstly, we shall show that $1-\tau\delta$ is an eigenvalue of $\mathscr{H}(\omega,\tau)$. Let $\lambda=1-\tau\delta$, so (3.6) reduces to

$$\begin{cases}\tau\delta(D-\omega L)x-\omega Ax=\omega By\ ,\\ (1-\tau\delta)B^Tx=0\ .\end{cases}\tag{3.7}$$

On multiplying the first equality in (3.7) from the left with $B^T$, we have

$$\tau\delta B^T(D-\omega L)x-\omega B^TAx=\omega B^TBy\ ,$$

which given the nonsingularity of $B^TB$ yields

$$y=\frac{1}{\omega}(B^TB)^{-1}B^T[\tau\delta(D-\omega L)-\omega A]x\ .$$

Combining this with (3.7), if we can find a nonzero vector $(x^T,y^T)^T$ satisfying

$$\begin{cases}x\in\mathscr{N}\{B^T\},\\ \tau\delta(D-\omega L)x-\omega Ax\in\mathscr{R}\{B\},\\ y=\frac{1}{\omega}(B^TB)^{-1}B^T[\tau\delta(D-\omega L)-\omega A]x\end{cases}$$

then $1-\tau\delta$ would be an eigenvalue of $\mathscr{H}(\omega,\tau)$.
On the other hand, if $\lambda \neq 1-\tau\delta$ it follows from the second equality in (3.6) that

$$y=\frac{\lambda\tau}{\lambda-1+\tau\delta}Q^{-1}B^{T}x\ ;$$

and substituting this into the first equality in (3.6) leads to

$$(1-\lambda)(D-\omega L)x-\omega Ax=\frac{\lambda\omega\tau}{\lambda-1+\tau\delta}BQ^{-1}B^{T}x\ ,$$

which can be simplified as

$$(\lambda-1)(\lambda-1+\tau\delta)(D-\omega L)x+(\lambda-1+\tau\delta)\omega Ax+\lambda\omega\tau BQ^{-1}B^{T}x=0\ .$$

On multiplying this equation from the left by $x^{*}/(x^{*}Dx)$, we have that

$$(\lambda-1)(\lambda-1+\tau\delta)(1-\omega\alpha)+(\lambda-1+\tau\delta)\omega\beta+\lambda\omega\tau\gamma=0\ ,$$

which can be reformulated as (3.3) to complete the proof. □

**Corollary 3.1.** *Under the same settings and conditions as in Theorem* 3.1*, if $\omega=\tau\neq 0$ then:*
*(a) when $C=0$, $\lambda$ satisfies the quadratic equation*

$$(1-\omega\alpha)\lambda^{2}+(2\omega\alpha+\omega\beta+\omega^{2}\gamma-2)\lambda+1-\omega\alpha-\omega\beta=0\ ;$$

*(b) when $C=\delta Q$ with $\delta\neq 0$ a real constant, either $\lambda=1-\delta\omega$ or $\lambda$ satisfies the quadratic equation*

$$(1-\omega\alpha)\lambda^{2}+\big[(2-\omega\delta)\omega\alpha+\omega\beta+\omega^{2}\gamma+\omega\delta-2\big]\lambda+(1-\omega\delta)(1-\omega\alpha-\omega\beta)=0\ .$$

**Corollary 3.2.** *Under the same settings and conditions as in Theorem* 3.1*, if $\omega=\tau=1$ then:*
*(a) when $C=0$, $\lambda$ satisfies the quadratic equation*

$$(1-\alpha)\lambda^{2}+(2\alpha+\beta+\gamma-2)\lambda+1-\alpha-\beta=0\ ;$$

*(b) when $C=\delta Q$ with $\delta\neq 0$ a real constant, either $\lambda=1-\delta$ or $\lambda$ satisfies the quadratic equation*

$$(1-\alpha)\lambda^{2}+[(2-\delta)\alpha+\beta+\gamma+\delta-2]\lambda+(1-\delta)(1-\alpha-\beta)=0\ .$$

From the above analysis, we now establish sufficiency conditions to guarantee $|\lambda|<1$ where $\lambda\in\Lambda(\mathscr{H}(\omega,\tau))$, presented in the following theorem.

**Theorem 3.2.** *Let $A\in R^{m\times m}$ and $Q\in R^{n\times n}$ be symmetric positive definite matrices, $C\in R^{n\times n}$ be symmetric positive semidefinite, and let $B\in R^{m\times n}$ have full column rank, with $m>n$. Assume that $\lambda$ and $z=(x^{T},y^{T})^{T}\in C^{m+n}$ with $x\in C^{m}$ and $y\in C^{n}$ are the respective eigenvalue and eigenvector of the iteration matrix $\mathscr{H}(\omega,\tau)$, and $\alpha=\alpha_{r}+\mathrm{i}\alpha_{\mathrm{i}}$, $\beta$ and $\gamma$ as* *defined in* (3.1).

*(a) When* $C=0$, $|\lambda|<1$ *provided that* $\omega$ *satisfies* $0<\omega<2$ *and* $\tau$ *satisfies the condition*

$$0<\tau<\frac{2\beta^2(2-\omega)}{\omega\gamma(\beta^2+4\alpha_{\mathrm{i}}^2)} \quad \textit{with} \quad x\notin\mathcal{N}\{B^T\}\,.$$

*(b) When* $C=\delta Q$ *with* $\delta\neq 0$, $|\lambda|<1$ *provided that* $\tau\delta=1$, *and*

$$0<\omega<\frac{2}{1+\tau\gamma}\,.$$

*Proof.* It follows from the proof of Theorem 3.1 that $x\neq 0$. Since $A$ and $Q$ are both symmetric positive definite and $B$ has full column rank, we have $\beta>0$ and $\gamma\geq 0$. Noticing that $0<\omega<2$, from Lemma 2.1 we know that $D-\omega L$ is positive definite. Then

$$x^*(D-\omega L)x=x^*Dx(1-\omega\alpha)\neq 0\,, \quad \text{hence } 1-\omega\alpha\neq 0\,.$$

We first consider the case when $C=0$. If $x\in\mathcal{N}\{B^T\}$, then it follows from (3.5) and $\lambda\neq 1$ that $(1-\lambda)(D-\omega L)x-\omega Ax=0$, which combining with (2.1) leads to

$$(D-\omega L)^{-1}[(1-\omega)D+\omega L^T]x=\lambda x.$$

As $A$ is symmetric positive definite and $0<\omega<2$, from [49], we can know that $|\lambda|<1$.

If $x\notin\mathcal{N}\{B^T\}$, then $\gamma>0$. From Theorem 3.1, we see that $\lambda$ satisfies the quadratic equation (3.2), which can be rewritten as

$$\lambda^2+\frac{2\omega\alpha+\omega\beta+\tau\omega\gamma-2}{1-\omega\alpha}\lambda+\frac{1-\omega\alpha-\omega\beta}{1-\omega\alpha}=0\,.$$

Together with Lemma 2.2, this yields $|\lambda|<1$ if and only if

$$\left|\frac{1-\omega\alpha-\omega\beta}{1-\omega\alpha}\right|<1, \tag{3.8}$$

$$\left|\frac{2\omega\alpha+\omega\beta+\tau\omega\gamma-2}{1-\omega\alpha}-\frac{(2\omega\bar{\alpha}+\omega\beta+\tau\omega\gamma-2)(1-\omega\alpha-\omega\beta)}{(1-\omega\bar{\alpha})(1-\omega\alpha)}\right| +\left|\frac{1-\omega\alpha-\omega\beta}{1-\omega\alpha}\right|^2<1\,. \tag{3.9}$$

Inequality (3.8) is equivalent to

$$|1-\omega\alpha-\omega\beta|^2<|1-\omega\alpha|^2\,,$$

and on substituting $\alpha=\alpha_r+\mathrm{i}\alpha_{\mathrm{i}}$ we have

$$(1-\omega\alpha_r-\omega\beta)^2+\omega^2\alpha_{\mathrm{i}}^2<(1-\omega\alpha_r)^2+\omega^2\alpha_{\mathrm{i}}^2\,,$$

whence for $\beta>0$

$$\omega[(\beta+2\alpha_r)\omega-2]<0\,. \tag{3.10}$$

We can directly check that

$$\begin{aligned}\beta+2\alpha_r &= \beta+\alpha+\bar{\alpha}\\ &= \frac{x^*Ax}{x^*Dx}+\frac{x^*Lx}{x^*Dx}+\frac{x^*L^Tx}{x^*Dx}\\ &= \frac{x^*(A+L+L^T)x}{x^*Dx}=\frac{x^*Dx}{x^*Dx}=1\,,\end{aligned}$$

and combining this with (3.10) we get $0<\omega<2$. On the other hand, it is easy to see that (3.9) is equivalent to

$$\begin{aligned}&|(1-\omega\bar{\alpha})(2\omega\alpha+\omega\beta+\tau\omega\gamma-2)-(2\omega\bar{\alpha}+\omega\beta+\tau\omega\gamma-2)(1-\omega\alpha-\omega\beta)|\\ &\quad+|1-\omega\alpha-\omega\beta|^2<|1-\omega\alpha|^2\,;\end{aligned}$$

and from straightforward computation we obtain

$$\begin{aligned}&|2\alpha_r\omega^2\beta+\omega^2\beta^2+\tau\omega^2\gamma\beta-2\omega\beta+2\mathrm{i}\alpha_\mathrm{i}\omega^2\tau\gamma|+|1-\omega\alpha_r-\omega\beta-\mathrm{i}\omega\alpha_\mathrm{i}|^2\\ <&|1-\omega\alpha_r-\mathrm{i}\omega\alpha_\mathrm{i}|^2\,,\end{aligned}$$

or equivalently

$$\begin{aligned}&\sqrt{(2\alpha_r\omega^2\beta+\omega^2\beta^2+\tau\omega^2\gamma\beta-2\omega\beta)^2+4\alpha_\mathrm{i}^2\omega^4\tau^2\gamma^2}+(1-\omega\alpha_r-\omega\beta)^2+\omega^2\alpha_\mathrm{i}^2\\ <&(1-\omega\alpha_r)^2+\omega^2\alpha_\mathrm{i}^2\,.\end{aligned}$$

Noticing that $\gamma>0$, $\beta+2\alpha_r=1$, from some simple computation we get

$$\tau[(\omega\gamma\beta^2+4\alpha_\mathrm{i}^2\omega\gamma)\tau-2\beta^2(2-\omega)]<0\,,$$

showing that

$$0<\tau<\frac{2\beta^2(2-\omega)}{\omega\gamma(\beta^2+4\alpha_\mathrm{i}^2)}\,,$$

and we know that if $0<\omega<2$ and $\tau$ satisfy this inequality then $|\lambda|<1$.
Next, we consider the case that $C=\delta Q$. If $x\in\mathcal{N}\{B^T\}$ and $\lambda\neq 1-\delta\tau$, from (3.6) we have $(1-\lambda)(D-\omega L)x-\omega Ax=0$. By the same way, we can prove that $|\lambda|<1$. If $x\notin\mathcal{N}\{B^T\}$, then $\gamma>0$. From Theorem 3.1, either $\lambda=1-\delta\tau$ or $\lambda$ satisfies the quadratic equation

$$\lambda^2+\frac{(2-\tau\delta)\omega\alpha+\omega\beta+\tau\omega\gamma+\tau\delta-2}{1-\omega\alpha}\lambda+\frac{(1-\tau\delta)(1-\omega\alpha-\omega\beta)}{1-\omega\alpha}=0\,,$$

which from Lemma 2.2 yields $|\lambda|<1$ if and only if

$$|1-\delta\tau|<1\,, \tag{3.11}$$

$$\left|\frac{(1-\tau\delta)(1-\omega\alpha-\omega\beta)}{1-\omega\alpha}\right|<1\,, \tag{3.12}$$

$$\left|\frac{(2-\tau\delta)\omega\alpha+\omega\beta+\tau\omega\gamma+\tau\delta-2}{1-\omega\alpha} - \frac{(2-\tau\delta)\omega\bar{\alpha}+\omega\beta+\tau\omega\gamma+\tau\delta-2}{1-\omega\bar{\alpha}}\cdot\frac{(1-\tau\delta)(1-\omega\alpha-\omega\beta)}{1-\omega\alpha}\right| + \left|\frac{(1-\tau\delta)(1-\omega\alpha-\omega\beta)}{1-\omega\alpha}\right|^2 < 1\,. \tag{3.13}$$

Inequalities (3.11) and (3.12) hold if $\tau\delta = 1$, and in that case (3.13) can be simplified to

$$|\omega\alpha+\omega\beta+\tau\omega\gamma-1| < |1-\omega\alpha|\,.$$

Using $\alpha = \alpha_r + \mathrm{i}\alpha_\mathrm{i}$, we have $(\omega\alpha_r+\omega\beta+\tau\omega\gamma-1)^2 < (1-\omega\alpha_r)^2$ , which is equivalent to

$$\omega(\beta+\tau\gamma)[(\beta+2\alpha_r+\tau\gamma)\omega-2] < 0\,,$$

and with $\beta+2\alpha_r = 1$ we obtain $\omega(\beta+\tau\gamma)[(1+\tau\gamma)\omega-2] < 0$. It is easy to see that $\tau > 0$, then the inequality immediately follows

$$0 < \omega < \frac{2}{1+\tau\gamma}.$$

Then combining these results with $0 < \omega < 2$, we complete the proof. □

Let us now derive some sufficient conditions for guaranteeing the convergence of the NSOR method (2.9). To this end, we introduce some notation:

$$\ell^2_{\max} = \max_{\|x\|=1}\left|\frac{1}{2\mathrm{i}}\left(\frac{x^*Lx}{x^*Dx} - \frac{x^*L^Tx}{x^*Dx}\right)\right|^2, \qquad \beta_{\min} = \min_{\|x\|=1}\frac{x^*Ax}{x^*Dx}\,,$$
$$\beta_{\max} = \max_{\|x\|=1}\frac{x^*Ax}{x^*Dx}\,,\quad \gamma_{\min} = \min_{\|x\|=1,x\notin\mathcal{N}\{B^T\}}\frac{x^*BQ^{-1}B^Tx}{x^*Dx}\,,\quad \gamma_{\max} = \max_{\|x\|=1}\frac{x^*BQ^{-1}B^Tx}{x^*Dx}\,.$$

As $A$ is symmetric positive definite and $D$ is symmetric positive definite, we have $\beta_{\min} > 0$, $\beta_{\max} > 0$, $\gamma_{\min} > 0$ and $\gamma_{\max} > 0$.

**Theorem 3.3.** *Let $A \in R^{m\times m}$ and $Q \in R^{n\times n}$ be symmetric positive definite matrices, $C \in R^{n\times n}$ symmetric positive semidefinite, and $B \in R^{m\times n}$ have full column rank, with $m > n$. Then the NSOR method* (2.9) *is convergent provided that:*

*(a) when $C = 0$, $\omega$ satisfies $0 < \omega < 2$ and $\tau$ satisfies the condition*

$$0 < \tau < \frac{2\beta^2_{\min}(2-\omega)}{\omega\gamma_{\max}(\beta^2_{\max}+4\ell^2_{\max})}\,;$$

*(b) when $C = \delta Q$ with $\delta \neq 0$, $\tau$ satisfies $\tau\delta = 1$ and $\omega$ satisfies the condition*

$$0 < \omega < \frac{2}{1+\tau\gamma_{\max}}\,.$$

*Proof.* The NSOR method is convergent if and only if the spectral radius of the iteration matrix $\mathscr{H}(\omega,\tau)$ is less than 1 — i.e. $|\lambda|<1$ holds for any $\lambda\in\Lambda(\mathscr{H}(\omega,\tau))$. Let $\lambda$ and $z=(x^T,y^T)^T\in C^{m+n}$ be the eigenvalue and eigenvector of $\mathscr{H}(\omega,\tau)$, respectively. From the proof in Theorem 3.1, it follows that $x\neq 0$, so without loss of generality we can assume $\|x\|=1$. For any $x\notin\mathscr{N}\{B^T\}$, $\alpha=\alpha_r+\mathrm{i}\alpha_{\mathrm{i}}$, $\beta$ and $\gamma$ defined in (3.1), we have

$$0\le\alpha_{\mathrm{i}}^2\le\ell_{\max}^2,\qquad\beta_{\min}\le\beta\le\beta_{\max},\qquad\gamma_{\min}\le\gamma\le\gamma_{\max}\,,$$

so given Theorem 3.2 and the arbitrariness of $\lambda$ the proof is complete. □

## 4. Numerical Experiments

In this section, we present the results of some numerical experiments to test the numerical feasibility and effectiveness of our new method. All experiments were run on a PC with Intel(R) Core(TM) i3 CPU M 370 @2.40GHz and RAM 2GBz, implemented in MATLAB R2013b. We report the number of required iterations (denoted by "Iter"), the required CPU time (denoted by "CPU"), and the relative error denoted and defined by

$$ERR:=\frac{\sqrt{\|x^k-\tilde{x}\|^2+\|y^k-\tilde{y}\|^2}}{\sqrt{\|\tilde{x}\|^2+\|\tilde{y}\|^2}}\,,$$

with $((x^k)^T,(y^k)^T)^T$ the final approximate solution and $((\tilde{x})^T,(\tilde{y})^T)^T$ the exact solution of the generalised saddle point problem (1.1).

In our implementation, we chose the right-hand-side vector $(b^T,-q^T)^T\in R^{m+n}$ such that the exact solution of the generalised saddle point problem (1.1) is $((\tilde{x})^T,(\tilde{y})^T)^T=(1,1,\cdots,1)^T\in R^{m+n}$, and $Q=I_n$. The iteration was halted when $ERR\le 10^{-5}$, and as the initial vector we chose $x^0=0$, $y^0=0$. We compared our NSOR method (2.9), with the successive overrelaxation iteration scheme (SOR) (2.3) and the parameterised inexact Uzawa method (denoted by "PIU") (see [16]).

**Example 4.1.** (cf. Ref. [10]) Consider the generalised saddle point problem (1.1), where

$$A\in R^{q\times q},\quad C\in R^{(n-q)\times(n-q)},\quad with\ 2q>n\,,$$

and the matrices $A=(a_{kj})$, $C=(c_{kj})$ and $B=(b_{kj})$ defined as follows:

$$a_{kj}=\begin{cases}k+1, & for\ j=k,\\ 1, & for\ |k-j|=1,\\ 0, & otherwise,\end{cases}\qquad k,\ j=1,2,\cdots,q\,,$$

$$c_{kj}=\begin{cases}k+1, & for\ j=k,\\ 1, & for\ |k-j|=1,\\ 0, & otherwise,\end{cases}\qquad k,\ j=1,2,\cdots,n-q\,,$$

$$b_{kj}=\begin{cases}j, & for\ k=j+2q-n,\\ 0, & otherwise,\end{cases}\qquad k=1,2,\cdots,q,\ j=1,2,\cdots,n-q\,.$$

Table 1: The optimal parameters for Example 4.1.

| | n | 100 | 400 | 800 | 1200 | 1600 |
|---|---|---|---|---|---|---|
| SOR | $\omega_{opt}$ | 0.1610 | 0.0470 | 0.0242 | 0.0162 | 0.0123 |
| NSOR | $\omega_{opt}$ | 0.6690 | 0.4271 | 0.0699 | 0.0750 | 0.0212 |
| | $\tau_{opt}$ | 0.1459 | 0.0449 | 0.0240 | 0.0162 | 0.0123 |
| PIU | $\omega_{opt}$ | 0.3100 | 0.4700 | 0.0891 | 0.0661 | 0.0191 |
| | $\tau_{opt}$ | 0.1491 | 0.0440 | 0.0240 | 0.0161 | 0.0101 |

Table 2: Numerical results for Example 4.1.

| n | SOR | | | NSOR | | | PIU | | |
|---|---|---|---|---|---|---|---|---|---|
| | Iter. | CPU | Err. | Iter. | CPU | Err. | Iter. | CPU | Err. |
| 100 | 94 | 0.0075 | 9.4212e-06 | 41 | 0.0030 | 9.2753e-06 | 40 | 0.0125 | 9.0362e-06 |
| 400 | 279 | 0.4176 | 9.9894e-06 | 130 | 0.1947 | 9.9664e-06 | 133 | 0.4914 | 9.7102e-06 |
| 800 | 512 | 3.1035 | 9.9792e-06 | 241 | 1.4125 | 9.9999e-06 | 242 | 5.3921 | 9.9673e-06 |
| 1200 | 745 | 9.6605 | 9.9906e-06 | 347 | 4.6147 | 9.8388e-06 | 347 | 22.5600 | 9.9920e-06 |
| 1600 | 967 | 21.7502 | 9.9786e-06 | 604 | 13.7452 | 9.9573e-06 | 606 | 90.4704 | 9.8650e-06 |

Table 3: The optimal parameters for Example 4.2.

| | p | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| SOR | $\omega_{opt}$ | 0.612 | 0.601 | 0.598 | 0.595 |
| NSOR | $\omega_{opt}$ | 0.5991 | 0.6200 | 0.6330 | 0.6570 |
| | $\tau_{opt}$ | 0.6749 | 0.5040 | 0.4188 | 0.3000 |
| PIU | $\omega_{opt}$ | 0.8220 | 0.8247 | 0.8201 | 0.8346 |
| | $\tau_{opt}$ | 0.3610 | 0.3683 | 0.3301 | 0.3411 |

In this example, we set $q = 0.9n$. Our numerical results are presented in Tables 1 and 2, and Figs. 1 and 2, for various dimensions $n$.

**Example 4.2.** (see [15])Consider the generalised saddle point problem (1.1), where

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in R^{2p^2 \times 2p^2},$$

$$B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2p^2 \times p^2}, \qquad C = 0 \in R^{p^2 \times p^2},$$

and

$$T = \frac{1}{h^2} \cdot \text{tridiag}(-1, 2, -1) \in R^{p \times p}, \qquad F = \frac{1}{h} \cdot \text{tridiag}(-1, 1, 0) \in R^{p \times p},$$

with $\otimes$ being the Kronecker product symbol and $h = 1/(p+1)$ the discretisation mesh sise.

The numerical results of Example 4.2 are presented in Tables 3 and 4, and Figs. 3 and 4, for various $p$.

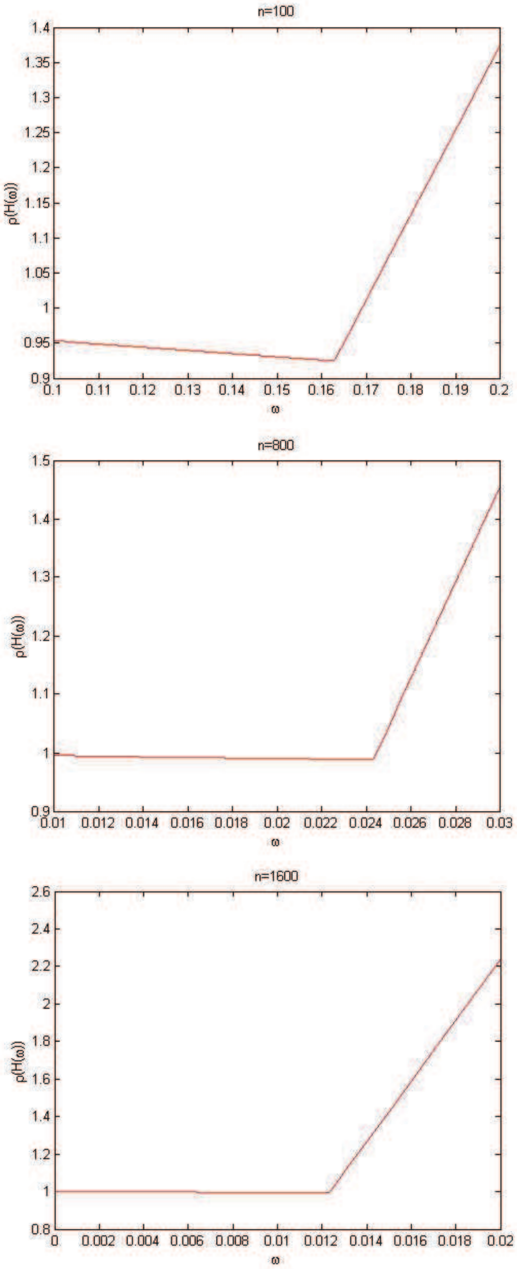


Figure 1: Curves of $\rho(H(\omega))$ vs. $\omega$ for the SOR iteration matrix for Example 4.1.

From Tables 2 and 4, it is seen that the NSOR method (2.9) is efficient for solving generalised saddle point problem (1.1); and from Figs. 1-4 that the new parameter $\tau$ plays an important role in the NSOR method (2.9). In addition, Figs 2 and 4 show that the minimum point of $\rho(H(\omega,\tau))$ is not the point where $\omega = \tau$. Moreover, $\tau$ should approach 0 when $\omega$ is close to 1 in order to guarantee that $\rho(H(\omega,\tau)) < 1$, and vice versa. This

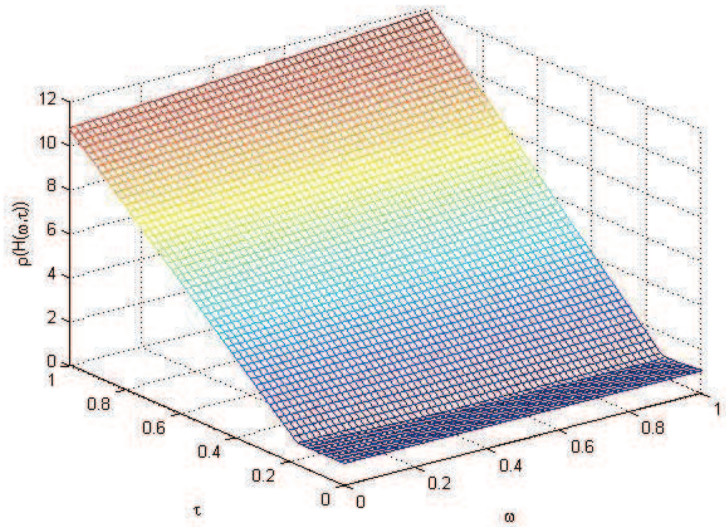


Figure 2: Curves of $\rho(H(\omega,\tau))$ vs. $(\omega,\tau)$ for the NSOR iteration matrix with $n = 100$ for Example 4.1.

Table 4: Numerical results for Example 4.2.

| p | SOR | | | NSOR | | | PIU | | |
|---|---|---|---|---|---|---|---|---|---|
| | Iter. | CPU | Err. | Iter. | CPU | Err. | Iter. | CPU | Err. |
| 8 | 666 | 0.0945 | 9.9564e-06 | 515 | 0.0617 | 9.9220e-06 | 132 | 0.0608 | 8.9490e-06 |
| 12 | 936 | 0.9003 | 9.4252e-06 | 875 | 0.8272 | 9.9093e-06 | 241 | 0.5356 | 9.8938e-06 |
| 16 | 1334 | 4.5647 | 9.5094e-06 | 1262 | 4.3504 | 9.9783e-06 | 350 | 3.5734 | 9.5898e-06 |
| 18 | 1773 | 14.9044 | 9.9084e-06 | 1668 | 13.8956 | 9.9425e-06 | 455 | 14.3540 | 9.8886e-06 |

implies that the new parameter $\tau$ provides a larger convergence range, so it is easier to choose the value of $\omega$.

## 5. Conclusion

A new method for solving the generalised saddle point problem (1.1) has been proposed. We have shown it converges under suitable restrictions on the iteration parameters, although a convergence analysis for optimal parameters has yet to be done.

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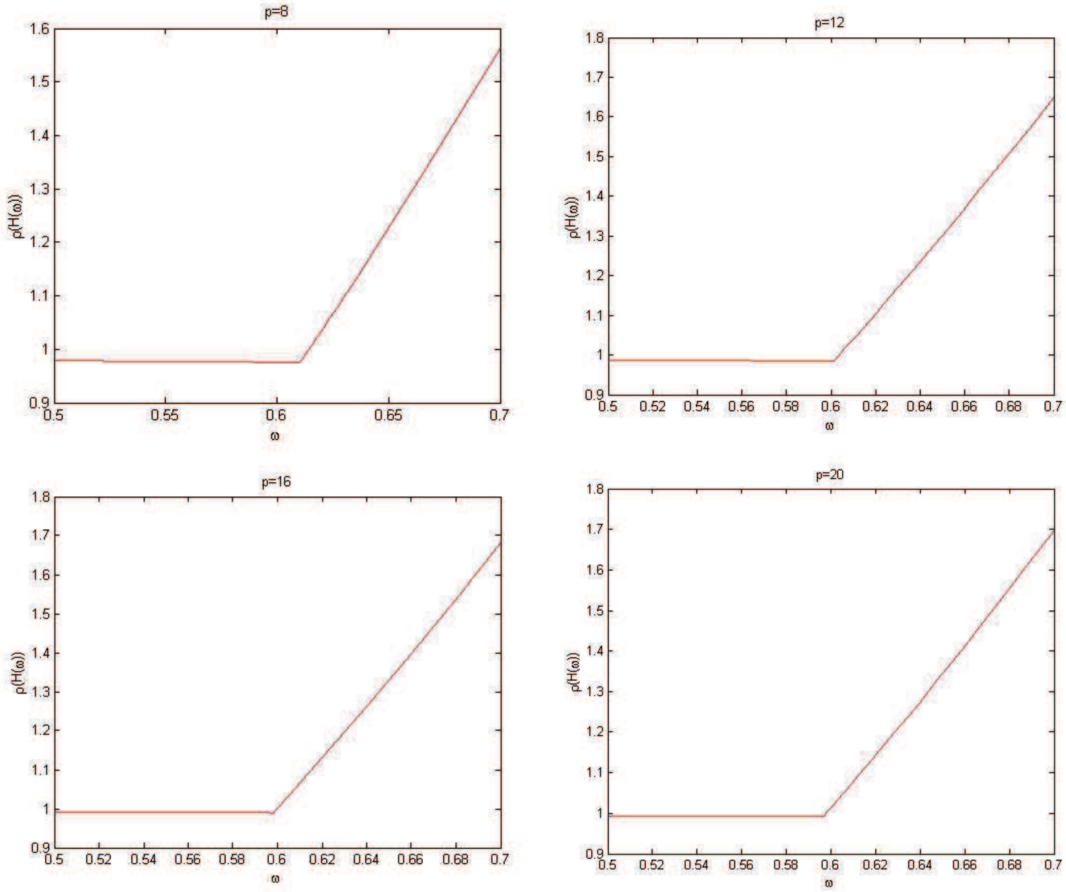


Figure 3: Curves of $\rho(H(\omega))$ vs. $\omega$ for the SOR iteration matrix for Example 4.2.

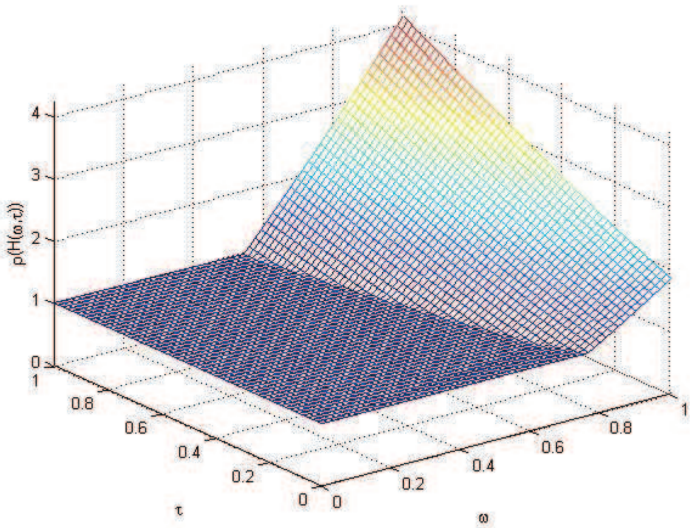


Figure 4: Curves of $\rho(H(\omega,\tau))$ vs. $(\omega,\tau)$ for the NSOR iteration matrix with $p=8$ for Example 4.2.

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