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A New Uzawa-Type Iteration Method for Non-Hermitian Saddle-Point Problems

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Abstract. Based on a preconditioned shift-splitting of the $(1,1)$-block of non-Hermitian saddle-point matrix and the Uzawa iteration method, we establish a new Uzawa-type iteration method. The convergence properties of this iteration method are analyzed. In addition, based on this iteration method, a preconditioner is proposed. The spectral properties of the preconditioned saddle-point matrix are also analyzed. Numerical results are presented to verify the robustness and the efficiency of the new iteration method and the preconditioner.

AMS subject classifications: 65F08, 65F10, 65F20

Key words: Saddle-point problems, Uzawa method, preconditioned shift-splitting, convergence, preconditioner.

1. Introduction

Consider the following large, sparse non-Hermitian saddle-point linear system

$$\mathscr{A}\, x := \begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} =: b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, i.e., its Hermitian part $H = (1/2)(A+A^*)$ is positive definite, $B \in \mathbb{C}^{n\times m}$ is a rectangular matrix of full column rank with $n \geq m$, and $b \in \mathbb{C}^{n+m}$ is a given vector. Here, $(\cdot)^*$ denotes the conjugate transpose of a matrix. This kind of systems of linear equations arises in a variety of scientific and engineering applications, such as computational fluid dynamics, optimal control, constrained optimization, weighted least-squares problems, electronic networks, computer graphic, mixed or hybrid finite element discretization of second-order elliptic problems and meshfree discretization of some partial differential equations; see [10, 13, 15, 17, 21, 24, 27, 28] and the references therein.

In order to solve the nonsingular saddle-point linear system (1.1), many effective iteration methods and preconditioning techniques have been proposed in the literature;

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see [3–7, 9, 11, 13, 19, 20, 23] and the references therein. As one of the most important iteration methods, Uzawa method [1] can be described as follows:

$$\begin{cases} u_{k+1} = A^{-1}(f - Bv_k), \\ v_{k+1} = v_k + \tau(B^* u_{k+1} - g), \end{cases} \tag{1.2}$$

where $\tau > 0$ is a relaxation parameter. Owing to its simple form, minimal computer memory requirements and good numerical performance, the Uzawa method has received wide attention in the literature. Many variants of the Uzawa method, including preconditioned Uzawa [19] and parameterized Uzawa [10] methods, have been proposed to improve the efficiency of the original Uzawa method.

In each step of the Uzawa method, a linear system with coefficient matrix A needs to be solved, which is the most expensive computation in the algorithms. With this in mind, an approximation matrix Q_A, i.e., a preconditioner of the matrix A, has been introduced in the inexact and the parameterized inexact Uzawa methods [11, 14]. Hence, the first iteration step of (1.2) becomes

$$u_{k+1} = u_k + Q_A^{-1}(f - Au_k - Bv_k). \tag{1.3}$$

From both the theoretical analysis and numerical results in Refs. [11, 14], we know that a good approximation matrix Q_A may lead to an efficient inexact Uzawa method. Hence, how to choose a good preconditioner Q_A for matrix A becomes an important problem.

When A is Hermitian positive definite, many efficient preconditioners have been discussed; see Refs. [31, 32] for the details. When A is non-Hermitian positive definite, Yang and Wu [29, 30] employed the Hermitian and skew-Hermitian splitting (HSS) preconditioner of matrix A to accelerate the convergence of Uzawa method, which leads to the following Uzawa–HSS method:

$$\begin{cases} u_{k+1} = u_k + 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}(f - Au_k - Bv_k), \\ v_{k+1} = v_k + \tau Q^{-1}(B^* u_{k+1} - g), \end{cases} \tag{1.4}$$

where $Q \in \mathbb{R}^{m\times m}$ is an Hermitian positive definite matrix, $H = (1/2)(A + A^*)$ and $S = (1/2)(A - A^*)$ are the Hermitian and skew-Hermitian parts of matrix A, respectively. A similar idea, i.e., using positive definite and skew-Hermitian splitting preconditioner to accelerate the convergence of Uzawa method, can be seen in Ref. [18]. In addition, a modified local HSS (MLHSS) iteration method and the corresponding MLHSS preconditioner proposed by Jiang and Cao [22] are also very efficient for solving the saddle-point problems. However, for these iteration methods referred above, the restrictions on the involved iteration parameters to ensure convergence are complicated. It is hard to verify whether the convergence conditions are satisfied.

In the saddle-point linear system (1.1) or the iteration scheme (1.2), matrix A is non-Hermitian positive definite. For the non-Hermitian positive definite system of linear equations $Az = c$, Bai et al. [12] introduced an efficient shift-splitting iteration method of the form

$$(\alpha I + A)z_{k+1} = (\alpha I - A)z_k + 2c, \tag{1.5}$$