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A Fast Shift-Splitting Iteration Method for Nonsymmetric Saddle Point Problems

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Abstract. Based on the shift-splitting technique and the idea of Hermitian and skew-Hermitian splitting, a fast shift-splitting iteration method is proposed for solving nonsingular and singular nonsymmetric saddle point problems in this paper. Convergence and semi-convergence of the proposed iteration method for nonsingular and singular cases are carefully studied, respectively. Numerical experiments are implemented to demonstrate the feasibility and effectiveness of the proposed method.

AMS subject classifications: 65M10, 78A48

Key words: Nonsymmetric saddle point problems, fast shift-splitting, convergence, semi-convergence.

1. Introduction

Consider the following nonsymmetric saddle point problems

$$\begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsymmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a rectangular matrix with $m \le n$, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are given vectors.

The saddle point problems (1.1) arise in a variety of scientific and engineering applications, such as computational fluid dynamics [13], mixed finite element approximation of elliptic partial differential equations [20] and Lagrange-type methods for constrained nonconvex optimization problems [27]. For a survey, we refer the readers to [13].

Since the matrices A and B are usually large and sparse, it may be more attractive to use iterative methods than direct methods for the solution of the saddle point problem (1.1). In the case that the matrix B has full row rank, many efficient iteration methods were proposed to solve the saddle point problems, for example, Uzawa-type methods [1, 2, 11, 15, 16], matrix splitting methods [5, 7–9, 23], residual algorithm [3], relaxation iterative methods [10, 22] and so on.

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When the matrix B is rank deficient, the coefficient matrix of the equation (1.1) is singular, and the linear system (1.1) is called singular saddle point problems. There exist a lot of methods for solving singular saddle point problems in the literature. Generalized successive overrelaxation method was studied in [34], and the semi-convergence of this method was proved when it is applied to solve singular saddle point problems. Minimum residual and conjugate gradient methods were proposed for solving the rank-deficient saddle point problems in [21,31], respectively. Inexact Uzawa method, which covers the Uzawa method, the preconditioned Uzawa method, and the parameterized method as special cases, was discussed for singular saddle point problems in [33], and the semi-convergence result under restrictions was proved by verifying two necessary and sufficient conditions. More numerical methods for singular saddle point problems could be found in [4, 19, 32] and the references therein.

In this paper, we construct a fast shift-splitting iteration method for nonsymmetric saddle point problems based on the ideas of the shift-splitting iteration method [12, 18] and the Hermitian and skew-Hermitian splitting technique [9,23,35]. The idea of shift-splitting iteration method was first proposed by Bai, Yin and Su in [12] for solving a class of non-Hermitian positive definite linear systems. Then, it was extended by Cao, Du and Niu in [17] to solve saddle point problems, and generalized by Salkuyeh for saddle point problems in [28]. After that, for nonsymmetric saddle point problems, Cao et al. in [18, 19] proposed the generalized shift-splitting (GSS) method

$$\frac{1}{2}\begin{pmatrix} \alpha I + A & B^T \\ -B & \beta I \end{pmatrix}\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha I - A & -B^T \\ B & \beta I \end{pmatrix}\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.2}$$

and Zhou et al. in [35] presented the modified shift-splitting (MSS) method

$$\frac{1}{2}\begin{pmatrix} \alpha I + 2H & B^T \\ -B & \alpha I \end{pmatrix}\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha I - 2S & -B^T \\ B & \alpha I \end{pmatrix}\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.3}$$

where α and β are two given positive constants, I is the identity matrix with appropriate dimension, and the matrices H and S are the symmetric (Hermitian) part and skew-symmetric (skew-Hermitian) part of the matrix A, respectively, i.e., $H = \frac{1}{2}(A + A^T)$, $S = \frac{1}{2}(A - A^T)$. Recently, Shen et al. applied the GSS iteration method to solve a broad class of nonsingular and singular generalized saddle point problems in [29]. In this paper, a fast shift-splitting iteration method is studied, which can be regarded as a special case of the BASI (block alternating splitting implicit) method proposed by Bai in [24]. Convergence and semi-convergence theories of this method for nonsingular and singular cases are carefully analyzed, respectively. Numerical experiments further show that the proposed method is efficient and feasible.

This paper is organized as follows. In Section 2, a fast shift-splitting iteration method for nonsymmetric saddle point problems is established. In Section 3, the convergence of the fast shift-splitting iteration method for nonsingular case is analyzed. In Section 4, semi-convergence of the fast shift-splitting iteration method for singular case is studied. In Section 5, numerical experiments are presented to illustrate the effectiveness and feasibility of the proposed method. Finally, a brief conclusion is given.