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Submatrix Constrained Inverse Eigenvalue Problem involving Generalised Centrohermitian Matrices in Vibrating Structural Model Correction

Wei-Ru Xu and Guo-Liang Chen*

Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, 200241, P. R. China.

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Abstract. Generalised centrohermitian and skew-centrohermitian matrices arise in a variety of applications in different fields. Based on the vibrating structure equation $M\ddot{x}+(D+G)\dot{x}+Kx=f(t)$ where M,D,G,K are given matrices with appropriate sizes and x is a column vector, we design a new vibrating structure mode. This mode can be discretised as the left and right inverse eigenvalue problem of a certain structured matrix. When the structured matrix is generalised centrohermitian, we discuss its left and right inverse eigenvalue problem with a submatrix constraint, and then get necessary and sufficient conditions such that the problem is solvable. A general representation of the solutions is presented, and an analytical expression for the solution of the optimal approximation problem in the Frobenius norm is obtained. Finally, the corresponding algorithm to compute the unique optimal approximate solution is presented, and we provide an illustrative numerical example.

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Key words: Left and right inverse eigenvalue problem, optimal approximation problem, generalised centrohermitian matrix, submatrix constraint.

1. Introduction

Generalised centrohermitian and skew-centrohermitian matrices arise in a variety of applications in fields such as information theory, linear system or estimate theory, signal processing, the numerical solution of differential equations and Markov processes — e.g. see Refs. [5, 7, 9–12, 17, 18, 21]. Here we consider vibrating structures such as bridges, highways, buildings and vehicles that are generally characterised by a linear second-order differential system

$$M\ddot{x}+(D+G)\dot{x}+Kx=f(t),$$

*Corresponding author. *Email addresses:* `weiruxu@foxmail.com` (W.-R. Xu), `glchen@math.ecnu.edu.cn` (G.-L. Chen)

where x is a column vector and M, D, G and K are matrices of appropriate size representing the mass (usually a diagonal matrix), damping, gyroscopic and stiffness, respectively. The general solution to the corresponding homogeneous equation $M\ddot{x} + (D+G)\dot{x} + Kx = 0$, on omitting the forcing function $f(t)$, plays an important role in the stability of the vibratory behaviour. In particular, we discuss the undamped non-gyroscopic model governed by

$$\begin{cases} M\ddot{x} + Kx = 0\,, \\ \ddot{y}^{\mathrm{H}}M + y^{\mathrm{H}}K = 0\,, \end{cases}$$

where y is a column vector of the same size as x and superscript H denotes the conjugate transpose (cf. below). The relevant solution form

$$\begin{cases} x(t) = \mathbf{u}\,e^{\lambda t}\,, \\ y(t) = \mathbf{v}\,e^{\mu t}\,, \end{cases}$$

for this linear system immediately leads to the two quadratic eigenvalue problems

$$\begin{cases} (\lambda^2 M + K)\,\mathbf{u} = 0\,, \\ \mathbf{v}^{\mathrm{H}}(\mu^2 M + K) = 0\,, \end{cases}$$

where $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ are their eigenpair solutions, respectively. Purely imaginary eigenvalues ($\lambda = \mathrm{i}\lambda_1\,,\ \mu = \mathrm{i}\mu_1$) define the natural frequency (λ_1 or μ_1) of the system and the corresponding natural mode $\mathbf{u}(\mathbf{v})$. Letting $\widetilde{\lambda} = \lambda_1^2$, $\widetilde{\mu} = \mu_1^2$, $A = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$, $z_1 = M^{\frac{1}{2}}\mathbf{u}$ and $z_2 = M^{\frac{1}{2}}\mathbf{v}$, we have

$$Az_1 = \widetilde{\lambda}z_1\,, \qquad z_2^{\mathrm{H}}A = \widetilde{\mu}z_2^{\mathrm{H}}\,. \tag{1.1}$$

The natural frequencies of the system and its associated natural modes are obviously determined by the stiffness matrix K or the mass matrix M. In practice, the stiffness matrix K is more complicated than the mass matrix M, and they are usually estimated by measurements or computed by some numerical methods (e.g. the finite element method). In engineering, some of the natural frequencies and natural modes can usually be identified in dynamic models, but there are often discrepancies between them and measured natural frequencies (natural modes). It is therefore often important to modify an approximate model such that the difference is minimised [13] — i.e. so the natural frequencies and natural modes in a corrected model are exactly the same as the identified natural frequencies and natural modes. In general, the stiffness or the mass matrix is corrected by vibration tests via nonlinear optimal optimisation techniques [3,4], but the existence and the uniqueness of the solution and the solution is not always optimal. Here we present a method to correct such an approximation model based on the left and right inverse eigenvalue problem (with spectral and structural constraint), where we find a matrix A of order n containing the given part of left and right eigenvalues and corresponding left and right eigenvectors. Prototypes of this problem also arise in the perturbation analysis of matrix eigenvalues [19] and in recursive processes [8], and has practical application in scientific computation and other engineering fields.

Throughout this article, $\mathscr{U}(n)$ denotes the set of n-by-n unitary matrices; and rank(A), A^{H} and $A^{\dagger}$ denote the rank, conjugate transpose and Moore-Penrose generalised inverse of any matrix $A \in \mathbb{C}^{n\times m}$, respectively. In addition, I_n, $\mathbf{0}$ and $\mathrm{i} = \sqrt{-1}$ respectively signify the identity matrix of order n, zero matrix or vector with appropriate size, and the familiar imaginary unit; $A(1:i, 1:j)$ denotes the $i \times j$ submatrix of a matrix A that lies in the rows $1, 2, \cdots, i$ and columns $1, 2, \cdots, j$; and tr(A) denotes its trace of any matrix $A \in \mathbb{C}^{n\times n}$. The inner product of matrices $A, B \in \mathbb{C}^{n\times m}$ is $\langle A, B\rangle = \mathrm{tr}(B^{\mathrm{H}}A)$; and the induced matrix norm is the Frobenius norm — i.e. $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{tr}(A^{\mathrm{H}}A)}$, such that $\mathbb{C}^{n\times m}$ is a Hilbert space. We now define generalised centrohermitian and skew-centrohermitian matrices as follows.

Definition 1.1. Given an involutory Hermitian matrix $\mathscr{P} \in \mathbb{C}^{k\times k}$, and an $n \times n$ matrix $K = \begin{pmatrix} \mathbf{0} & \mathscr{P} \\ \mathscr{P} & \mathbf{0} \end{pmatrix}$ for $n = 2k$ (even) and $K = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathscr{P} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathscr{P} & \mathbf{0} & \mathbf{0} \end{pmatrix}$ for $n = 2k+1$ (odd), then:

1. $A \in \mathbb{C}^{n\times n}$ is a generalised centrohermitian matrix if $A = K\bar{A}K$, and the set of all $n \times n$ generalised centrohermitian matrices is denoted by $\mathscr{GC}^{n\times n}(\mathscr{P})$; and
2. $A \in \mathbb{C}^{n\times n}$ is a generalised skew-centrohermitian matrix if $A = -K\bar{A}K$, and the set of all $n \times n$ generalised skew-centrohermitian matrices is denoted by $\mathscr{GSC}^{n\times n}(\mathscr{P})$.

If $\mathscr{P}$ is the cross-identity matrix of order k (i.e. with ones along the secondary diagonal and zeros elsewhere), then $\mathscr{GC}(\mathscr{P})$ and $\mathscr{GSC}(\mathscr{P})$ in the real number field reduce to the well-known sets of centrosymmetric and skew-centrosymmetric matrices, respectively.

Although inverse eigenvalue problems with one equality constraint involving centrosymmetric and skew-centrosymmetric matrices have been solved [1, 2, 15, 20, 24], the left and right inverse eigenvalue problem for generalised centrohermitian matrices with a submatrix constraint has not been analysed previously. Let X and Y be the identified natural mode matrices and let Λ and Δ be the natural frequency matrices. In reference to the system of linear matrix equations (1.1), we assume $M = I_n$ and A is generalised centrohermitian with an identified submatrix C_0 constraint. The corrected version of the model can be mathematically formulated via the following two problems involving generalised centrohermitian matrices.

Problem I Given partial eigeninformation $X \in \mathbb{C}^{n\times m}$, $Y \in \mathbb{C}^{n\times l}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, $\Delta = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_l) \in \mathbb{C}^{l\times l}$ and a matrix $C_0 \in \mathbb{C}^{f\times f}$, find $A \in \Omega$ satisfying

$$AX = X\Lambda, \quad Y^{\mathrm{H}}A = \Delta Y^{\mathrm{H}}, \quad A(1:f, 1:f) = C_0,$$

such that A maintains the eigeninformation, where Ω is the set $\mathscr{GC}^{n\times n}(\mathscr{P})$.

If there is no submatrix C_0 constraint, this problem corresponding to different classes of structured matrices has been solved — e.g. Liang & Dai [14] considered generalised reflexive and anti-reflexive matrices, Yin & Huang [23] considered (R, S)-symmetric and (R, S)-skew symmetric matrices. If there is a submatrix constraint in the inverse eigenvalue problem with one equality constraint, Bai [1] solved the case of centrosymmetric matrices and Yin *et al.* [22] discussed the case of (R, S)-symmetric matrices.

The second is the corrected optimal approximation problem, as follows.

Problem II Given $A^{\#} \in \mathbb{C}^{n\times n}$, find $A^{*} \in \mathscr{S}$ such that

$$\|A^{\#} - A^{*}\| = \min_{A\in\mathscr{S}} \|A^{\#} - A\| \,,$$

where $\mathscr{S}$ is the solution set of Problem I.

In Section 2, we introduce the properties of generalised centrohermitian matrices and obtain the necessary and sufficient conditions such that Problem I is solvable. Furthermore, we present the general representation of the solutions for Problem I. In Section 3, the existence and uniqueness of the solution for Problem II is proven. Finally, we give the algorithm to compute the unique optimal approximate solution, and present an illustrative numerical example in Section 4. Our conclusions are summarised in Section 5.

2. Solvability Conditions for Problem I

We first state the following lemma without proof, given the generalised results in Ref. [24].

Lemma 2.1. *Let $A \in \mathscr{GC}^{n\times n}(\mathscr{P})$. Then*

$$A = \begin{pmatrix} \mathscr{M} & \mathscr{H}\mathscr{P} \\ \mathscr{P}\mathscr{H} & \mathscr{P}\mathscr{M}\mathscr{P} \end{pmatrix}, \quad \mathscr{M}, \mathscr{H} \in \mathbb{C}^{k\times k}$$

for $n = 2k$, or

$$A = \begin{pmatrix} \mathscr{N} & \mathbf{u} & \mathscr{H}\mathscr{P} \\ \mathbf{v}^{\mathrm{H}} & \alpha & \mathbf{v}^{\mathrm{H}}\mathscr{P} \\ \mathscr{P}\mathscr{H} & \mathscr{P}\mathbf{u} & \mathscr{P}\mathscr{N}\mathscr{P} \end{pmatrix}, \quad \mathscr{N}, \mathscr{H} \in \mathbb{C}^{k\times k}, \ \mathbf{u}, \mathbf{v} \in \mathbb{C}^{k\times 1}, \ \alpha \in \mathbb{C}^{1\times 1}$$

for $n = 2k+1$. Furthermore, $A \in \mathscr{GC}^{n\times n}(\mathscr{P})$ if and only if

$$A = D_{2k} \begin{pmatrix} \mathscr{M} + \mathscr{H} & \mathbf{0} \\ \mathbf{0} & \mathscr{M} - \mathscr{H} \end{pmatrix} D_{2k}^{\mathrm{H}} \tag{2.1}$$

for $n = 2k$, where

$$D_{2k} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ \mathscr{P} & -\mathscr{P} \end{pmatrix} \in \mathscr{U}(n)\,; \tag{2.2}$$

or

$$A = D_{2k+1} \begin{pmatrix} \mathscr{N} + \mathscr{H} & \sqrt{2}\mathbf{u} & \mathbf{0} \\ \sqrt{2}\mathbf{v}^{\mathrm{H}} & \alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathscr{N} - \mathscr{H} \end{pmatrix} D_{2k+1}^{\mathrm{H}} \tag{2.3}$$

for $n = 2k+1$, where

$$D_{2k+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & \mathbf{0} & I_k \\ \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathscr{P} & \mathbf{0} & -\mathscr{P} \end{pmatrix} \in \mathscr{U}(n)\,. \tag{2.4}$$

Now consider $X,Z\in\mathbb{C}^{n\times m}$ and $Y,W\in\mathbb{C}^{n\times l}$, and let D_n be defined by (2.2) or (2.4). Set

$$\begin{aligned} D_n^{\mathrm{H}}X&=\begin{pmatrix} X_1\\ X_2\end{pmatrix}, && D_n^{\mathrm{H}}Z=\begin{pmatrix} Z_1\\ Z_2\end{pmatrix},\\ D_n^{\mathrm{H}}Y&=\begin{pmatrix} Y_1\\ Y_2\end{pmatrix}, && D_n^{\mathrm{H}}W=\begin{pmatrix} W_1\\ W_2\end{pmatrix}, \end{aligned}\tag{2.5}$$

where $X_1,Z_1\in\mathbb{C}^{\lfloor\frac{n+1}{2}\rfloor\times m}$,$X_2,Z_2\in\mathbb{C}^{k\times m}$, $Y_1,W_1\in\mathbb{C}^{\lfloor\frac{n+1}{2}\rfloor\times l}$ and $Y_2,W_2\in\mathbb{C}^{k\times l}$; and let the respective singular value decompositions (SVDs) of X_1,X_2,Y_1 and Y_2 be

$$\begin{aligned} X_1&=U\begin{pmatrix}\Sigma_1&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{pmatrix}V^{\mathrm{H}}, && Y_1=P\begin{pmatrix}\Gamma_1&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{pmatrix}Q^{\mathrm{H}},\\ X_2&=R\begin{pmatrix}\Sigma_2&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{pmatrix}S^{\mathrm{H}}, && Y_2=L\begin{pmatrix}\Gamma_2&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{pmatrix}\widetilde{M}^{\mathrm{H}}, \end{aligned}\tag{2.6}$$

where

$$\begin{aligned} &U=(U_1,U_2),\ P=(P_1,P_2)\in\mathscr{U}\left(\lfloor\frac{n+1}{2}\rfloor\right); \qquad R=(R_1,R_2),\ L=(L_1,L_2)\in\mathscr{U}(k);\\ &V=(V_1,V_2),\ S=(S_1,S_2)\in\mathscr{U}(m); \qquad Q=(Q_1,Q_2),\ \widetilde{M}=(\widetilde{M}_1,\widetilde{M}_2)\in\mathscr{U}(l);\\ &U_1\in\mathbb{C}^{\lfloor\frac{n+1}{2}\rfloor\times r},\ \ P_1\in\mathbb{C}^{\lfloor\frac{n+1}{2}\rfloor\times s},\ \ R_1\in\mathbb{C}^{k\times t},\ \ L_1\in\mathbb{C}^{k\times p},\\ &r=\mathrm{rank}(X_1),\ \ s=\mathrm{rank}(Y_1),\ \ t=\mathrm{rank}(X_2),\ \ p=\mathrm{rank}(Y_2),\\ &\Sigma_1=\mathrm{diag}(\xi_1,\xi_2,\cdots,\xi_r),\ \ \xi_i>0,\ 1\le i\le r,\\ &\Sigma_2=\mathrm{diag}(\eta_1,\eta_2,\cdots,\eta_t),\ \ \eta_i>0,\ 1\le i\le t,\\ &\Gamma_1=\mathrm{diag}(\sigma_1,\sigma_2,\cdots,\sigma_s),\ \ \sigma_i>0,\ 1\le i\le s,\\ &\Gamma_2=\mathrm{diag}(\delta_1,\delta_2,\cdots,\delta_p),\ \ \delta_i>0,\ 1\le i\le p. \end{aligned}$$

We immediately obtain the following Lemma, which contributes to solving Problem I later.

Lemma 2.2. *Given $X,Z\in\mathbb{C}^{n\times m}$, $Y,W\in\mathbb{C}^{n\times l}$ and D_n as defined in* (2.2) *or* (2.4). *Let $D_n^{\mathrm{H}}X$, $D_n^{\mathrm{H}}Y$, $D_n^{\mathrm{H}}Z$, $D_n^{\mathrm{H}}W$ be as given in* (2.5)*, and the SVDs of X_1, X_2, Y_1 and Y_2 be the same as* (2.6)*. Then $\varphi(A):=\|AX-Z\|^2+\|Y^{\mathrm{H}}A-W^{\mathrm{H}}\|^2=\min$ is solvable in $\mathscr{GC}^{n\times n}(\mathscr{P})$, and its general solution can be expressed as*

$$A=D_n\begin{pmatrix}A_{11}&\mathbf{0}\\ \mathbf{0}&A_{22}\end{pmatrix}D_n^{\mathrm{H}},\tag{2.7}$$

where

$$A_{11}=P\begin{pmatrix}\Phi*(P_1^{\mathrm{H}}Z_1V_1\Sigma_1+\Gamma_1Q_1^{\mathrm{H}}W_1^{\mathrm{H}}U_1) & \Gamma_1^{-1}Q_1^{\mathrm{H}}W_1^{\mathrm{H}}U_2\\ P_2^{\mathrm{H}}Z_1V_1\Sigma_1^{-1} & E_1\end{pmatrix}U^{\mathrm{H}},\ \ E_1\in\mathbb{C}^{(\lfloor\frac{n+1}{2}\rfloor-s)\times(\lfloor\frac{n+1}{2}\rfloor-r)},\tag{2.8}$$

$$A_{22}=L\begin{pmatrix}\Psi*(L_1^{\mathrm{H}}Z_2S_1\Sigma_2+\Gamma_2\widetilde{M}_1^{\mathrm{H}}W_2^{\mathrm{H}}R_1) & \Gamma_2^{-1}\widetilde{M}_1^{\mathrm{H}}W_2^{\mathrm{H}}R_2\\ L_2^{\mathrm{H}}Z_2S_1\Sigma_2^{-1} & E_2\end{pmatrix}R^{\mathrm{H}},\ \ E_2\in\mathbb{C}^{(k-p)\times(k-t)},\tag{2.9}$$

and

$$\Phi = \left(\frac{1}{\sigma_i^2 + \xi_j^2} \right)_{s\times r}, \quad 1 \le i \le s, \ \ 1 \le j \le r,$$
$$\Psi = \left(\frac{1}{\delta_i^2 + \eta_j^2} \right)_{p\times t}, \quad 1 \le i \le p, \ \ 1 \le j \le t. \tag{2.10}$$

Proof. We separately consider n even or odd, as follows.

When $n = 2k$, for any matrix $A \in \mathscr{GC}^{n\times n}(\mathscr{P})$ we obtain from (2.1) and (2.5):

$$\begin{aligned}\varphi(A) =& \|(\mathscr{M} + \mathscr{H})X_1 - Z_1\|^2 + \|Y_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H}) - W_1^{\mathrm{H}}\|^2 \\ &+ \|(\mathscr{M} - \mathscr{H})X_2 - Z_2\|^2 + \|Y_2^{\mathrm{H}}(\mathscr{M} - \mathscr{H}) - W_2^{\mathrm{H}}\|^2 .\end{aligned}$$

Because $\varphi(A)$ is a convex, continuous and differentiable function with respect to $\mathscr{M}$ and $\mathscr{N}$, if $\varphi(A) = \min$ then

$$\begin{cases} \dfrac{\partial\varphi(A)}{\partial\mathscr{M}} = (\mathscr{M} + \mathscr{H})X_1X_1^{\mathrm{H}} + (\mathscr{M} - \mathscr{H})X_2X_2^{\mathrm{H}} + Y_1Y_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H}) + Y_2Y_2^{\mathrm{H}}(\mathscr{M} - \mathscr{H}) \\ \qquad - Z_1X_1^{\mathrm{H}} - Z_2X_2^{\mathrm{H}} - Y_1W_1^{\mathrm{H}} - Y_2W_2^{\mathrm{H}} = 0\,, \\ \dfrac{\partial\varphi(A)}{\partial\mathscr{N}} = (\mathscr{M} + \mathscr{H})X_1X_1^{\mathrm{H}} - (\mathscr{M} - \mathscr{H})X_2X_2^{\mathrm{H}} + Y_1Y_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H}) - Y_2Y_2^{\mathrm{H}}(\mathscr{M} - \mathscr{H}) \\ \qquad - Z_1X_1^{\mathrm{H}} + Z_2X_2^{\mathrm{H}} - Y_1W_1^{\mathrm{H}} + Y_2W_2^{\mathrm{H}} = 0\,, \end{cases}$$

and we obtain the following equivalent form:

$$\begin{cases} (\mathscr{M} + \mathscr{H})X_1X_1^{\mathrm{H}} + Y_1Y_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H}) = Z_1X_1^{\mathrm{H}} + Y_1W_1^{\mathrm{H}}\,, \\ (\mathscr{M} - \mathscr{H})X_2X_2^{\mathrm{H}} + Y_2Y_2^{\mathrm{H}}(\mathscr{M} - \mathscr{H}) = Z_2X_2^{\mathrm{H}} + Y_2W_2^{\mathrm{H}}\,. \end{cases} \tag{2.11}$$

From (2.6), the first equality in (2.11) is equivalent to

$$\begin{aligned} &P^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U\begin{pmatrix} \Sigma_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \Gamma_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} P^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U \\ =& P^{\mathrm{H}}Z_1V\begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} Q^{\mathrm{H}}W_1^{\mathrm{H}}U\,, \end{aligned}$$

hence

$$\begin{cases} P_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U_1\Sigma_1^2 + \Gamma_1^2P_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U_1 = P_1^{\mathrm{H}}Z_1V_1\Sigma_1 + \Gamma_1Q_1^{\mathrm{H}}W_1^{\mathrm{H}}U_1\,, \\ P_2^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U_1\Sigma_1^2 = P_2^{\mathrm{H}}Z_1V_1\Sigma_1\,, \\ \Gamma_1^2P_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U_2 = \Gamma_1Q_1^{\mathrm{H}}W_1^{\mathrm{H}}U_2\,. \end{cases}$$

so that

$$\begin{cases} P_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U_1 = \Phi * (P_1^{\mathrm{H}}Z_1V_1\Sigma_1 + \Gamma_1Q_1^{\mathrm{H}}W_1^{\mathrm{H}}U_1)\,, \\ P_2^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U_1 = P_2^{\mathrm{H}}Z_1V_1\Sigma_1^{-1}\,, \\ P_1^{\mathrm{H}}(\mathscr{M} + \mathscr{H})U_2 = \Gamma_1^{-1}Q_1^{\mathrm{H}}W_1^{\mathrm{H}}U_2\,, \end{cases} \tag{2.12}$$

where Φ is given by (2.10).
Similarly, from the second equality in (2.11) we obtain

$$\begin{cases} L_1^{\mathrm{H}}(\mathscr{M}-\mathscr{H})R_1 = \Psi * (L_1^{\mathrm{H}} Z_2 S_1 \Sigma_2 + \Gamma_2 \widetilde{M}_1^{\mathrm{H}} W_2^{\mathrm{H}} R_1)\,, \\ L_2^{\mathrm{H}}(\mathscr{M}-\mathscr{H})R_1 = L_2^{\mathrm{H}} Z_2 S_1 \Sigma_2^{-1}\,, \\ L_1^{\mathrm{H}}(\mathscr{M}-\mathscr{H})R_2 = \Gamma_2^{-1} \widetilde{M}_1^{\mathrm{H}} W_2^{\mathrm{H}} R_2\,, \end{cases} \tag{2.13}$$

where Ψ is given by (2.10). From (2.12), we obtain $A_{11} = \mathscr{M} + \mathscr{H}$ in (2.8), and similarly from (2.13) we obtain $A_{22} = \mathscr{M} - \mathscr{H}$ in (2.9), hence we get (2.7).

When $n = 2k+1$, from (2.3) we may set $A_{11} = \begin{pmatrix} \mathscr{N}+\mathscr{H} & \sqrt{2}\mathbf{u} \\ \sqrt{2}\mathbf{v}^{\mathrm{H}} & \alpha \end{pmatrix}$ and $A_{22} = \mathscr{N} - \mathscr{H}$, and the proof is similar to that for the case $n = 2k$. So the proof is complete. □

In the above lemma, from (2.6) we also know that

$$\begin{aligned} \varphi(A) =& \|A_{11}X_1 - Z_1\|^2 + \|Y_1^{\mathrm{H}} A_{11} - W_1^{\mathrm{H}}\|^2 + \|A_{22}X_2 - Z_2\|^2 + \|Y_2^{\mathrm{H}} A_{22} - W_2^{\mathrm{H}}\|^2 \\ =& \|P_1^{\mathrm{H}} A_{11} U_1 \Sigma_1 - P_1^{\mathrm{H}} Z_1 V_1\|^2 + \|\Gamma_1 P_1^{\mathrm{H}} A_{11} U_1 - Q_1^{\mathrm{H}} W_1^{\mathrm{H}} U_1\|^2 \\ &+ \|P_2^{\mathrm{H}} A_{11} U_1 \Sigma_1 - P_2^{\mathrm{H}} Z_1 V_1\|^2 + \|\Gamma_1 P_1^{\mathrm{H}} A_{11} U_2 - Q_1^{\mathrm{H}} W_1^{\mathrm{H}} U_2\|^2 \\ &+ \|P^{\mathrm{H}} Z_1 V_2\|^2 + \|Q_2^{\mathrm{H}} W_1^{\mathrm{H}} U\|^2 + \|L^{\mathrm{H}} Z_2 S_2\|^2 + \|\widetilde{M}_2^{\mathrm{H}} W_2^{\mathrm{H}} R\|^2 \\ &+ \|L_1^{\mathrm{H}} A_{22} R_1 \Sigma_2 - L_1^{\mathrm{H}} Z_2 S_1\|^2 + \|\Gamma_2 L_1^{\mathrm{H}} A_{22} R_1 - \widetilde{M}_1^{\mathrm{H}} W_2^{\mathrm{H}} R_1\|^2 \\ &+ \|L_2^{\mathrm{H}} A_{22} R_1 \Sigma_2 - L_2^{\mathrm{H}} Z_2 S_1\|^2 + \|\Gamma_2 L_1^{\mathrm{H}} A_{22} R_2 - \widetilde{M}_1^{\mathrm{H}} W_2^{\mathrm{H}} R_2\|^2\,, \end{aligned}$$

where

$$\begin{aligned} &A_{11} = \mathscr{M} + \mathscr{H}\,, && A_{22} = \mathscr{M} - \mathscr{H} && \text{for } n = 2k\,, \\ &A_{11} = \begin{pmatrix} \mathscr{N}+\mathscr{H} & \sqrt{2}\mathbf{u} \\ \sqrt{2}\mathbf{v}^{\mathrm{H}} & \alpha \end{pmatrix}, && A_{22} = \mathscr{N} - \mathscr{H} && \text{for } n = 2k+1\,. \end{aligned}$$

Hence $\varphi(A) = 0$ if and only if

$$\begin{cases} P_1^{\mathrm{H}} Z_1 V_1 \Sigma_1^{-1} = \Gamma_1^{-1} Q_1^{\mathrm{H}} W_1^{\mathrm{H}} U_1\,, \\ Z_1 V_2 = 0\,, \\ Q_2^{\mathrm{H}} W_1^{\mathrm{H}} = 0\,, \end{cases} \quad \text{and} \quad \begin{cases} L_1^{\mathrm{H}} Z_2 S_1 \Sigma_2^{-1} = \Gamma_2^{-1} \widetilde{M}_1^{\mathrm{H}} W_2^{\mathrm{H}} R_1\,, \\ Z_2 S_2 = 0\,, \\ \widetilde{M}_2^{\mathrm{H}} W_2^{\mathrm{H}} = 0\,. \end{cases}$$

From (2.6), the above conditions are equivalent to

$$\begin{cases} Y_1^{\mathrm{H}} Z_1 = W_1^{\mathrm{H}} X_1\,, \\ Z_1 = Z_1 X_1^{\dagger} X_1\,, \\ W_1 = W_1 Y_1^{\dagger} Y_1\,, \end{cases} \quad \text{and} \quad \begin{cases} Y_2^{\mathrm{H}} Z_2 = W_2^{\mathrm{H}} X_2\,, \\ Z_2 = Z_2 X_2^{\dagger} X_2\,, \\ W_2 = W_2 Y_2^{\dagger} Y_2\,, \end{cases} \tag{2.14}$$

whence

$$
\begin{aligned}
A_{11} =&P\begin{pmatrix} P_1^{\mathrm{H}}Z_1V_1\Sigma_1^{-1} & \Gamma_1^{-1}Q_1^{\mathrm{H}}W_1^{\mathrm{H}}U_2 \\ P_2^{\mathrm{H}}Z_1V_1\Sigma_1^{-1} & E_1 \end{pmatrix}U^{\mathrm{H}} \\
=&Z_1X_1^{\dagger}+(Y_1^{\dagger})^{\mathrm{H}}W_1^{\mathrm{H}}(I_{\lfloor\frac{n+1}{2}\rfloor}-X_1X_1^{\dagger})+P_2E_1U_2^{\mathrm{H}} \quad\text{where}\quad E_1\in\mathbb{C}^{(\lfloor\frac{n+1}{2}\rfloor-s)\times(\lfloor\frac{n+1}{2}\rfloor-r)}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
A_{22} =&L\begin{pmatrix} L_1^{\mathrm{H}}Z_2S_1\Sigma_2^{-1} & \Gamma_2^{-1}\widetilde{M}_1^{\mathrm{H}}W_2^{\mathrm{H}}R_2 \\ L_2^{\mathrm{H}}Z_2S_1\Sigma_2^{-1} & E_2 \end{pmatrix}R^{\mathrm{H}} \\
=&Z_2X_2^{\dagger}+(Y_2^{\dagger})^{\mathrm{H}}W_2^{\mathrm{H}}(I_k-X_2X_2^{\dagger})+L_2E_2R_2^{\mathrm{H}} \quad\text{where}\quad E_2\in\mathbb{C}^{(k-p)\times(k-t)}\,.
\end{aligned}
$$

By substituting A_{11} and A_{22} into (2.7), we get the general expression for the solution of $\varphi(A)=0$:

$$
A=A_0+D_n\begin{pmatrix} P_2E_1U_2^{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & L_2E_2R_2^{\mathrm{H}} \end{pmatrix}D_n^{\mathrm{H}}\,, \tag{2.15}
$$

where

$$
A_0=D_n\begin{pmatrix} Z_1X_1^{\dagger}+(Y_1^{\dagger})^{\mathrm{H}}W_1^{\mathrm{H}}(I_{\lfloor\frac{n+1}{2}\rfloor}-X_1X_1^{\dagger}) & \mathbf{0} \\ \mathbf{0} & Z_2X_2^{\dagger}+(Y_2^{\dagger})^{\mathrm{H}}W_2^{\mathrm{H}}(I_k-X_2X_2^{\dagger}) \end{pmatrix}D_n^{\mathrm{H}}\,.
$$

Then letting

$$
(I_f,\mathbf{0})D_n=(T_1,T_2)\,,\qquad T_1\in\mathbb{C}^{f\times\lfloor\frac{n+1}{2}\rfloor},\ \ T_2\in\mathbb{C}^{f\times k}.
$$

The generalised singular value decomposition (GSVD) of the matrix pair $(P_2^{\mathrm{H}}T_1^{\mathrm{H}},L_2^{\mathrm{H}}T_2^{\mathrm{H}})$ described in Ref. [16] is

$$
P_2^{\mathrm{H}}T_1^{\mathrm{H}}=U_P\Sigma_P\widehat{M}\quad\text{and}\quad L_2^{\mathrm{H}}T_2^{\mathrm{H}}=U_L\Sigma_L\widehat{M}\,, \tag{2.16}
$$

where $\widehat{M}\in\mathbb{C}^{f\times f}$ is a nonsingular matrix, $U_P\in\mathscr{U}(\lfloor\frac{n+1}{2}\rfloor-s)$, $U_L\in\mathscr{U}(k-p)$, and

$$
\Sigma_P=\begin{array}{c}\begin{pmatrix} I_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda_P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \\ \begin{matrix} l_1 & g_1 & g-l_1-g_1 & f-g \end{matrix}\end{array},\qquad
\Sigma_L=\begin{array}{c}\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Gamma_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{g-l_1-g_1} & \mathbf{0} \end{pmatrix} \\ \begin{matrix} l_1 & g_1 & g-l_1-g_1 & f-g \end{matrix}\end{array},
$$

with $g=\mathrm{rank}(T_1P_2,T_2L_2)$, $l_1=\mathrm{rank}(T_1P_2,T_2L_2)-\mathrm{rank}(T_2L_2)$, $g_1=\mathrm{rank}(T_1P_2)+\mathrm{rank}(T_2L_2)-\mathrm{rank}(T_1P_2,T_2L_2)$, and $\Lambda_P=\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{g_1})$, $\Gamma_L=\mathrm{diag}(\beta_1,\beta_2,\cdots,\beta_{g_1})$ with $1>\alpha_1\geq\cdots\geq\alpha_{g_1}>0$ and $0<\beta_1\leq\cdots\leq\beta_{g_1}<1$, $\Lambda_P^2+\Gamma_L^2=I_{g_1}$.

Similarly, the GSVD of the matrix pair $(U_2^{\mathrm{H}}T_1^{\mathrm{H}},R_2^{\mathrm{H}}T_2^{\mathrm{H}})$ is given by

$$
U_2^{\mathrm{H}}T_1^{\mathrm{H}}=U_U\Pi_UN\quad\text{and}\quad R_2^{\mathrm{H}}T_2^{\mathrm{H}}=U_R\Pi_RN\,, \tag{2.17}
$$

where $N \in \mathbb{C}^{f\times f}$ is a nonsingular matrix, $U_U \in \mathscr{U}(\lfloor\frac{n+1}{2}\rfloor - r)$, $U_R \in \mathscr{U}(k-t)$, and

$$\Pi_U = \begin{array}{c} \begin{pmatrix} I_{l_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda_U & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \\ \begin{matrix} l_2 & g_2 & h-l_2-g_2 & f-h \end{matrix} \end{array}, \qquad \Pi_R = \begin{array}{c} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Gamma_R & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{h-l_2-g_2} & \mathbf{0} \end{pmatrix} \\ \begin{matrix} l_2 & g_2 & h-l_2-g_2 & f-h \end{matrix} \end{array},$$

with $h = \mathrm{rank}(T_1U_2, T_2R_2)$, $l_2 = \mathrm{rank}(T_1U_2, T_2R_2) - \mathrm{rank}(T_2R_2)$, $g_2 = \mathrm{rank}(T_1U_2) + \mathrm{rank}(T_2R_2) - \mathrm{rank}(T_1U_2, T_2R_2)$, $\Lambda_U = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_{g_2})$, $\Gamma_R = \mathrm{diag}(\nu_1, \nu_2, \ldots, \nu_{g_2})$ with $1 > \omega_1 \geq \cdots \geq \omega_{g_2} > 0$ and $0 < \nu_1 \leq \cdots \leq \nu_{g_2} < 1$, and $\Lambda_U^2 + \Gamma_R^2 = I_{g_2}$.

Furthermore, we compatibly partition the nonsingular matrices

$$\begin{aligned} \widehat{M}^{-1} &= \begin{array}{c} \begin{pmatrix} \widehat{M}_1, & \widehat{M}_2, & \widehat{M}_3 & \widehat{M}_4 \end{pmatrix} \\ \begin{matrix} l_1 & g_1 & g-l_1-g_1 & f-g \end{matrix} \end{array} \\ N^{-1} &= \begin{array}{c} \begin{pmatrix} N_1, & N_2, & N_3 & N_4 \end{pmatrix} \\ \begin{matrix} l_2 & g_2 & h-l_2-g_2 & f-h \end{matrix} \end{array} \end{aligned} \tag{2.18}$$

with the block column partitioning of Σ_P and Π_U, respectively.

We may now proceed to solving Problem I over the matrices in $\mathscr{GC}^{n\times n}(\mathscr{P})$ as follows.

Theorem 2.1. *Consider $X \in \mathbb{C}^{n\times m}$, $Y \in \mathbb{C}^{n\times l}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, $\Delta = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_l) \in \mathbb{C}^{l\times l}$, $C_0 \in \mathbb{C}^{f\times f}$ and D_n as described in* (2.2) *or* (2.4)*; and let $D_n^{\mathrm{H}}X$, $D_n^{\mathrm{H}}Y$ be as given in* (2.5) *and the SVDs of X_1, X_2, Y_1 and Y_2 be the same as in* (2.6)*. Set*

$$(I_f, \mathbf{0})D_n = (T_1, T_2), \quad T_1 \in \mathbb{C}^{f\times\lfloor\frac{n+1}{2}\rfloor},\ T_2 \in \mathbb{C}^{f\times k}, \tag{2.19}$$

and

$$\begin{aligned} C = C_0 &- T_1\left[X_1\Lambda X_1^{\dagger} + (Y_1\Delta^{\mathrm{H}}Y_1^{\dagger})^{\mathrm{H}}(I_{\lfloor\frac{n+1}{2}\rfloor} - X_1X_1^{\dagger})\right]T_1^{\mathrm{H}} \\ &- T_2\left[X_2\Lambda X_2^{\dagger} + (Y_2\Delta^{\mathrm{H}}Y_2^{\dagger})^{\mathrm{H}}(I_k - X_2X_2^{\dagger})\right]T_2^{\mathrm{H}}. \end{aligned} \tag{2.20}$$

Assume the respective GSVDs of the matrix pairs $(P_2^{\mathrm{H}}T_1^{\mathrm{H}}, L_2^{\mathrm{H}}T_2^{\mathrm{H}})$ and $(U_2^{\mathrm{H}}T_1^{\mathrm{H}}, R_2^{\mathrm{H}}T_2^{\mathrm{H}})$ are given by (2.16) *and* (2.17)*; and denote*

$$\widehat{M}^{-\mathrm{H}}CN^{-1} = (C_{ij})_{4\times4} \ \text{ with } \ C_{ij} = \widehat{M}_i^{\mathrm{H}}CN_j, \quad i,j = 1,2,3,4, \tag{2.21}$$

where $\widehat{M}_i$ and N_j are given by (2.18) *for all $i,j = 1,2,3,4$. Then Problem I is solvable in $\mathscr{GC}^{n\times n}(\mathscr{P})$ if and only if*

$$\begin{cases} Y_1^{\mathrm{H}}X_1\Lambda = \Delta Y_1^{\mathrm{H}}X_1\,, \\ X_1\Lambda = X_1\Lambda X_1^{\dagger}X_1\,, \\ Y_1\Delta^{\mathrm{H}} = Y_1\Delta^{\mathrm{H}}Y_1^{\dagger}Y_1\,, \end{cases} \qquad \begin{cases} Y_2^{\mathrm{H}}X_2\Lambda = \Delta Y_2^{\mathrm{H}}X_2\,, \\ X_2\Lambda = X_2\Lambda X_2^{\dagger}X_2\,, \\ Y_2\Delta^{\mathrm{H}} = Y_2\Delta^{\mathrm{H}}Y_2^{\dagger}Y_2\,, \end{cases} \tag{2.22}$$

and

$$C_{13} = \mathbf{0}, \quad C_{31} = \mathbf{0}, \quad (C_{14}^{\mathrm{H}}, \quad C_{24}^{\mathrm{H}}, \quad C_{34}^{\mathrm{H}}) = \mathbf{0}, \quad (C_{41}, C_{42}, C_{43}, C_{44}) = \mathbf{0} \tag{2.23}$$

hold. Moreover, its general solution can be expressed as

$$A = A_0 + D_n \begin{pmatrix} P_2 E_1 U_2^{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & L_2 E_2 R_2^{\mathrm{H}} \end{pmatrix} D_n^{\mathrm{H}} \tag{2.24}$$

with

$$A_0 = D_n \begin{pmatrix} X_1 \Lambda X_1^{\dagger} + (Y_1 \Delta^{\mathrm{H}} Y_1^{\dagger})^{\mathrm{H}} (I_{\lfloor \frac{n+1}{2} \rfloor} - X_1 X_1^{\dagger}) & \mathbf{0} \\ \mathbf{0} & X_2 \Lambda X_2^{\dagger} + (Y_2 \Delta^{\mathrm{H}} Y_2^{\dagger})^{\mathrm{H}} (I_k - X_2 X_2^{\dagger}) \end{pmatrix} D_n^{\mathrm{H}}, \tag{2.25}$$

$$E_1 = U_P \begin{pmatrix} C_{11} & C_{12} \Lambda_U^{-1} & X_{13} \\ \Lambda_P^{-1} C_{21} & \Lambda_P^{-1} (C_{22} - \Gamma_L Y_{22} \Gamma_R) \Lambda_U^{-1} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} U_U^{\mathrm{H}} \tag{2.26}$$

and

$$E_2 = U_L \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & \Gamma_L^{-1} C_{23} \\ Y_{31} & C_{32} \Gamma_R^{-1} & C_{33} \end{pmatrix} U_R^{\mathrm{H}}, \tag{2.27}$$

where X_{13}, X_{23}, X_{31}, X_{32}, X_{33}, Y_{11}, Y_{12}, Y_{13}, Y_{21}, Y_{22} and Y_{31} are arbitrary block matrices with appropriate sizes.

Proof. Problem I is solvable if and only if there exists a matrix $A \in \mathscr{G}\mathscr{C}^{n \times n}(\mathscr{P})$ such that the system of matrix equations

$$AX = X\Lambda, \qquad Y^{\mathrm{H}} A = \Delta Y^{\mathrm{H}} \tag{2.28}$$

is consistent and

$$(I_f, \mathbf{0}) A (I_f, \mathbf{0})^{\mathrm{H}} = C_0. \tag{2.29}$$

Firstly, set $Z = X\Lambda$ and $W = Y\Delta^{\mathrm{H}}$, from (2.14) and (2.15) we know that (2.28) is consistent if and only if the conditions in (2.22) hold. Furthermore, its general solution is given by (2.24). Then from (2.19), (2.20), (2.24) and (2.25), we have that Eq. (2.29) is equivalent to find the matrices $E_1 \in \mathbb{C}^{(\lfloor \frac{n+1}{2} \rfloor - s) \times (\lfloor \frac{n+1}{2} \rfloor - r)}$ and $E_2 \in \mathbb{C}^{(k-p)\times(k-t)}$ such that

$$T_1 P_2 E_1 U_2^{\mathrm{H}} T_1^{\mathrm{H}} + T_2 L_2 E_2 R_2^{\mathrm{H}} T_2^{\mathrm{H}} = C \tag{2.30}$$

is consistent. From (2.16), (2.17), (2.21) and [6, Theorem 3.1] it follows that Eq. (2.30) is consistent if and only if the conditions in (2.23) are satisfied. In this case, its general solution pair (E_1, E_2) can be given by (2.26) and (2.27). Problem I is therefore solvable if and only if the conditions in (2.22) and (2.23) hold, and then substituting (2.26) and (2.27) into (2.24) yields the general solution of Problem I. $\square$

3. The Unique Optimal Approximate Solution of Problem II

If the solution set $\mathscr{S}$ of Problem I is nonempty, then Problem II has a unique solution as follows.

Lemma 3.1. *For any matrix $A \in \mathbb{C}^{n\times n}$, there exist unique matrices $A_1 \in \mathscr{G}\mathscr{C}^{n\times n}(\mathscr{P})$ and $A_2 \in \mathscr{G}\mathscr{S}\mathscr{C}^{n\times n}(\mathscr{P})$ such that*

$$A = A_1 + A_2, \quad \langle A_1, A_2\rangle = 0,$$

where

$$A_1 = \frac{1}{2}(A + KAK), \quad A_2 = \frac{1}{2}(A - KAK).$$

Because this result is similar to that of Ref. [24], we omit its proof of this lemma.

Given Theorem 2.1, we have the following theorem for the generalised centrohermitian solution of Problem II.

Theorem 3.1. *Consider $A^{\#} \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times m}$, $Y \in \mathbb{C}^{n\times l}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, $\Delta = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_l) \in \mathbb{C}^{l\times l}$ and D_n as described in* (2.2) *or* (2.4)*; and let $D_n^{\mathrm{H}}X$, $D_n^{\mathrm{H}}Y$ be as given in* (2.5) *and the SVDs of X_1, X_2, Y_1 and Y_2 be the same as in* (2.6). *Set*

$$\begin{aligned} A_{11}^0 &= X_1\Lambda X_1^{\dagger} + \left(Y_1\Delta^{\mathrm{H}}Y_1^{\dagger}\right)^{\mathrm{H}}\left(I_{\lfloor\frac{n+1}{2}\rfloor} - X_1X_1^{\dagger}\right), \\ A_{22}^0 &= X_2\Lambda X_2^{\dagger} + \left(Y_2\Delta^{\mathrm{H}}Y_2^{\dagger}\right)^{\mathrm{H}}\left(I_k - X_2X_2^{\dagger}\right) \end{aligned} \tag{3.1}$$

and

$$H_1 = D_n\left(1:n, 1:\lfloor\frac{n+1}{2}\rfloor\right), \qquad H_2 = D_n\left(1:n, \lfloor\frac{n+1}{2}\rfloor + 1:n\right). \tag{3.2}$$

Partition the matrices

$$U_P^{\mathrm{H}}P_2^{\mathrm{H}}(H_1^{\mathrm{H}}A_1H_1 - A_{11}^0)U_2U_U = \begin{array}{c} \left(\begin{array}{ccc} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{array}\right) \begin{array}{c} l_1 \\ g_1 \\ \lfloor\frac{n+1}{2}\rfloor - s - l_1 - g_1 \end{array} \\ \begin{array}{ccc} l_2 & g_2 & \lfloor\frac{n+1}{2}\rfloor - r - l_2 - g_2 \end{array} \end{array}, \tag{3.3}$$

$$U_L^{\mathrm{H}}L_2^{\mathrm{H}}(H_2^{\mathrm{H}}A_1H_2 - A_{22}^0)R_2U_R = \begin{array}{c} \left(\begin{array}{ccc} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & Y_{22}^* & Y_{23}^* \\ Y_{31}^* & Y_{32}^* & Y_{33}^* \end{array}\right) \begin{array}{c} k - p - g + l_1 \\ g_1 \\ g - l_1 - g_1 \end{array} \\ \begin{array}{ccc} k - t - h + l_2 & g_2 & h - l_2 - g_2 \end{array} \end{array}, \tag{3.4}$$

where $A_1 = (A^{\#} + KA^{\#}K)/2$. If the solution set $\mathscr{S}$ of Problem I over the matrices in $\mathscr{G}\mathscr{C}^{n\times n}(\mathscr{P})$ is nonempty, then Problem II has a unique solution

$$A^* = D_n\begin{pmatrix} A_{11}^0 + P_2E_1^*U_2^{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & A_{22}^0 + L_2E_2^*R_2^{\mathrm{H}} \end{pmatrix}D_n^{\mathrm{H}} \tag{3.5}$$

with

$$E_1^* = U_P \begin{pmatrix} C_{11} & C_{12}\Lambda_U^{-1} & X_{13}^* \\ \Lambda_P^{-1}C_{21} & \Lambda_P^{-1}(C_{22}-\Gamma_L\widehat{Y}_{22}\Gamma_R)\Lambda_U^{-1} & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix} U_U^{\mathrm{H}} \tag{3.6}$$

and

$$E_2^* = U_L \begin{pmatrix} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & \widehat{Y}_{22} & \Gamma_L^{-1}C_{23} \\ Y_{31}^* & C_{32}\Gamma_R^{-1} & C_{33} \end{pmatrix} U_R^{\mathrm{H}}\,, \tag{3.7}$$

where

$$\begin{aligned} &\widehat{Y}_{22} = \Theta * \left(\Lambda_P^2 Y_{22}^*\Lambda_U^2 + \Gamma_L C_{22}\Gamma_R - \Lambda_P\Gamma_L X_{22}^*\Gamma_R\Lambda_U\right)\,, \\ &\Theta = (\theta_{ij}) = \left(\frac{1}{\alpha_i^2\omega_j^2 + \beta_i^2\nu_j^2}\right) \in \mathbb{R}^{g_1\times g_2}\,. \end{aligned}$$

Proof. Assuming the solution set $\mathscr{S}$ of Problem I is nonempty, it is apparent that $\mathscr{S}$ is a closed convex set and forms an affine subspace, therefore Problem II has a unique solution [8, pp. 209]. Now for any given matrix $A^\# \in \mathbb{C}^{n\times n}$, from Lemma 3.1 there exist unique matrices $A_1 = \frac{1}{2}(A^\# + KA^\# K) \in \mathscr{GC}^{n\times n}(\mathscr{P})$ and $A_2 = \frac{1}{2}(A^\# - KA^\# K) \in \mathscr{GSC}^{n\times n}(\mathscr{P})$ such that $A^\# = A_1 + A_2$ and $\langle A_1, A_2\rangle = 0$. Then for any matrix $A \in \mathscr{S} \subseteq \mathscr{GC}^{n\times n}(\mathscr{P})$ given by (2.24), we have

$$\|A^\# - A\|^2 = \|A_1 + A_2 - A\|^2 = \|A_1 - A\|^2 + \|A_2\|^2\,.$$

From the unitary invariance of the Frobenius norm and Eqs. (3.1) and (3.2), we have

$$\begin{aligned} \|A - A_1\|^2 =& \|D_n^{\mathrm{H}} A D_n - D_n^{\mathrm{H}} A_1 D_n\|^2 \\ =& \|H_1^{\mathrm{H}} A_1 H_2\|^2 + \|H_2^{\mathrm{H}} A_1 H_1\|^2 + \|A_{11}^0 + P_2 E_1 U_2^{\mathrm{H}} - H_1^{\mathrm{H}} A_1 H_1\|^2 \\ &+ \|A_{22}^0 + L_2 E_2 R_2^{\mathrm{H}} - H_2^{\mathrm{H}} A_1 H_2\|^2 \\ =& \|H_1^{\mathrm{H}} A_1 H_2\|^2 + \|H_2^{\mathrm{H}} A_1 H_1\|^2 + \|E_1 - P_2^{\mathrm{H}}(H_1^{\mathrm{H}} A_1 H_1 - A_{11}^0)U_2\|^2 \\ &+ \|H_1^{\mathrm{H}} A_1 H_1 - A_{11}^0 - P_2 P_2^{\mathrm{H}}(H_1^{\mathrm{H}} A_1 H_1 - A_{11}^0)U_2 U_2^{\mathrm{H}}\|^2 \\ &+ \|E_2 - L_2^{\mathrm{H}}(H_2^{\mathrm{H}} A_1 H_2 - A_{22}^0)R_2\|^2 \\ &+ \|H_2^{\mathrm{H}} A_1 H_2 - A_{22}^0 - L_2 L_2^{\mathrm{H}}(H_2^{\mathrm{H}} A_1 H_2 - A_{22}^0)R_2 R_2^{\mathrm{H}}\|^2\,. \end{aligned}$$

The problem $\min_{A\in\mathscr{S}} \|A^\# - A\|^2$ is therefore equivalent to the problem

$$\|E_1 - P_2^{\mathrm{H}}(H_1^{\mathrm{H}} A_1 H_1 - A_{11}^0)U_2\|^2 + \|E_2 - L_2^{\mathrm{H}}(H_2^{\mathrm{H}} A_1 H_2 - A_{22}^0)R_2\|^2 = \min$$

for any $(E_1, E_2) \in \mathbb{C}^{(\lfloor\frac{n+1}{2}\rfloor - s)\times(\lfloor\frac{n+1}{2}\rfloor - r)} \times \mathbb{C}^{(k-p)\times(k-t)}$ given by (2.26) and (2.27). From the unitary invariance of the Frobenius norm and Eqs. (3.3) and (3.4),

$$
\begin{aligned}
&\|E_1 - P_2^{\mathrm{H}}(H_1^{\mathrm{H}}A_1H_1 - A_{11}^0)U_2\|^2 + \|E_2 - L_2^{\mathrm{H}}(H_2^{\mathrm{H}}A_1H_2 - A_{22}^0)R_2\|^2 \\
=&\|U_P^{\mathrm{H}}E_1U_U - U_P^{\mathrm{H}}P_2^{\mathrm{H}}(H_1^{\mathrm{H}}A_1H_1 - A_{11}^0)U_2U_U\|^2 + \|U_L^{\mathrm{H}}E_2U_R - U_L^{\mathrm{H}}L_2^{\mathrm{H}}(H_2^{\mathrm{H}}A_1H_2 - A_{22}^0)R_2U_R\|^2 \\
=&\left\|\begin{pmatrix} C_{11} - X_{11}^* & C_{12}\Lambda_U^{-1} - X_{12}^* & X_{13} - X_{13}^* \\ \Lambda_P^{-1}C_{21} - X_{21}^* & \Lambda_P^{-1}(C_{22} - \Gamma_L Y_{22}\Gamma_R)\Lambda_U^{-1} - X_{22}^* & X_{23} - X_{23}^* \\ X_{31} - X_{31}^* & X_{32} - X_{32}^* & X_{33} - X_{33}^* \end{pmatrix}\right\|^2 \\
&+\left\|\begin{pmatrix} Y_{11} - Y_{11}^* & Y_{12} - Y_{12}^* & Y_{13} - Y_{13}^* \\ Y_{21} - Y_{21}^* & Y_{22} - Y_{22}^* & \Gamma_L^{-1}C_{23} - Y_{23}^* \\ Y_{31} - Y_{31}^* & C_{32}\Gamma_R^{-1} - Y_{32}^* & C_{33} - Y_{33}^* \end{pmatrix}\right\|^2.
\end{aligned}
$$

Consequently, $\|A^{\#} - A\|^2 = \min,\ \forall A \in \mathscr{S}$ if and only if

$$
\begin{aligned}
&X_{13} = X_{13}^*, \quad X_{23} = X_{23}^*, \quad X_{31} = X_{31}^*, \quad X_{32} = X_{32}^*, \quad X_{33} = X_{33}^*, \\
&Y_{11} = Y_{11}^*, \quad Y_{12} = Y_{12}^*, \quad Y_{13} = Y_{13}^*, \quad Y_{21} = Y_{21}^*, \quad Y_{31} = Y_{31}^*,
\end{aligned} \tag{3.8}
$$

and

$$
\varphi(Y_{22}) := \|\Lambda_P^{-1}(C_{22} - \Gamma_L Y_{22}\Gamma_R)\Lambda_U^{-1} - X_{22}^*\|^2 + \|Y_{22} - Y_{22}^*\|^2 = \min, \quad \forall Y_{22} \in \mathbb{C}^{g_1\times g_2}. \tag{3.9}
$$

On using matrix differential calculus,

$$
\frac{\partial\varphi(Y_{22})}{\partial Y_{22}} = 2\left[\Lambda_P^{-2}\Gamma_L^2Y_{22}\Gamma_R^2\Lambda_U^{-2} + Y_{22} + \Lambda_P^{-1}\Gamma_L(X_{22}^* - \Lambda_P^{-1}C_{22}\Lambda_U^{-1})\Gamma_R\Lambda_U^{-1} - Y_{22}^*\right],
$$

and setting $\partial\varphi(Y_{22})/\partial Y_{22} = 0$, the solution of (3.9) is

$$
\widehat{Y}_{22} = \Theta * \left(\Lambda_P^2 Y_{22}^*\Lambda_U^2 + \Gamma_L C_{22}\Gamma_R - \Lambda_P\Gamma_L X_{22}^*\Gamma_R\Lambda_U\right), \tag{3.10}
$$

where

$$
\Theta = (\theta_{ij}) = \left(\frac{1}{\alpha_i^2\omega_j^2 + \beta_i^2\nu_j^2}\right) \in \mathbb{R}^{g_1\times g_2}.
$$

Substituting (3.8) and (3.10) into (2.26) and (2.27), we get (3.6) and (3.7) respectively. Finally, substituting (3.6) and (3.7) into (2.24) yields the unique optimal approximate solution A^* as described in (3.5). □

4. Numerical Algorithm and Experiment

Based on Theorem 3.1, we establish the following algorithm for computing the optimal approximate generalised centrohermitian solution of Problem II.

Algorithm 4.1.
Input: $X \in \mathbb{C}^{n\times m}$, $Y \in \mathbb{C}^{n\times l}$, $\Lambda \in \mathbb{C}^{m\times m}$, $\Delta \in \mathbb{C}^{l\times l}$, $C_0 \in \mathbb{C}^{f\times f}$, $\mathscr{P} \in \mathbb{C}^{k\times k}$ and $A^{\#} \in \mathbb{C}^{n\times n}$.
Output: $A^* \in \mathbb{C}^{n\times n}$.
Begin

1. Judge $n=2k$ or $n=2k+1$.
2. Decide $K=\begin{pmatrix} \mathbf{0} & \mathscr{P} \\ \mathscr{P} & \mathbf{0} \end{pmatrix}$ or $K=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathscr{P} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathscr{P} & \mathbf{0} & \mathbf{0} \end{pmatrix}$.
3. Determine D_n by (2.2) or (2.4).
4. Calculate X_1, X_2, Y_1 and Y_2 by (2.5).
5. Compute the SVDs of X_1, X_2, Y_1 and Y_2 by (2.6).
6. Determine T_1, T_2 and C by (2.19) and (2.20), respectively.
7. Construct the GSVDs of the matrix pairs $(P_2^{\mathrm{H}}T_1^{\mathrm{H}}, L_2^{\mathrm{H}}T_2^{\mathrm{H}})$ and $(U_2^{\mathrm{H}}T_1^{\mathrm{H}}, R_2^{\mathrm{H}}T_2^{\mathrm{H}})$ by (2.16) and (2.17), respectively.
8. Partition the matrix $\widehat{M}^{-\mathrm{H}}CN^{-1}=(C_{ij})_{4\times 4}$ by (2.21).
9. If all the conditions in (2.22) and (2.23) hold, then continue. Otherwise, we stop.
10. Compute A_{11}^0, A_{22}^0 by (3.1) and H_1, H_2 by (3.2).
11. Calculate $A_1=\frac{1}{2}(A^{\#}+KA^{\#}K)$.
12. Partition the matrices $U_P^{\mathrm{H}}P_2^{\mathrm{H}}(H_1^{\mathrm{H}}A_1H_1-A_{11}^0)U_2U_U$ and $U_L^{\mathrm{H}}L_2^{\mathrm{H}}(H_2^{\mathrm{H}}A_1H_2-A_{22}^0)R_2U_R$ as given in (3.3) and (3.4), respectively.
13. Compute $\widehat{Y}_{22}$ by (3.10).
14. Calculate E_1^* and E_2^* as described in (3.6) and (3.7), respectively.
15. Compute the solution A^* of Problem II by (3.5).

End

We now present a numerical example to verify Algorithm 4.1, using MATLAB R2013a with a machine precision 10^{-15}.

Example 4.1. Let X, Y, Λ, Δ, C_0, $\mathscr{P}$, $A^{\#}$ be given as follows:

$$X=\begin{pmatrix} 0.2500+0.2500\mathrm{i} & 0.0426-0.0000\mathrm{i} \\ 0.2500+0.2500\mathrm{i} & 0.1423+0.0000\mathrm{i} \\ 0.2500+0.2500\mathrm{i} & 0.3239+0.0000\mathrm{i} \\ 0.2500+0.2500\mathrm{i} & 0.6108+0.0000\mathrm{i} \\ 0.3536-0.0000\mathrm{i} & -0.0301+0.4319\mathrm{i} \\ 0.3536+0.0000\mathrm{i} & -0.1006+0.2290\mathrm{i} \\ -0.3536+0.0000\mathrm{i} & 0.2290-0.1006\mathrm{i} \\ -0.3536-0.0000\mathrm{i} & 0.4319-0.0301\mathrm{i} \end{pmatrix},$$

$$Y = \begin{pmatrix} 0.3536+0.0000\mathrm{i} & 0.0426+0.0000\mathrm{i} & 0.1041+0.0000\mathrm{i} \\ 0.3536+0.0000\mathrm{i} & 0.1423-0.0000\mathrm{i} & -0.2455+0.0000\mathrm{i} \\ 0.3536-0.0000\mathrm{i} & 0.3239+0.0000\mathrm{i} & -0.3869-0.0000\mathrm{i} \\ 0.3536+0.0000\mathrm{i} & 0.6108+0.0000\mathrm{i} & 0.5284+0.0000\mathrm{i} \\ 0.2500-0.2500\mathrm{i} & -0.0301+0.4319\mathrm{i} & 0.0736-0.3736\mathrm{i} \\ 0.2500-0.2500\mathrm{i} & -0.1006+0.2290\mathrm{i} & -0.1736+0.2736\mathrm{i} \\ -0.2500+0.2500\mathrm{i} & 0.2290-0.1006\mathrm{i} & 0.2736-0.1736\mathrm{i} \\ -0.2500+0.2500\mathrm{i} & 0.4319-0.0301\mathrm{i} & -0.3736+0.0736\mathrm{i} \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 34.0000 & 0 \\ 0 & 26.3047 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 34.0000 & 0 & 0 \\ 0 & 26.3047 & 0 \\ 0 & 0 & -8.9443 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 9.3794+0.1756\mathrm{i} & 0.4480-0.2361\mathrm{i} & 1.9763+0.0378\mathrm{i} & 7.1963+0.0227\mathrm{i} \\ 3.1800+0.0359\mathrm{i} & 6.2847-0.0483\mathrm{i} & 6.4951+0.0077\mathrm{i} & 6.0402+0.0046\mathrm{i} \\ 4.4402-0.1118\mathrm{i} & 5.6697+0.1503\mathrm{i} & 6.0151-0.0241\mathrm{i} & 10.8750-0.0144\mathrm{i} \\ 2.0004-0.0998\mathrm{i} & 9.5976+0.1341\mathrm{i} & 12.5135-0.0215\mathrm{i} & 10.3885-0.0129\mathrm{i} \end{pmatrix},$$

$$\mathscr{P} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 0 & 0 & -\mathrm{i} \\ 0 & 1 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & -1 & 0 \\ \mathrm{i} & 0 & 0 & -1 \end{pmatrix}$$

and

$$A^{\#} = 10 \cdot hilb(8) + \frac{\mathrm{i}}{10} \cdot magic(8)\,,$$

where *hilb*(8) and *magic*(8) are the Hilbert matrix and magic matrix of order 8, respectively.

Using Algorithm 4.1, we obtain the matrix

$$C = \begin{pmatrix} -1.5393+0.1763\mathrm{i} & 1.8913+0.0862\mathrm{i} & -2.6859-0.2384\mathrm{i} & 3.0807-0.0240\mathrm{i} \\ 3.9297-0.3114\mathrm{i} & -2.8519+0.0072\mathrm{i} & 1.6479+0.2452\mathrm{i} & -1.9146+0.0589\mathrm{i} \\ -1.9738+0.1816\mathrm{i} & 2.3760-0.0460\mathrm{i} & 0.3990-0.0969\mathrm{i} & -0.3396-0.0387\mathrm{i} \\ 0.3302-0.0465\mathrm{i} & -0.6044-0.0474\mathrm{i} & 1.1005+0.0900\mathrm{i} & -1.3121+0.0038\mathrm{i} \end{pmatrix}.$$

Then we compute the GSVDs of the matrix pairs $(P_2^{\mathrm{H}}T_1^{\mathrm{H}}, L_2^{\mathrm{H}}T_2^{\mathrm{H}})$ and $(U_2^{\mathrm{H}}T_1^{\mathrm{H}}, R_2^{\mathrm{H}}T_2^{\mathrm{H}})$ by (2.16) and (2.17), respectively. It follows that

$$\Lambda_P = \Gamma_L = \Lambda_U = \Gamma_R = \left(\frac{1}{\sqrt{2}}\right).$$

Now, we can verify that the conditions in (2.22) and (2.23) hold. Continue to use

Algorithm 4.1, we obtain the matrix

$$A^* = \left(\begin{array}{cccc} 9.3794+0.1756i & 0.4480-0.2361i & 1.9763+0.0378i & 7.1963+0.0227i \\ 3.1800+0.0359i & 6.2847-0.0483i & 6.4951+0.0077i & 6.0402+0.0046i \\ 4.4402-0.1118i & 5.6697+0.1503i & 6.0151-0.0241i & 10.8750-0.0144i \\ 2.0004-0.0998i & 9.5976+0.1341i & 12.5135-0.0215i & 10.3885-0.0129i \\ 8.0302-0.3288i & -3.3354-4.9113i & 1.3308-0.9703i & 4.5810+6.5641i \\ -2.9047-4.3882i & 8.9511+1.7445i & 1.6072-2.0692i & 0.8317-0.2367i \\ -5.4060-2.1618i & 1.5942+8.8708i & 0.1918+0.2670i & -1.3299+1.5093i \\ -0.0752+8.0514i & -4.4639-3.5523i & -2.2036+1.6431i & 7.0962+4.4645i \end{array}\right.$$

$$\left.\begin{array}{cccc} 8.3068+4.3959i & -4.0605+1.2633i & -1.7107+2.8272i & -4.6495-7.7747i \\ -3.0798+1.6794i & 10.0414+1.0122i & -0.8620-7.7804i & -0.6616+1.9867i \\ 5.2359+0.4118i & -2.3394+0.9789i & -0.8986+0.9993i & -1.1548-4.5583i \\ 0.1437-6.8407i & 4.8437+1.6954i & -1.4785-4.5315i & 6.8195-0.2602i \\ 9.8227+2.6792i & 6.5289-3.9395i & -5.8943+5.9468i & -4.5041+0.5430i \\ 6.9693+0.8107i & 6.2212+0.3765i & -6.0945-0.2138i & -5.2150+3.9011i \\ -5.2654-3.7940i & -6.0703+0.0558i & 6.0786-0.4489i & 7.0857-0.7893i \\ -4.6925-0.4661i & -5.6796-6.1187i & 6.4326+3.6818i & 9.9452-2.5166i \end{array}\right).$$

and $\|A^{\#}-A^*\| = 59.0511$. Then we have

$$\|A^* - KA^*K\| = 1.6245e-14$$

and

$$\|A^*X - X\Lambda\| = 3.4822e-14, \qquad \|Y^{\mathrm{H}}A^* - \Delta Y^{\mathrm{H}}\| = 2.4008e-14,$$

indicating that $A^* \in \mathscr{GC}^{8\times 8}(\mathscr{P})$ is the solution of the system of the matrix equations $AX = X\Lambda$, $Y^{\mathrm{H}}A = \Delta Y^{\mathrm{H}}$. Consequently, A^* is the unique optimal approximate solution of Problem II.

Conclusions

In this paper, we discuss an undamped non-gyroscopic model which can be discretized as the left and right inverse eigenvalue problem of a certain structured matrix. When the structured matrix A in Eq. (1.1) is generalised centrohermitian and has a constrained submatrix C_0, we design Algorithm 4.1 to correct the model by using algebraic method. Meanwhile, Example 4.1 verifies that Algorithm 4.1 can be effective and feasible to obtain the unique optimal approximate solution of Problem II. When $Y = \mathbf{0}$ and $\mathscr{P}$ is the cross-identity matrix of order k in Problem I, our results extend previous results in the real number field of Bai [1]. In addition, by using the SVD and GSVD we can similarly solve Problems I and II for the case of generalised skew-centrohermitian matrices.

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