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A Multi-Exposure Variational Method for Retinex

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Abstract. Retinex theory explains how the human visual system perceives colors. The goal of retinex is to decompose the reflectance and the illumination from the given images and thereby compensating for non-uniform lighting. The existing methods for retinex usually use a single image with a fixed exposure to restore the reflectance of the image. In this paper, we propose a variational model for retinex problem by utilizing multi-exposure images of a given scene. The existence and uniqueness of the solutions of the proposed model have been elaborated. An alternating minimization method is constructed to solve the proposed model and its convergence is also demonstrated. The experimental results show that the proposed method is effective for reflectance recovery in retinex problem.

AMS subject classifications: 65K10, 68U10, 91-08

Key words: Retinex, reflectance, illumination, multi-exposure images, alternating minimization algorithm.

1. Introduction

Retinex theory was first proposed by Edwin H. Land in [11] which explains how the human visual system perceives colors. Upon this theory, color sensations correlate with the intrinsic reflectance of objects and are independent of the radiance values captured by eyes. Therefore, human visual system (HVS) can identify the same colors of a given scene under varying illumination conditions, which is commonly regarded as the color constancy, see, for instance, [11–13]. Based on retinex theory, eyes can see colors correctly when light is low, while cameras and video cameras can not manage this well. Images taken under different illumination levels may shift the real color of the object. In retinex theory, it is assumed that the observed image intensity S can be decomposed as pixel-wise product of two components, they are reflectance function R and illumination function L as

$$S = RL. \tag{1.1}$$

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In order to compensate for the non-uniform lighting in a given image and enhance the contrast of images, the primary task of retinex is to find efficient methods to separate the reflectance R from the observed intensity S. The retinex theory is widely applied in image editing [19], shadow removal [5], multi-spectral image fusion [22], image and video fusion for context enhancement [20] and high dynamic range compression [4]. There have been many methods for retinex proposed in the literature. For example, the well-known path-based algorithms which were put forward by Land in [12, 13] and the one put forward by Brainard et al. in [2]. Path-based algorithms require to tune many parameters and the implementation is very complicate. A recursive matrix calculation was designed to replace the path computation and recursive algorithms were proposed in [6, 7]. The single scale retinex model was proposed in [8], though the optimized single scale retinex result is short of human observation, it succeeds in producing the correct beige scene color and some dynamic range compression of the shadow. The multi-scale retinex was proposed in [9] which more closely approaches the performance of human vision. In 2009, Bertalmo, Caselles, and Provenzi [1] proposed a kernel-based retinex method in which the main computation is to get the expectation value of a suitable random variable weighted by a kernel function. The partial differential equation based algorithms are important methods for retinex. For example, the methods proposed by Morel et al. in [16, 17] utilize fast Fourier transformation to perform the computation cheaply to get the decomposition of reflectance and illumination in recorded images. Morel et al. also further demonstrated that the random walk method and the partial differential equation formulation are equivalent. Efficient variational methods for retinex have surged in recent years. Ma et al. [14] proposed a model in which the L_1 regularization is used to recover sharp edges and boundaries of the reflectance component and a fast approach based on Bregman iteration is designed to solve the model. Kimmel et al. [10] presented a variational method based on H_1-norm regularization for the reflectance function. Ma and Osher established a total variation and nonlocal TV regularized model in [15]. In [18], Ng and Wang proposed a model for retinex which consists of a data-fidelity term, a total variation term for reflectance function, an H_1-norm regularization term for illumination function. An alternating minimization method is designed to solve this problem. In this method, due to the blurring recovering effect of the recovered R from the model, $R' = S/L$ instead of R is used in image enhancement and illumination compensation. In 2014, Zosso et al. in [26] and Wang, Ng in [24] constructed methods in which the nonlocal total variation regularization of the reflectance function is used in order to improve the reflectance recovering effect. Wang and He [25] proposed a variational model which has the same data-fidelity term as the one used in [18], but two barrier functions are added. The details and edges of the recovered reflectance R from this model is clearer and sharper than the one got from [18]. In [3], Chang et al. used sparse and redundant representations of the reflectance component in the retinex model over a learned dictionary and more details are revealed in the low-light part.

In the above mentioned methods, only single image with a fixed exposure is used to restore the reflectance of the image. Although reflectance is a constant property and related to the physical characteristics of the material object, in practice, certain parts of image details may be lost in the saturated or over-dark regions due to the different exposure times, see, for instance, [21]. In this paper, we develop a variational model for retinex problem by utilizing multi-exposure images. The proposed method is constructed based on the variational model plus barrier functions for retinex proposed in [25]. We expect that the reflectance can preserve image details by using the information of the image details got under different exposures.

The paper is organized as follows: In Section 2, we present the proposed model and give some theoretical analysis to show the existence and the uniqueness of the solutions of the model; in Section 3, we introduce an alternating iterative method to solve the proposed model and discuss the convergence of the algorithm; in Section 4, some numerical experiments are given to demonstrate the effectiveness of the proposed method; finally, we end the paper in Section 5 by giving some concluding remarks.

2. The Proposed Model

Assume that S_1, S_2, $\cdots$, S_K are intensities of recorded images of a same scene taken under different exposures. From (1.1), S_i $(i=1,\cdots,K)$ are the pixel-wise product of the illumination function L_i $(i=1,\cdots,K)$ and the reflectance function R, that is,

$$S_i = L_i R, \quad i=1,\cdots,K. \tag{2.1}$$

Where $0 < R \leq 1$ (reflectivity) and then $0 < S_i \leq L_i < \infty$ $(i=1,\cdots,K)$. Let $s_i = \log(S_i), r=\log(R), l_i=\log(L_i)$ $(i=1,\cdots,K)$. We perform logarithm operation on both sides of (2.1) and the product expression is converted to the following additive expression of new variables r and l_i.

$$s_i = r + l_i, \quad i=1,\cdots,K.$$

Same as the variational method proposed in [25], we assume that the reflectance function R and the illumination functions L_i are spatial smoothness for each $i=1,\cdots,K$, such that r, $l_i \in W^{1,2}(\Omega)$ and $s_i \in W^{1,2}(\Omega)$, $l_i + r$ is close to s_i for each $i=1,\cdots,K$. For each input image S_i, since the reflectance $0<R\leq 1$, it holds that $r \leq 0$ and $l_i \geq s_i$ $(i=1,\cdots,K)$.

Based on these assumptions, we propose the following minimization model for retinex problem by utilizing multiple exposure images.

$$\begin{aligned} &\min \tilde{E}_{\mu,\nu}(r,l_1,\cdots,l_K), \quad r,l_1,\cdots,l_K \in W_0^{1,2}(\Omega), \\ &\text{Subject to } \ r\leq 0 \ \text{ and } \ l_i \geq s_i \ (i=1,\cdots,K). \end{aligned} \tag{2.2}$$

Energy function $\tilde{E}_{\mu,\nu}(r,l_1,\cdots,l_K)$ is defined as

$$\begin{aligned} \tilde{E}_{\mu,\nu}(r,l_1,\cdots,l_K) =& \alpha \int_\Omega |\nabla r|^2 + \beta \sum_{i=1}^K \int_\Omega (l_i + r - s_i)^2 \\ &+ \sum_{i=1}^K \int_\Omega |\nabla l_i|^2 + \mu \sum_{i=1}^K \int_\Omega \frac{2}{l_i - s_i} - \nu \int_\Omega \frac{2}{r}. \end{aligned} \tag{2.3}$$

Where α and β are real positive parameters, μ and ν are small positive barrier parameters. The change of space from $W^{1,2}(\Omega)$ to $W_0^{1,2}(\Omega)$ which r and l_i $(i=1,\cdots,K)$ belong to is just for theoretical analysis. We note that $\int_\Omega \frac{2}{r}$ and $\int_\Omega \frac{2}{l_i-s_i}$ $(i=1,\cdots,K)$ are terms by borrowing the idea of barrier methods to meet the constraints $r\le 0$ and $l_i\ge s_i$ $(i=1,\cdots,K)$. But now we should use the constraints $r<0$ and $l_i>s_i$, $i=1,\cdots,K$ a.e. (almost everywhere) to avoid the last two terms in (2.3) being infinite. The new constraints are practical in view of the physical characteristics of the material object. By introducing variable transformations $\gamma=-r$ and $\omega_i=l_i-s_i$ $(i=1,\cdots,K)$, we get the equivalent minimization problem to (2.2) as follows:

$$\begin{aligned}&\min E_{\mu,\nu}(\gamma,\omega_1,\cdots,\omega_K),\quad \gamma,\omega_1,\cdots,\omega_K\in W_0^{1,2}(\Omega),\\&\text{Subject to }\ \gamma>0\ \text{ and }\ \omega_i>0\ (i=1,\cdots,K).\end{aligned}\tag{2.4}$$

Where

$$\begin{aligned}E_{\mu,\nu}(\gamma,\omega_1,\cdots,\omega_K)=&\alpha\int_\Omega|\nabla\gamma|^2+\sum_{i=1}^K\int_\Omega|\nabla(\omega_i+s_i)|^2\\&+\beta\sum_{i=1}^K\int_\Omega(\omega_i-\gamma)^2+\mu\sum_{i=1}^K\int_\Omega\frac{2}{\omega_i}+\nu\int_\Omega\frac{2}{\gamma}.\end{aligned}\tag{2.5}$$

In the rest of this section, we will give some theoretical analysis about the solutions of the minimization problem (2.4) and numerical algorithm for solving it will be demonstrated in the next section.

Theorem 2.1. *Let μ and ν be any fixed positive constants, then the function $E_{\mu,\nu}(\gamma,\omega_1,\cdots,\omega_K)$ in* (2.5) *is strictly convex in $\{(\gamma,\omega_1,\cdots,\omega_K)|\gamma,\omega_1,\cdots,\omega_K\in W_0^{1,2}(\Omega),\gamma,\omega_1,\cdots,\omega_K>0\ a.e.\}$.*

Proof. Since both functions $\int_\Omega|\nabla(\cdot)|^2$ and $\int_\Omega(\cdot)^2$ are convex, the function $\int_\Omega\frac{2}{(\cdot)}$ is strictly convex, the conclusion is obvious. ☐

Theorem 2.2. *Suppose for $i=1,\cdots,K$, $s_i\in W_0^{1,2}(\Omega)$, then the problem* (2.4) *has a unique solution.*

Proof. It is clear that the energy function $E_{\mu,\nu}(\gamma,\omega_1,\omega_2,\cdots,\omega_K)$ is nonnegative and proper. Suppose $\{(\gamma^n,\omega_1^n,\omega_2^n,\cdots,\omega_K^n)\}$ $(n=1,2,\cdots)$ is a minimizing sequence of the problem (2.4), then there exists a constant C, such that

$$E_{\mu,\nu}(\gamma^n,\omega_1^n,\omega_2^n,\cdots,\omega_K^n)\le C.\tag{2.6}$$

For $i=1,\cdots,K$, according to the formula of $E_{\mu,\nu}(\cdot)$ in (2.5), we can get that

$$\int_\Omega|\nabla\gamma^n|^2\le C\alpha^{-1},\qquad \int_\Omega|\nabla\omega_i^n|^2\le\left(\sqrt{C}+\sqrt{\int_\Omega|\nabla s_i|^2}\right)^2,\tag{2.7}$$

and

$$\int_\Omega \frac{2}{\gamma^n} \le C\nu^{-1}, \quad \int_\Omega \frac{2}{\omega_i^n} \le C\mu^{-1}. \tag{2.8}$$

Consider the classical Poincaré inequality and the constraints $\gamma^n, \omega_i^n \in W_0^{1,2}(\Omega)$, we derive that

$$\int_\Omega |\gamma^n|^2 \le M \int_\Omega |\nabla\gamma^n|^2 \ \text{ and } \ \int_\Omega |\omega_i^n|^2 \le M \int_\Omega |\nabla\omega_i^n|^2 \ \text{ for } \ i = 1,\cdots,K, \tag{2.9}$$

with $M > 0$ being a constant. Because of the boundedness of $\int_\Omega |\nabla\gamma^n|^2$ and $\int_\Omega |\nabla\omega_i^n|^2$ shown in (2.7) and boundedness of $\int_\Omega |\gamma^n|^2$ and $\int_\Omega |\omega_i^n|^2$ shown in (2.9), for any $i = 1,\cdots,K$, it follows that

$$\|\gamma^n\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega \left(|\gamma^n|^2 + |\nabla\gamma^n|^2\right) \le C(1+M)\alpha^{-1},$$

and

$$\|\omega_i^n\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega \left(|\omega_i^n|^2 + |\nabla\omega_i^n|^2\right) \le C(1+M).$$

Therefore, for $i = 1,\cdots,K$, the sequences $\{\gamma^n\}$ and $\{\omega_i^n\}$ are bounded in $W_0^{1,2}(\Omega)$, we can find subsequences (which are still denoted by $\{\gamma^n\},\{\omega_i^n\} \in W_0^{1,2}(\Omega)$ for the simplicity of description) and points $\gamma^*, \omega_i^* \in W_0^{1,2}(\Omega)$ such that

$$\gamma^n \rightharpoonup \gamma^*, \quad \omega_i^n \rightharpoonup \omega_i^* \ \text{ in } \ W_0^{1,2}(\Omega) \ \text{ as } \ n \to \infty. \tag{2.10}$$

Since $W_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$, we have

$$\gamma^n \to \gamma^*, \quad \omega_i^n \to \omega_i^* \ \text{ in } \ L_2(\Omega) \ \text{ as } \ n \to \infty. \tag{2.11}$$

From (2.10) and (2.11), it derives that

$$\omega_i^n + s_i \rightharpoonup \omega_i^* + s_i \ \text{ in } \ W_0^{1,2}(\Omega) \ \text{ as } \ n \to \infty, \tag{2.12}$$

and

$$\gamma^n - \omega_i^n \to \gamma^* - \omega_i^* \ \text{ in } \ L_2(\Omega) \ \text{ as } \ n \to \infty. \tag{2.13}$$

For $i = 1,\cdots,K$, $\gamma^*, \omega_i^* > 0$, otherwise,

$$E_{\mu,\nu}(\gamma^n, \omega_1^n, \omega_2^n, \cdots, \omega_K^n) \ge \mu \sum_{i=1}^K \int_\Omega \frac{2}{\omega_i^n} + \nu \int_\Omega \frac{2}{\gamma^n} \to +\infty.$$

This is contrary to the fact that $\{(\gamma^n, \omega_1^n, \omega_2^n, \cdots, \omega_K^n)\}$ is a minimizing sequence of the problem (2.4). By utilizing the weakly lower semicontinuity of $\int_\Omega |\nabla(\cdot)|^2$ and the relations between (2.10) and (2.12), it follows that

$$\liminf_{n\to\infty} \alpha \int_\Omega |\nabla\gamma^n|^2 \ge \alpha \int_\Omega |\nabla\gamma^*|^2,$$

and

$$\liminf_{n\to\infty}\int_\Omega |\nabla(\omega_i^n+s_i)|^2 \geq \int_\Omega |\nabla(\omega_i^*+s_i)|^2 \text{ for } i=1,\cdots,K.$$

Similarly, since $L^2(\Omega)$ norm is lower semi-continuous, by (2.13), for $i=1,\cdots,K$, it follows that

$$\liminf_{n\to\infty}\beta\int_\Omega(\omega_i^n-\gamma^n)^2 \geq \beta\int_\Omega(\omega_i^*-\gamma^*)^2.$$

Based on Lebesgue's dominated convergence theorem and the boundedness of $\int_\Omega \frac{2}{\gamma^n}$ and $\int_\Omega \frac{2}{\omega_i^n}$ shown in (2.8), by (2.11), for $i=1,\cdots,K$, we have

$$\lim_{n\to\infty}\nu\int_\Omega\frac{2}{\gamma^n}=\nu\int_\Omega\frac{2}{\gamma^*} \text{ and } \lim_{n\to\infty}\mu\int_\Omega\frac{2}{\omega_i^n}=\mu\int_\Omega\frac{2}{\omega_i^*}.$$

By considering the definition of the energy functional $E_{\mu,\nu}(\cdot)$ in (2.5), we can derive that

$$\begin{aligned}&\min_{\gamma,\omega_i\in W_0^{1,2}(\Omega),\gamma,\omega_i>0} E_{\mu,\nu}(\gamma,\omega_1,\omega_2,\cdots,\omega_K)\\ =&\liminf_{n\to\infty}E_{\mu,\nu}(\gamma^n,\omega_1^n,\omega_2^n,\cdots,\omega_K^n)\geq E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*).\end{aligned} \tag{2.14}$$

Therefore, $(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)$ is a solution of the optimization problem (2.4). Based on Theorem 2.1, it is easily obtained that the minimization problem (2.4) has a unique solution $(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)$. □

3. The Alternating Minimization Algorithm

In this section, an alternating minimization algorithm is designed to solve (2.4). The following minimization subproblems (3.1) with respect to γ and (3.2) with respect to ω_i, $i=1,\cdots,K$, are solved.

$$\begin{aligned}&\min_\gamma E_\nu(\gamma)=\alpha\int_\Omega|\nabla\gamma|^2+\sum_{i=1}^K\beta\int_\Omega(\omega_i-\gamma)^2+\nu\int_\Omega\frac{2}{\gamma},\\ &\text{Subject to } \gamma>0 \text{ and } \gamma\in W_0^{1,2}(\Omega).\end{aligned}\tag{3.1}$$

For $i=1,\cdots,K$,

$$\begin{aligned}&\min_{\omega_i}E_\mu(\omega_i)=\beta\int_\Omega(\omega_i-\gamma)^2+\int_\Omega\left|\nabla(\omega_i+s_i)\right|^2+\mu\int_\Omega\frac{2}{\omega_i},\\ &\text{Subject to } \omega_i>0 \text{ and } \omega_i\in W_0^{1,2}(\Omega).\end{aligned}\tag{3.2}$$

The Euler-Lagrange equation of (3.1) is as follows:

$$\frac{\partial E_\nu}{\partial\gamma}=K\beta(\gamma-\overline{\omega})-\alpha\Delta\gamma-\frac{\nu}{\gamma^2}=0.$$

Where $\overline{\omega} = (\sum_{i=1}^{K} \omega_i)/K$. The steepest descent method is applied to solve the problem (3.1), and the corresponding gradient descent flow equation is

$$\frac{\partial \gamma}{\partial t} = \alpha\Delta\gamma - K\beta(\gamma - \overline{\omega}) + \frac{\nu}{\gamma^2}. \tag{3.3}$$

Implicit scheme is used to discrete (3.3), and we obtain

$$\frac{\gamma^{j+1} - \gamma^j}{\tau_\gamma} = \alpha\Delta\gamma^j - K\beta(\gamma^j - \overline{\omega}) + \frac{\nu}{(\gamma^{j+1})^2},$$

which implies that

$$(\gamma^{j+1})^3 + \big(K\tau_\gamma\beta(\gamma^j - \overline{\omega}) - \gamma^j - \tau_\gamma\alpha\Delta\gamma^j\big)(\gamma^{j+1})^2 - \tau_\gamma\nu = 0. \tag{3.4}$$

Therefore, γ can be obtained by solving the cubic equation (3.4) pixel by pixel. In the same way, we can have that all ω_i $(i = 1, \cdots, K)$ of the problems (3.2) can be solved by the following cubic equations pixel by pixel

$$(\omega_i^{j+1})^3 + \Big(\tau_\omega\beta(\omega_i^j - \gamma) - \omega_i^j - \tau_\omega\Delta(\omega_i + s_i)^j\Big)(\omega_i^{j+1})^2 - \tau_\omega\mu = 0. \tag{3.5}$$

The following theorem shows that both equations (3.4) and (3.5) have only one positive solution.

Theorem 3.1. *There exists a unique positive solution of* (3.4) *and* (3.5).

Proof. Let $F(x) = (x-x_1)(x-x_2)(x-x_3) = x^3-(x_1+x_2+x_3)x^2+(x_1x_2+x_1x_3+x_2x_3)x-x_1x_2x_3$ be a cubic function. In cubic equation $F(x) = 0$, it satisfies $x_1x_2 + x_1x_3 + x_2x_3 = 0$, and $x_1x_2x_3 > 0$, which implies that $F(0) < 0, F(\infty) > 0$, and then $F(x)$ has one positive root or three positive roots. Clearly, only one of the three solutions x_1, x_2, x_3 is positive, otherwise, there is a contradiction in the fact that $x_1x_2 + x_1x_3 + x_2x_3 = 0$. That is to say that both (3.4) and (3.5) have only one positive root. □

The whole computation of the alternating minimization procedure for solving (2.4) can be performed through the following Algorithm 3.1.

Algorithm 3.1. Set initial values γ^0, ω_i^0 $(i = 1, \cdots, K)$, the maximum iteration numbers of outer iteration N and inner iteration n, the tolerances ϵ_ω, ϵ_γ. Denote $\overline{\omega^0} = (\sum_{i=1}^{K} \omega_i^0)/K$ and t being the index of the outer iteration, set $t = 0$.

1 Set $\gamma^{1,t} = \gamma^t$, update γ^{t+1} by solving the problem (3.1). For $j = 1$ to n, find the positive solution of the following equation by the root formula of the cubic equation.

$$(\gamma^{j+1,t})^3 + \big(K\tau_\gamma\beta(\gamma^{j,t} - \overline{\omega^t}) - \gamma^{j,t} - \tau_\gamma\alpha\Delta\gamma^{j,t}\big)(\gamma^{j+1,t})^2 - \tau_\gamma\nu = 0. \tag{3.6}$$

If the positive solution of (3.6) $\gamma^{j+1,t}$ satisfies

$$\frac{\|\gamma^{j+1,t} - \gamma^{j,t}\|}{\|\gamma^{j,t}\|} \le \epsilon_\gamma,$$

then $\gamma^{t+1} = \gamma^{j+1,t}$, break;
Set $\gamma^{t+1} = \gamma^{j+1,t}$.

2 For $i=1,\cdots,K$, set $\omega_i^{1,t}=\omega_i^t$, update ω_i^{t+1} by solving the problem (3.2). For $j=1$ to n, find the positive solution of the following equation by the root formula of the cubic equation.

$$(\omega_i^{j+1,t})^3+\Big(\tau_\omega\beta(\omega_i^{j,t}-\gamma^{t+1})-\omega_i^{j,t}-\tau_\omega\Delta(\omega_i+s_i)^{j,t}\Big)(\omega_i^{j+1,t})^2-\tau_\omega\mu=0. \quad (3.7)$$

If the positive solution of (3.7) $\omega_i^{j+1,t}$ satisfies

$$\frac{\|\omega_i^{j+1,t}-\omega_i^{j,t}\|}{\|\omega_i^{j,t}\|}\le\epsilon_\omega.$$

then $\omega_i^{t+1}=\omega_i^{j+1,t}$, break;
Set $\omega_i^{t+1}=\omega_i^{j+1,t}$.

3 Update $\overline{\omega^{t+1}}=(\sum_{i=1}^K\omega_i^{t+1})/K$.

4 If

$$t=N \quad \text{or} \quad \frac{\|\gamma^{t+1}-\gamma^t\|}{\|\gamma^t\|}\le\epsilon_\gamma, \quad \frac{\|\omega_i^{t+1}-\omega_i^t\|}{\|\omega_i^t\|}\le\epsilon_\omega, \quad i=1,\cdots,K.$$

break; otherwise set $t=t+1$, go to step 1.
Output $\gamma=\gamma^{t+1},\omega_i=\omega_i^{t+1}$ $(i=1,\cdots,K)$.

The convergence of the Algorithm 3.1 can be guaranteed by the following theorem.

Theorem 3.2. *Let* $\{(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\}$ *be a sequence generated by the Algorithm* 3.1. *Then* $\{(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\}$ *converges to* $(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)$ *(up to a subsequence), which is in* $L_2(\Omega)\times L_2(\Omega)\times\cdots\times L_2(\Omega)$*, as* $t\to\infty$*. And,* $E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)$ *converges to* $E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)$*. For* $i=1,\cdots,K$*,* γ*,* $\omega_i\in W_0^{1,2}(\Omega)$*, and* γ*,* $\omega_i>0$ *, we have*

$$\begin{aligned}
E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*) &\le E_{\mu,\nu}(\gamma,\omega_1^*,\omega_2^*,\cdots,\omega_K^*),\\
E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*) &\le E_{\mu,\nu}(\gamma^*,\omega_1,\omega_2^*,\cdots,\omega_K^*),\\
&\vdots\\
E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*) &\le E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K).
\end{aligned}$$

Proof. It is easy to deduce the following inequality from the Algorithm 3.1.

$$\begin{aligned}
&E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\
\le&E_{\mu,\nu}(\gamma^{t+1},\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\le E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t).
\end{aligned}$$

That is, $E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\le E_{\mu,\nu}(\gamma^0,\omega_1^0,\omega_2^0,\cdots,\omega_K^0)$ and $E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)$ decreases with t, similar to the proof of Theorem 2.2 (from (2.6) to (2.14)), for $i=1,\cdots,K$, we can find subsequences (which are still noted by $\{\gamma^t\}, \{\omega_i^t\}$ for the simplicity of description) and points γ^*, $\omega_i^* \in W_0^{1,2}(\Omega)$, γ^*, $\omega_i^* > 0$, satisfying the formulas:

$$\begin{aligned}&\gamma^t \rightharpoonup \gamma^*, \quad \omega_i^t \rightharpoonup \omega_i^* \;\text{ in }\; W_0^{1,2}(\Omega) \;\text{ as }\; t\to\infty,\\ &\gamma^t \to \gamma^*, \quad \omega_i^t \to \omega_i^* \;\text{ in }\; L_2(\Omega) \;\text{ as }\; t\to\infty,\end{aligned}$$

and

$$\liminf_{t\to\infty} E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t) \geq E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*). \tag{3.8}$$

Recall that $E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)$ is nonnegative and decreasing with t, then there exists $e>0$ such that

$$e = \lim_{t\to\infty} E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t). \tag{3.9}$$

From (3.8) and (3.9), it follows that

$$e = \liminf_{t\to\infty} E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t) \geq E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*). \tag{3.10}$$

Based on the following inequalities

$$\begin{aligned}&E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\ \leq &E_{\mu,\nu}(\gamma^{t+1},\omega_1^t,\omega_2^t,\cdots,\omega_K^t) \leq E_{\mu,\nu}(\gamma^*,\omega_1^t,\omega_2^t,\cdots,\omega_K^t),\end{aligned}$$

and

$$\begin{aligned}&E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\ \leq &E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t) \leq E_{\mu,\nu}(\gamma^t,\omega_1^*,\omega_2^*,\cdots,\omega_K^*),\end{aligned}$$

we have

$$\begin{aligned}&2E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\ \leq &E_{\mu,\nu}(\gamma^*,\omega_1^t,\omega_2^t,\cdots,\omega_K^t) + E_{\mu,\nu}(\gamma^t,\omega_1^*,\omega_2^*,\cdots,\omega_K^*).\end{aligned} \tag{3.11}$$

Bringing the energy functions to the right-hand side of (3.11), we derive that

$$\begin{aligned}&E_{\mu,\nu}(\gamma^*,\omega_1^t,\omega_2^t,\cdots,\omega_K^t) + E_{\mu,\nu}(\gamma^t,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)\\ =&\alpha\int_\Omega |\nabla\gamma^*|^2 + \beta\sum_{i=1}^K\int_\Omega(\omega_i^t-\gamma^*)^2 + \sum_{i=1}^K\int_\Omega \left|\nabla(\omega_i^t+s_i)\right|^2\\ &+\alpha\int_\Omega |\nabla\gamma^t|^2 + \beta\sum_{i=1}^K\int_\Omega(\omega_i^*-\gamma^t)^2 + \sum_{i=1}^K\int_\Omega \left|\nabla(\omega_i^*+s_i)\right|^2\\ &+\mu\sum_{i=1}^K\int_\Omega\frac{2}{\omega_i^t} + \mu\sum_{i=1}^K\int_\Omega\frac{2}{\omega_i^*} + \nu\int_\Omega\frac{2}{\gamma^*} + \nu\int_\Omega\frac{2}{\gamma^t}.\end{aligned} \tag{3.12}$$

Rewrite the sum

$$\begin{aligned}&\beta\sum_{i=1}^K\int_\Omega(\omega_i^t-\gamma^*)^2 + \beta\sum_{i=1}^K\int_\Omega(\omega_i^*-\gamma^t)^2\\ =&\beta\sum_{i=1}^K\int_\Omega(\omega_i^t-\gamma^t)^2 + \beta\sum_{i=1}^K\int_\Omega(\omega_i^*-\gamma^*)^2 + 2\beta\sum_{i=1}^K\int_\Omega(\omega_i^*-\omega_i^t)(\gamma^*-\gamma^t),\end{aligned}$$

and bring it into (3.12), we have

$$\begin{aligned}
&E_{\mu,\nu}(\gamma^*,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)+E_{\mu,\nu}(\gamma^t,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)\\
=&E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)+E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\\
&+2\beta\sum_{i=1}^{K}\int_\Omega(\omega_i^*-\omega_i^t)(\gamma^*-\gamma^t).
\end{aligned}$$

Thus, by (3.11), we obtain that

$$\begin{aligned}
&2E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\
\le&E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)+E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\\
&+2\beta\sum_{i=1}^{K}\int_\Omega(\omega_i^*-\omega_i^t)(\gamma^*-\gamma^t).
\end{aligned}\tag{3.13}$$

If $t\to+\infty$ in (3.13), we can derive that

$$2e\le E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)+e,$$

which implies that $E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)\ge e$. Thus, by (3.10), we get $e=E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)$. For any $\gamma\in W_0^{1,2}(\Omega),\gamma>0$, by the Algorithm 3.1, we have

$$\begin{aligned}
&E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\
\le&E_{\mu,\nu}(\gamma^{t+1},\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\le E_{\mu,\nu}(\gamma,\omega_1^t,\omega_2^t,\cdots,\omega_K^t),
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\
\le&E_{\mu,\nu}(\gamma^t,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\le E_{\mu,\nu}(\gamma^t,\omega_1^*,\omega_2^*,\cdots,\omega_K^*).
\end{aligned}\tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\begin{aligned}
&2E_{\mu,\nu}(\gamma^{t+1},\omega_1^{t+1},\omega_2^{t+1},\cdots,\omega_K^{t+1})\\
\le&E_{\mu,\nu}(\gamma^t,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)+E_{\mu,\nu}(\gamma,\omega_1^t,\omega_2^t,\cdots,\omega_K^t)\\
&+2\beta\sum_{i=1}^{K}\int_\Omega(\omega_i^*-\omega_i^t)(\gamma-\gamma^t).
\end{aligned}$$

If $t\to+\infty$, we obtain

$$E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)\le E_{\mu,\nu}(\gamma,\omega_1^*,\omega_2^*,\cdots,\omega_K^*).$$

Similarly, for $i=1,\cdots,K,\omega_i\in W_0^{1,2}(\Omega)$ and $\omega_i>0$, we have

$$\begin{aligned}
E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)&\le E_{\mu,\nu}(\gamma^*,\omega_1,\omega_2^*,\cdots,\omega_K^*),\\
E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)&\le E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2,\cdots,\omega_K^*),\\
&\vdots\\
E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K^*)&\le E_{\mu,\nu}(\gamma^*,\omega_1^*,\omega_2^*,\cdots,\omega_K).
\end{aligned}$$

□

4. Experimental Results

In this section, we present the experimental results to demonstrate the performance of the proposed method. We compare the recovered reflectance results with those obtained by the method proposed in [25]. Suppose there are K images of a same scene with different exposures. In Algorithm 3.1, the initials are set as $\omega_i^0 = \log(256) - s_i (i = 1, \cdots, K)$, $\gamma^0 = 1$. We select the parameters $\alpha = 0.2 * K$, $\beta = 2.8$, $\mu = 1e-5$, $\nu = (1e-5) * K$, $\tau_\omega = 0.006$, $\tau_\gamma = 0.006/K$, the maximum outer iteration number $N = 5$ and inner iteration number $n = 100$, the stop criteria $\epsilon_\omega = \epsilon_\gamma = 1e-3$. Color channels (R, G, and B) are handled separately. After γ is obtained by applying Algorithm 3.1 in the logarithmic domain, the recovered reflectance is computed by $R = \exp(-\gamma)$.

In the first step and second step of Algorithm 3.1, based on the result of Theorem 3.1, the only positive solution of the following cubic polynomial equation

$$x^3 + bx^2 + d = 0$$

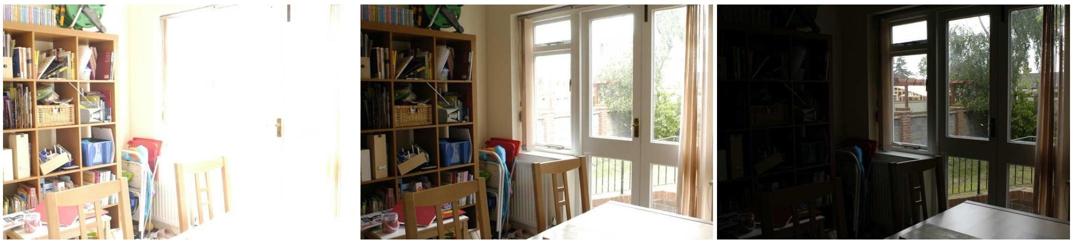
with $b \neq 0$ and $d < 0$ need to be determined. Denoting

$$\begin{aligned} \delta =& \sqrt[3]{\sqrt{(\frac{b^3}{27} + \frac{d}{2})^2 - \frac{b^6}{729}} - \frac{d}{2} - \frac{b^3}{27}} \\ =& \sqrt[3]{\sqrt{d(\frac{b^3}{27} + \frac{d}{4})} - \frac{d}{2} - \frac{b^3}{27}}. \end{aligned} \tag{4.1}$$

We note that all solutions of such cubic polynomial equation can be obtained from formulas

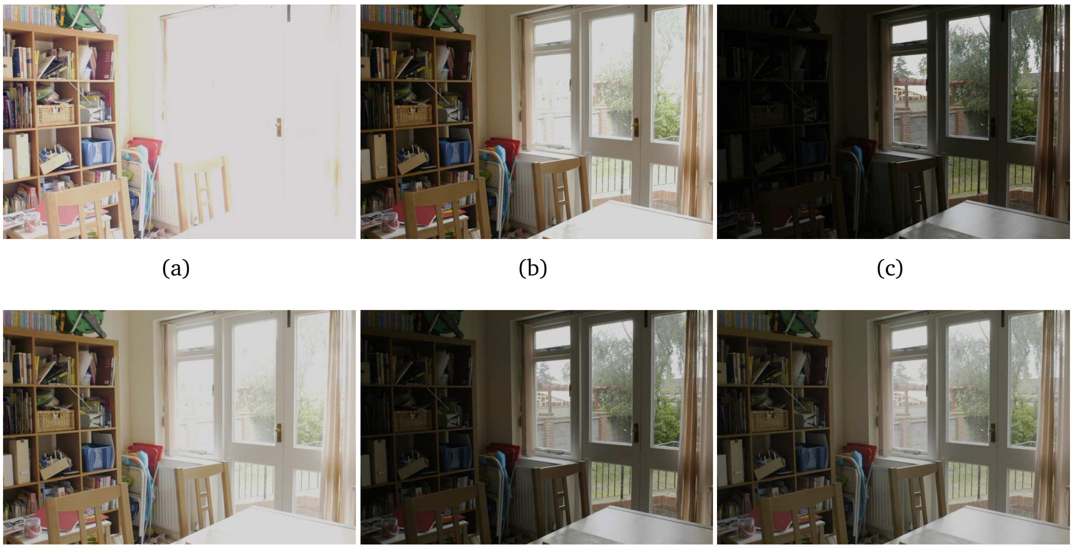
$$\begin{aligned} x_1 =& \delta + \frac{b^2}{9\delta} - \frac{b}{3}, \\ x_2 =& -\frac{b}{3} - \frac{\delta}{2} - \frac{b^2}{18\delta} + \frac{\sqrt{3}i}{2}\left(\delta - \frac{b^2}{9\delta}\right), \\ x_3 =& -\frac{b}{3} - \frac{\delta}{2} - \frac{b^2}{18\delta} - \frac{\sqrt{3}i}{2}\left(\delta - \frac{b^2}{9\delta}\right). \end{aligned} \tag{4.2}$$

See, for instance, [23]. We consider cubic equation (3.4), where the quadratic term coefficient $b_\gamma = K\tau_\gamma\beta(\gamma^j - \overline{\omega}) - \gamma^j - \tau_\gamma\alpha\Delta\gamma^j$. Since all parameters are greater than zero, $\overline{\omega} > 0$, $\gamma^j > 0$ and $\Delta\gamma^j \geq -4\gamma^j$, we have $b_\gamma \leq (K\tau_\gamma\beta - 1 + 4\tau_\gamma\alpha)\gamma^j$. Bringing the parameters τ_γ, β and α we use in the experiments into the above estimated bound of b_γ, we estimate that $b_\gamma \leq (0.006 \times 2.8 - 1 + 4 \times 0.006 \times 0.2)\gamma^j = -0.9784\gamma^j < 0$. The constant term of (3.4) is $d_\gamma = -\tau_\gamma\nu < 0$. Therefore, it is obvious that δ is a real number and x_1 in (4.2) is the unique positive root of equation (3.4). Our numerical results also show that the solution x_1 of equation (3.4) is always positive, x_2 and x_3 are a pair of conjugate complex numbers. Such results are consistent with our analyses shown above. For equation (3.5), we compute all the solutions x_1, x_2 and x_3 in (4.2), and we find that x_1 is always the only positive solution of cubic equation (3.5).



(a) E1 (b) E2 (c) E3

Figure 1: The original three "house" images with different exposures.



(a) (b) (c)

(d) (e) (f)

Figure 2: The reflectance images, by using images in Fig. 1, from left to right, first line: using E1, E2, E3, respectively; second line: using E1+ E2, E2+ E3, E1+ E2+ E3, respectively.

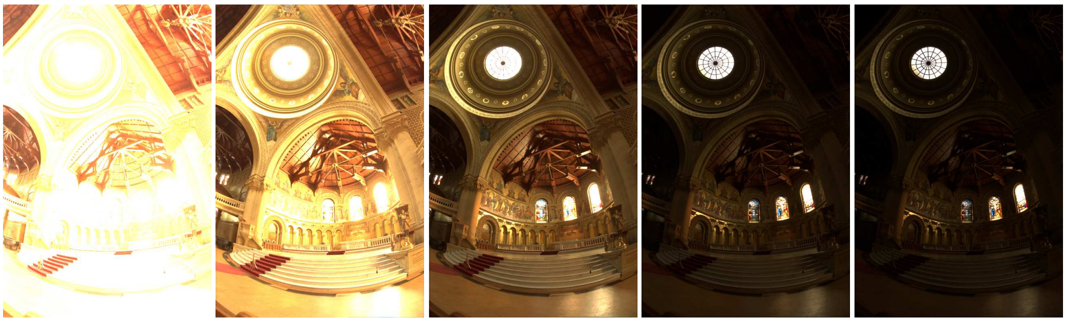
Two experiments are performed. Each experiment contains several images of a same scene with different exposures. Base on retinex theory, all images in an image sequence should have the same reflectance image, but in practice, some image details may be lost in the saturated or over-dark regions due to the different exposure times. Therefore, the reflectances recovered from images of a same scene with different exposures may be different. We employ the proposed model to get a reflectance image of the image sequence. We expect that the obtained reflectance should be closer to the real reflectance by taking advantage of the information got under different exposures.

Three "House" images with different exposures shown in Figs. 1 are used in the first experiment. We name the first, second and third images as "E1", "E2" and "E3", respectively. The recovered reflectances are shown in Fig. 2. Images (a)-(c) in the first line of Fig. 2 are the results obtained by using single "E1", single "E2" and single "E3", respectively. That is, the first line is the results got by using the method proposed in [25]. Images (d)-(f) in the second line of Fig. 2 are the results obtained by applying the proposed method when using "E1" and "E2", "E2" and "E3", "E1" and "E2" and "E3", respectively. We simply denote them as "E1+E2", "E2+E3" and "E1+E2+E3", respectively. From Fig. 2, we can find that the recovered reflectance (f) is the best in terms of the visual quality and details recovery. The illumination of reflectance (f) is more natural. Reflectance (a) is over-exposed. Reflectance (b) is brighter, but details of walls and trees outside the window in (b) are weaker than those in (f). The illumination of shelf, things on shelf and chairs in (f) are not brighter than that in (b), but the details of these parts in both reflectances are almost the same. The shelf part of reflectance (c) is over-dark. Reflectance (d) is a bit over-exposed, but (d) has much more details than (a) does. The window and trees part of (e) is a bit better than those of (f), but the details of the things on the shelf and chairs are not better than those in (f). By summarizing the above comparison, (f) is the best when considering the reflectance recovery effects of the whole image. The results of Fig. 2 show that the multiple exposed reflectance recovery method is superior to the single exposed reflectance recovery method.

Five "Church" images with different exposures shown in Figs. 3 are used in the second experiment. Same as in the first experiment, we name the images from left to right as "E1", "E2", "E3", "E4" and "E5", respectively. The recovered reflectances are shown in Fig. 4. Images (a)-(e) in the first line of Fig. 4 are the results obtained by using single "E1", single "E2", single "E3", single "E4" and single "E5", respectively. Images (f)-(j) in the second line of Fig. 4 are the results obtained by applying the proposed method when using "E1" and "E2", "E1" and "E2" and "E3", "E1" and "E2" and "E3" and "E4", "E2" and "E3" and "E4" and "E5", "E1" and "E2" and "E3" and "E4" and "E5", respectively. We simply denote them as "E1+E2", "E1+E2+E3", "E1+E2+E3+E4", "E2+E3+E4+E5" and "E1+E2+E3+E4+E5", respectively. From Fig. 4, we can find that the recovered reflectance (j) is the best. The illumination of reflectance (j) is more natural. Reflectance (a) and (b) are over-exposed. Reflectance (c) is bright, but details of the painting on the window and the ceiling lamp in (c) are weaker than those in (j). Both reflectances (d) and (e) are over-dark in the upper-left part. Reflectances (f) and (g) are a bit over-exposed, but they both have much more details than (a) does. Reflectance (h) is brighter, but details of the ceiling lamp and the painting on the window in (j) are weaker than those in (h). Reflectance (i) is darker than (j). The illumination of fresco on wall and pillar in (i) are brighter than those in (j), but the details of these parts in both reflectances are almost the same. By summarizing the above comparison, (j) is the best when considering the reflectance recovery effects of the whole image. The multiple exposed reflectance recovery method is superior to the single exposed reflectance recovery method.

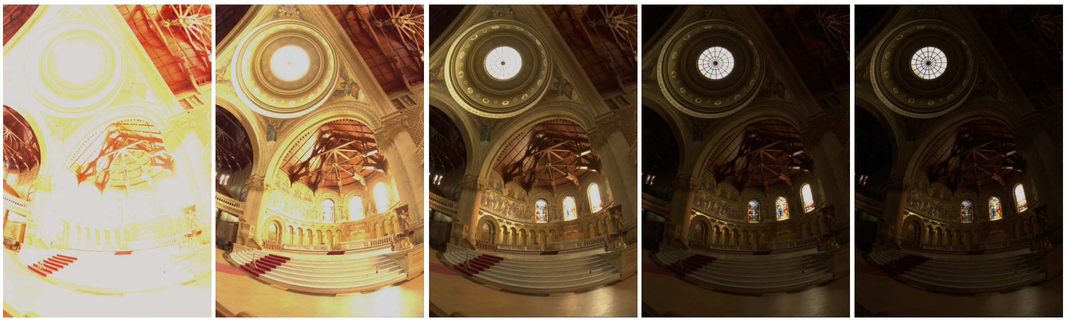
5. Concluding Remarks

Retinex theory has a wide application areas. This theory explains color perception, brightness perception, and constancies, theorizing that the color of an object is not decided by the material property of the object. The main problem in retinex is to decompose the reflectance function and illumination function. In this paper, a variational model for retinex problem by utilizing multiple exposured images is constructed. The existence and

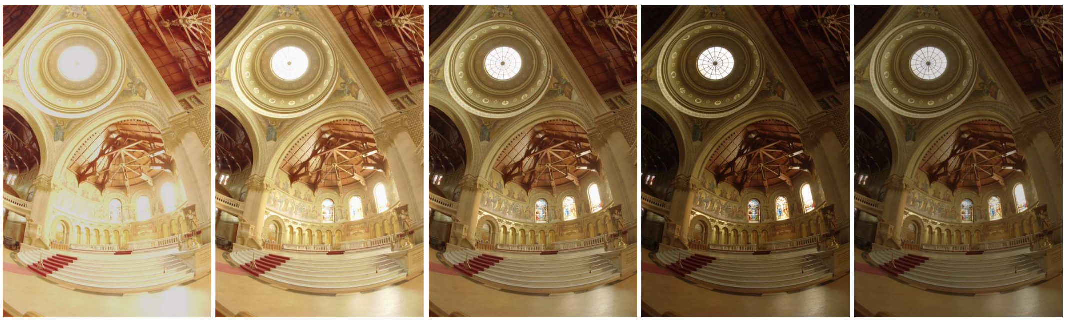


(a) E1 (b) E2 (c) E3 (d) E4 (e) E5

Figure 3: The original five "Church" images with different exposures.



(a) (b) (c) (d) (e)



(f) (g) (h) (i) (j)

Figure 4: The reflectance images, by using images in Fig. 3, from left to right, first line: using E1, E2, E3, E4, E5, respectively; second line: using E1+ E2, E1+ E2+ E3, E1+ E2+ E3 +E4, E2+ E3+ E4 +E5, E1+ E2+ E3 +E4 +E5, respectively.

uniqueness of the solutions of the proposed model has been demonstrated. An alternating minimization method is designed to solve the proposed model and its convergence is also demonstrated. Experimental results further show that the reflectance restored effects by the proposed method are much better than single exposure retinex method in terms of image detail preservation and visual quality.

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