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New Perturbation Bounds Analysis of a Kind of Generalized Saddle Point Systems

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Abstract. In this paper we consider new perturbation bounds analysis of a kind of generalized saddle point systems. We provide perturbation upper bounds for the solutions of generalized saddle point systems, which extend the corresponding results in [W.-W. Xu, W. Li, *New perturbation analysis for generalized saddle point systems*, Calcolo., 46(2009), pp. 25-36] to more general cases.

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1. Introduction

The saddle point system appears in scientific and engineering applications, such as, aeronautics, the mixed finite element solution of the Navier-Stokes, the Maxwell equations, electromagnetics and data fitting et. al. Numerical methods and perturbation bounds analysis for solving the saddle point system studied in some literatures. For details, please see [2-15] and the references therein. Recently, Xu et. al. in [1] considered perturbation bounds of the following generalized saddle point systems:

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $A\in\mathscr{R}^{m\times m}$, $B\in\mathscr{R}^{n\times m}$, and $C\in\mathscr{R}^{n\times n}$, $n\le m$ (possibly $n\ll m$). This kind of system arises in many application problems, e.g., see [1]. As we know, a number of literatures deal with the solvers of the saddle point problem (1.1) with $C\neq 0$. Due to practical applications, perturbation analysis of the saddle point problem (1.1) should be discussed and the perturbation bounds and condition numbers for the system (1.1) are derived.

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In this paper we will extend System (1.1) to the more generalized saddle point system and consider perturbation upper bound for the solutions of this system:

$$\begin{pmatrix} A & D \\ B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.2}$$

where $A \in \mathscr{R}^{m\times m}$, $B \in \mathscr{R}^{n\times m}$, $D \in \mathscr{R}^{m\times n}$ and $C \in \mathscr{R}^{n\times n}$, $x \in \mathscr{R}^{m}$, $y \in \mathscr{R}^{n}$, $n \le m$ (possibly $n \ll m$). Let $\mathscr{A}$ be the coefficient matrix of (1.2) and assume that $\mathscr{A}$ is nonsingular. The non-singularity conditions of $\mathscr{A}$ can be referred in Lemma 2.1 of [15]. Obviously, when $D = B^T$ in (1.2), System (1.2) reduces to System (1.1). We note that the perturbation bounds analysis for the solutions x and y of the system (1.2) have not discussed so far. By this motivation, we will consider this problem in the paper.

Let the perturbed system of (1.2) be as follows:

$$(\mathscr{A}+\Delta\mathscr{A})\begin{pmatrix} x+\Delta x \\ y+\Delta y \end{pmatrix}=\begin{pmatrix} A+\Delta A & D+\Delta D \\ B+\Delta B & C+\Delta C \end{pmatrix}\begin{pmatrix} x+\Delta x \\ y+\Delta y \end{pmatrix}=\begin{pmatrix} f+\Delta f \\ g+\Delta g \end{pmatrix}.$$

Throughout the paper, we always assume that

$$\begin{aligned} &\|\Delta A\|_F \le \epsilon\mathscr{D}_1, \qquad \|\Delta B\|_F \le \epsilon\mathscr{D}_2, \qquad \|\Delta C\|_F \le \epsilon\mathscr{D}_3, \\ &\|\Delta D\|_F \le \epsilon\sigma_1, \qquad \|\Delta f\|_2 \le \epsilon\mathscr{D}_4, \qquad \|\Delta g\|_2 \le \epsilon\mathscr{D}_5, \end{aligned} \tag{1.3}$$

and let

$$\delta = (\delta_1,\ \delta_2,\ \delta_3)^T, \qquad \hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2)^T, \tag{1.4}$$

where

$$\begin{aligned} \epsilon > 0, \quad &\delta_1 = \sqrt{\mathscr{D}_1^2+\mathscr{D}_2^2}, \qquad \delta_2 = \sqrt{\sigma_1^2+\mathscr{D}_3^2}, \qquad \delta_3 = \sqrt{\mathscr{D}_4^2+\mathscr{D}_5^2}, \\ &\hat{\delta}_1 = \sqrt{\mathscr{D}_1^2+\mathscr{D}_2^2+\sigma_1^2+\mathscr{D}_3^2}, \qquad \hat{\delta}_2 = \sqrt{\mathscr{D}_4^2+\mathscr{D}_5^2}. \end{aligned}$$

Here $\|\cdot\|_F$ denotes the Frobinus-norm.

The rest of the paper is organized as follows. In Section 2 we give some definitions, notations and useful lemmas to deduce the main results. In Section 3 we give perturbation bounds for the solutions of a kind of generalized saddle point systems. In Section 4 we give numerical examples to illustrate our results.

2. Preliminaries

We briefly give some useful lemmas in order to deduce our main results.

Lemma 2.1. *If $\mathscr{A}$ is nonsingular, then*

i) $$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \mathscr{H}\theta + \mathscr{A}^{-1}(P,Q)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix},$$

ii) $$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \bar{\mathscr{H}}\bar{\theta} + \mathscr{A}^{-1}\Delta\mathscr{A}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix},$$