East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.210815.281215a

Vol. **6**, No. 1, pp. 80-88
February 2016

Traffic Lights or Roundabout? Analysis using the Modified Kinematic LWR Model

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Received 21 August 2015; Accepted (in revised version) 28 December 2015

Abstract. Traffic flow is treated as a continuum governed by the kinematic LWR model and the Greenshield flux function. The model is modified to describe traffic flow on a road with traffic lights or a roundabout. Parameters introduced determine the traffic flow behaviour and queue formation, and numerical simulations based on the Godunov method are carried out. The numerical procedure is shown to converge, and produces results consistent with previous analytic results.

AMS subject classifications: 65M08, 76M12, 35L65

Key words: Kinematic LWR model, Riemann solutions, Godunov method.

1. Introduction

Many roads in urban areas suffer from heavy traffic, and even on modern highways there can be “stop and go" traffic. In particular, traffic congestion often occurs in the approach to an intersection with traffic lights or a roundabout. Sometimes the congestion might be reduced by applying some new traffic regulation — e.g. by making a section of two-way road one-way, or by replacing traffic lights with a roundabout as discussed here. Simulations can help to assess whether the consequent modified traffic behaviour results in better traffic management. Traffic simulation packages have been developed since the 1960s [14], but research on traffic modelling and simulation is ongoing. Specific issues for traffic flow engineering include the study of highway junctions where there is merging [9, 10, 13], and the effects of entrances and exits [1]. The efficiency of a roundabout versus traffic lights at an intersection that we address is another.

Traffic may often be treated as a continuum in the macroscopic modelling of traffic flow. The kinematic Lighthill-Whitham-Richards (LWR) model [7,8] is often adopted, where the number of vehicles is assumed to be conserved and the flow is described in terms of traffic

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density and vehicular speed (flux). Driver behaviour is modelled by some specified prescribed flux to density relationship, and various such flux functions have been used [6].

The Godunov finite difference method [11, 12] with upwinding is an especially appropriate numerical approach when discontinuities are to be expected, and it is used here. Another procedure is the supply-demand method [3–5], which is also based on conservation of the number of vehicles. The analysis of traffic flow simulation using cellular automata for one-lane or multi-lane roundabouts has also been explored — e.g. see Refs. [2, 16, 17].

The rest of this article is organised as follows. In Section 2, the kinematic LWR model and its discrete formulation are discussed. In Section 3, we propose a mathematical model for traffic flow at an intersection with traffic lights or a roundabout, and numerical calculations are conducted to compare their relative performance. The length of the queue formed behind the intersection is discussed in Section 4, and our conclusions and discussion are in Section 5.

2. The Kinematic LWR model and its Discrete Formulation

We consider a road with heavy traffic moving in one direction. Let n vehicles per kilometre denote the traffic density (the number of vehicles in a unit length of road), and f vehicles per hour denote the flux (the number of vehicles passing in a unit length of time). The flux and density are related by

$$f = n \times v \,,$$

where v is the mean velocity of the vehicles. In reality, the velocities depend on aspects such as individual driver characteristics, traffic density and road condition. However, here we adopt the commonly used Greenshield model, where the average velocity depends linearly on traffic density as follows:

$$v(n) = V_m \left(1 - \frac{n}{N_m}\right) , \tag{2.1}$$

where N_m (vehicles/km) and V_m (km/hour) are the maximum traffic density and velocity, respectively. This Greenshield velocity and flux are depicted in Fig. 1. In applications, the parameter V_m depends on road conditions — e.g. V_m is much larger on highways than along city or suburban roads. Although Eq. (2.1) is linear, it captures two important aspects of traffic flow — viz. when the road is nearly empty $n \to 0$ so the mean velocity tends to its maximum ($v(n) \to V_m$), whereas if the road is nearly full $n \to N_m$ so the mean velocity tends to zero ($v(n) \to 0$) when vehicles can hardly move. The underlying conservation principle that governs traffic dynamics is

$$n_t + f_x = 0 \,, \quad \text{with} \quad f(n) = nV_m \left(1 - \frac{n}{N_m}\right). \tag{2.2}$$

In this model, the maximum flux is given by the ordinate of the vertex parabola $f(n)$, which is $f_m = N_m V_m / 4$. Eq. (2.2) can be rewritten $n_t + f'(n) n_x = 0$, where $f'(n)$ is the signal speed. As indicated in Fig. 1 (right), the signal speed is positive when $n(x,t) < N_m/2$,

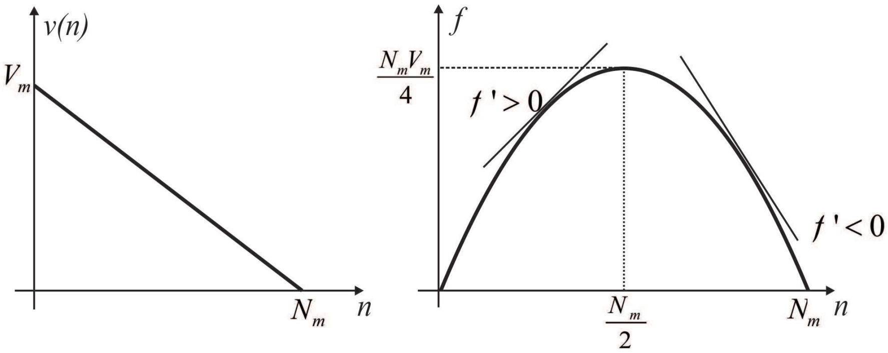

Figure 1: Curves in the Greenshield model for both the velocity $v(n)$ (left) and the flux $f(n) = nv(n)$ (right).

which means information is moving to the right, whereas information is moving to the left when $n(x,t) > N_m/2$ (i.e. when the traffic density exceeds half of the maximum density).

Let us now discuss the Godunov method for Eq. (2.2). Consider the computational domain $[0, L]$ divided into N_x cells of homogeneous length Δx in a staggered way, with partition points $x_{1/2} = 0$, $x_{3/2} = \Delta x, \cdots, x_{i+1/2} = i\Delta x, \cdots, x_{N_x+1/2} = N_x \Delta x = L$. Within the time interval Δt, the number of vehicles in the cell $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ changes according to the net flux from its left and right boundaries as follows:

$$n(x_i, t_{n+1})\Delta x = n(x_i, t_n)\Delta x - \Delta t \Big(f(x_{i-\frac{1}{2}}, t_n) - f(x_{i+\frac{1}{2}}, t_n) \Big). \tag{2.3}$$

This is the conservation principle for vehicles in cell C_i in discrete form, and the formulation (2.3) is the Godunov method. At every time step, the traffic density at cell C_i is updated using the net flux from both boundaries $x_{i\pm\frac{1}{2}}$. The upwind approximation of the flux at staggered points is discussed below.

Since the signal speed associated with the continuum model (2.2) is given by $f'(n)$, the flux $f_{i+\frac{1}{2}}$ needs to be evaluated at the left end (f_i) if information is propagating forward (i.e. if $f'_{i+\frac{1}{2}} > 0$), and at the right end (f_{i+1}) if information is travelling upstream (i.e. if $f'_{i+\frac{1}{2}} < 0$). In summary, with $f'_{i+\frac{1}{2}} \approx \frac{1}{2}(f'_i + f'_{i+1})$ the upwind approximation is

$$f_{i+\frac{1}{2}} \approx \begin{cases} f_i, & \text{if } \frac{1}{2}(f'_i + f'_{i+1}) > 0, \\ f_{i+1}, & \text{if } \frac{1}{2}(f'_i + f'_{i+1}) < 0. \end{cases} \tag{2.4}$$

In the next section, the scheme defined by Eqs. (2.3) and (2.4) is used to analyse the traffic flow when there is a set of traffic lights or a roundabout.

3. Models for Traffic Lights and Roundabout

Before arriving at an intersection, the drivers often encounter a queue of vehicles. There can be a lengthy wait in a long queue at a traffic light until it turns green, when at least some

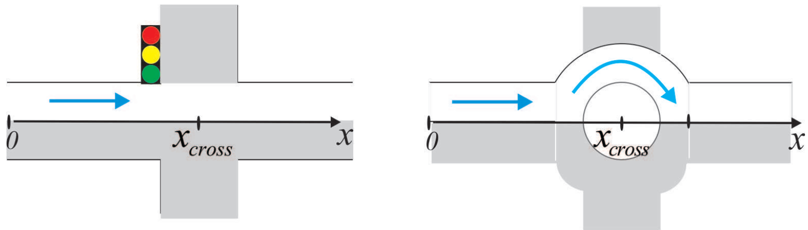

Figure 2: Schematic of an intersection with a traffic lights (left) or a roundabout (right).

of the leading vehicles can pass the intersection. We now analyse the relative performance of traffic lights and a roundabout — i.e. if traffic lights are replaced by a roundabout, will more of the traffic proceed through the intersection in a given time? To model traffic dynamics in a road segment with a set of traffic lights, we consider the kinematic LWR (2.2) in a spatial domain with an intersection located at x_{cross} — cf. Fig. 2. From an analogy in hydrodynamics, the intersection can be regarded as an obstacle that will affect traffic dynamics upstream and downstream. Our interest is then the upstream part of the intersection, and in particular the length of the queue that has formed behind the traffic lights or roundabout.

We adopt a simplified traffic light model by assuming that a light turns between red and green — i.e. we neglect an intermediate yellow phase. The presence of the traffic light changes the flux at x_{cross} as no vehicle may legally pass through the intersection during a red light (i.e. the flux is then zero), so that

$$v(n(x_{cross},t)) = \begin{cases} 0\,, & \text{for time } t \text{ during a red light}\,, \\ v(n)\,, & \text{for time } t \text{ during a green light}\,, \end{cases} \tag{3.1}$$

where $v(n)$ is given in Eq. (2.1). If T_r, T_g denote the respective duration of a red and green light, a traffic light parameter measuring the relative green light time is

$$\alpha \equiv \frac{T_g}{T_{cycle}}\,, \text{ where } T_{cycle} = T_r + T_g\,. \tag{3.2}$$

For most traffic lights at the intersection of two roadways, a typical range for this parameter is $1/4 < \alpha < 1/3$. The value $\alpha = 1/4$ means that each of the four legs of the two roads has an equal green light period, whereas when $\alpha = 1/3$ two legs of the roads share the same green light period. Incidentally, the parameter α in (3.2) is defined on the assumption that any time lost during the initiation and termination of a green phase may be neglected.

Vehicles approaching roundabouts usually slow down, as drivers reduce speed due to their circular roadways and merging traffic from other legs. Observations in the USA [15] suggest this reduction is proportional to the square root of the roundabout radius, but here we simply assume that the reduced velocity is $\beta v(n)$ where $\beta < 1$. Thus the averaged velocity in a road segment with a roundabout is modelled by

$$v(n(x,t)) = \begin{cases} \beta v(n)\,, & \text{when } x = x_{cross}\,, \\ v(n)\,, & \text{otherwise}\,, \end{cases} \tag{3.3}$$

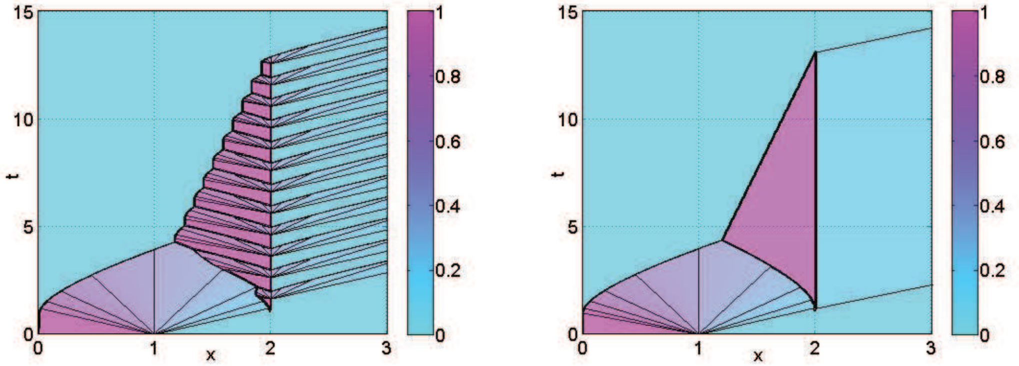

Figure 3: Left: Traffic through the traffic light where $\alpha = 1/3$ is periodically stopped, and the queue is only partly cleared in each green light phase during a time of $t \approx 14$ units. Right: Traffic through the roundabout where $\beta = 1/3$ proceeds with a reduced flow, and the queue is cleared after the time of $t \approx 14$ units.

with $v(n)$ given in Eq. (2.1). This reduced velocity implies a reduction in the traffic flux, although if the roundabout has a large diameter drivers barely need to reduce speed such that β is close to one.

We contemplated two particular situations in examining the relative performance of a traffic light and roundabout. The kinematic LWR model (2.2) with parameters N_m and V_m can be normalised to one, and we considered a computational domain $[0,3]$ with an intersection located at $x_{cross} = 2$. As the initial condition, we assumed that a unit length of road behind the intersection is fully occupied — i.e.

$$n(x,0) = \begin{cases} N_m, & \text{when } 0 < x < 1, \\ 0, & \text{for } 1 < x. \end{cases}$$

Given that no vehicle may enter from the left, we have the boundary condition $n(0,t) = 0$, but our right boundary condition allows open entry. For $\alpha = 1/3$, we take the normalised time $T_{cycle} = 1$ with red and green light periods $\frac{2}{3}T_{cycle}$ and $\frac{1}{3}T_{cycle}$, respectively. We used $\Delta t = \Delta x = 0.01$ for this computation, and Fig. 3 shows the resulting traffic densities (in contour plots) passing through the traffic light and roundabout.

At earlier times, Figs. 3 (left) and (right) both show expected rarefaction waves emerging from $x = 1$, the location of the discontinuous initial density. As time passes, the vehicles move forward to the intersection $x_{cross} = 2$. There is clearly "stop and go" traffic for the traffic light at x_{cross} in Fig. 3 (left), whereas for the roundabout the traffic is constrained but moves continuously in Fig. 3 (right). The time needed for all of the initial traffic to pass the intersection x_{cross} (the clearance time) is an important criterion that can be formulated as follows. Without either obstacle, the maximum flux is $f_m = V_m N_m/4$ and the vehicles can pass x_{cross} within $1/f_m = 4/V_m N_m = 4$ time units. The traffic light and roundabout reduce the flux by the respective factors α and β, so that the initial queue passes the intersection x_{cross} in $1/(\alpha f_m) = 1/(\beta f_m)$ time units.

We also undertook computation for other green light proportions (different α) and

Table 1: Calculated values of t_{clear} for a traffic light with parameter α and a roundabout with parameter β when $f_m = V_m N_m/4 = 1/4$, and the errors as percentages.

Traffic light			Roundabout		
α	$\left\lvert t_{clear} - \frac{1}{\alpha f_m} \right\rvert$	error (%)	β	$\left\lvert t_{clear} - \frac{1}{\beta f_m} \right\rvert$	error (%)
1/2	0.25	3.12	1/2	0.37	4.63
1/3	0.57	4.75	1/3	0.20	1.67
1/4	0.73	4.56	1/4	0.24	1.50

different radii roundabouts (different β), to evaluate their corresponding clearance times t_{clear}. Since there is initially no traffic on $x > 1$, the time t_{clear} can be regarded as the interval when the traffic density at $x = 3$ is nonzero — cf. Fig. 3. There $t_{clear} = 14.3-2.87 = 11.43$ for the traffic light, and $t_{clear} = 14.18-2.38 = 11.8$ for the roundabout. The clearance times t_{clear} for the other α and β are summarised in Table 1, where we see that the clearance times for the traffic light and roundabout are comparable and agree with analytical predictions. We conclude that operating an intersection with either a traffic light or a roundabout is comparable if the parameters α and β are the same, so In our discussion below an intersection is simply treated as an obstacle with reduced flux βf_m.

4. Queue Length and the Shock Line

We focus on the traffic dynamics behind an intersection, to determine the length of the queue that forms behind it for comparison with the queue length determined from the corresponding analytical shock line.

Let us assume the initial traffic density is n_A corresponding to a flux $f(n_A)$, as indicated by the point A in Fig. 4. An obstacle at x_{cross} with the reduced flux βf_m is associated with two traffic densities — i.e. low and high densities. High density is more appropriate, since the obstacle obstructs the flow. On the flux curve, the obstructed condition is depicted as point B in Fig. 4, and the quantitative effect on the initial traffic with density n_A is described by the line segment AB. The gradient of the segment AB is precisely the shock speed from the Rankine-Hugoniot formula

$$m_{AB} = \frac{[f(n)]_A^B}{[n]_A^B}.$$

We can directly obtain the queue length behind the obstacle from the shock line, on noting the analytical solution

$$n(x,t) = \begin{cases} n_A, & \text{for } (x - x_{cross})/t < m_{AB}, \\ n_B, & \text{for } (x - x_{cross})/t > m_{AB}. \end{cases} \tag{4.1}$$

The reduced flux for $\beta = 0.5$ is $N_m V_m/8$, which corresponds to the high density flux

$$n_B = \left(\frac{1}{2} + \frac{1}{4}\sqrt{2}\right) N_m \approx 0.8536 N_m$$

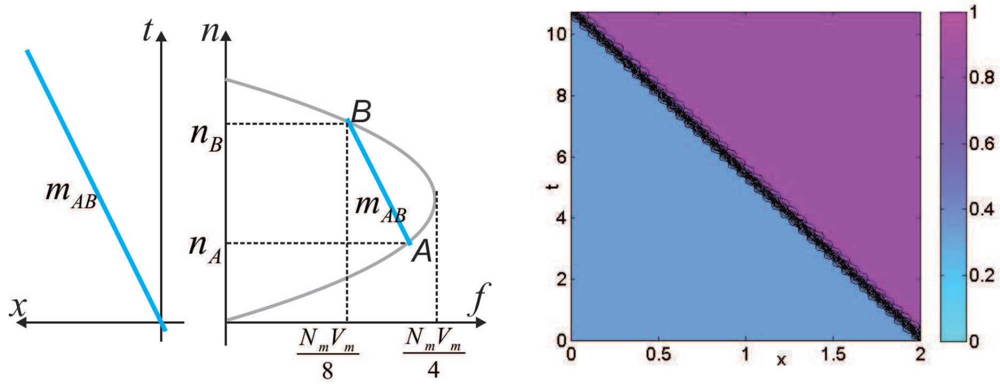

Figure 4: (Left) The flux diagram to determine the shock line. (Right) Contour plot of the computed traffic density on the upstream side of an intersection.

Table 2: The L^1 error and the convergence rate τ of the numerical scheme.

N_x	L^1 error	τ
40	0.026058	/
80	0.013052	0.997422
120	0.008717	0.995600
160	0.006549	0.993810

such that

$$m_{AB} = V_m(1-(n_A+n_B)/N_m) = -0.1869V_m\,,$$

on assuming an initial traffic density $n_A = N_m/3$. The shock line plotted on the xt—plane in Fig. 4 (left) separates two different traffic densities (a high density n_B and a low density n_A), and directly determines the queue length as a function of time.

We solved the model (2.2) on a computational domain $[0, x_{cross}]$ with a roundabout located at $x_{cross} = 2$, and initial traffic density $n(x,0) = \frac{1}{3}N_m$ with normalised parameters $V_m = 1$, $N_m = 1$. The spatial domain was divided into $N_x = 40$ cells with $\Delta x = 0.05$ and we took $\Delta t = 0.05$, yielding the result shown in Fig. 4. The contour plot clearly shows two traffic densities $n_A = \frac{1}{3}N_m$ and $n_B = 0.8536N_m$, separated by a line with gradient that conforms to the analytical shock line $m_{AB} = -0.1869V_m$.

Finally, we compared our numerical results with the analytical solution (4.1). The error between the numerical and analytical solutions was measured at time $T = 10.5$ using the L^1—norm. Computations for several spatial grid cells N_x were performed, keeping the Courant number $V_m\Delta t/\Delta x$ equal to one, and the resulting errors are presented in Table 2. The numerical results show good convergence as the number of cells N_x increases, and the convergence rate $\tau = \log(\text{error}_2/\text{error}_1)/\log(N_{x1}/N_{x2})$, confirms order one accuracy for the method — cf. the right column in Table 2.

5. Conclusions and discussion

The kinematic LWR model has been modified to account for traffic lights or roundabouts. The Godunov scheme successfully simulates “stop and go" traffic due to a traffic light and reduced traffic due to a roundabout. With the same parameters, their performance is comparable, as several numerical computations demonstrated. Moreover, the growth of the traffic queue in our simulation conformed with the gradient of the analytical shock line.

At an intersection connecting four legs, a typical parameter value for the traffic light is less than 1/3 and very likely around 1/4, while vehicle speed due to a roundabout is often roughly halved. Our work tends to explain why a traffic light can cause more traffic congestion than a roundabout, which seems the better option given the queues that form behind an intersection. However, decision-makers should also consider other aspects such as traffic volumes and the site of the intersection.

Acknowledgments

The first author thanks Neville Fowkes for useful discussions during the early stage of this research. This project is supported by the Desentralisasi ITB and Riset KK-ITB 2016 research funds.

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