On Preconditioners Based on HSS for the Space Fractional CNLS Equations

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Abstract. The space fractional coupled nonlinear Schrödinger (CNLS) equations are discretized by an implicit conservative difference scheme with the fractional centered difference formula, which is unconditionally stable. The coefficient matrix of the discretized linear system is equal to the sum of a complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and a symmetric Toeplitz-plus-diagonal matrix. In this paper, we present new preconditioners based on Hermitian and skew-Hermitian splitting (HSS) for such Toeplitz-like matrix. Theoretically, we show that all the eigenvalues of the resulting preconditioned matrices lie in the interior of the disk of radius 1 centered at the point \((1,0)\). Thus Krylov subspace methods with the proposed preconditioners converge very fast. Numerical examples are given to illustrate the effectiveness of the proposed preconditioners.

AMS subject classifications: 65F10, 65F15

Key words: The space fractional Schrödinger equations, Toeplitz matrix, Hermitian and skew-Hermitian splitting, preconditioner, Krylov subspace methods.

1. Introduction

The classical Schrödinger equations describe the evolution of microscopic particles, and they can be derived from the path integral over the Brownian motion. Laskin \cite{12} generalized the path integral method from the Brownian motion to the Lévy-\(\alpha\) process to obtain the space fractional Schrödinger equations \cite{13}.

In this paper, we consider the space fractional coupled nonlinear Schrödinger (CNLS) equations

\[
\begin{align*}
&iu_t + \gamma (-\Delta)^{\frac{\alpha}{2}} u + \rho (|u|^2 + \beta |v|^2) u = 0, \\
i v_t + \gamma (-\Delta)^{\frac{\alpha}{2}} v + \rho (|v|^2 + \beta |u|^2) v = 0,
\end{align*}
\]

\((1.1)\)

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with the initial boundary value conditions
\[
\begin{align*}
&\begin{cases}
  u(x,0) = u_0(x), & \alpha \leq x \leq \beta, \\
  v(x,0) = v_0(x), & \alpha \leq x \leq \beta,
\end{cases} \\
&\begin{cases}
  u(a,t) = 0, & 0 \leq t \leq T, \\
  v(a,t) = 0, & 0 \leq t \leq T,
\end{cases}
\end{align*}
\]
where \( i = \sqrt{-1}, \ 1 < \alpha < 2 \) and the parameters \( \gamma > 0, \ \rho > 0, \ \beta \geq 0 \) are constants. The fractional Laplacian \([11]\) can be characterized as
\[
(-\Delta)^{\alpha} u(x,t) = \mathcal{F}^{-1}\left( |\xi|^\alpha \mathcal{F}(u(x,t)) \right),
\]
where \( \mathcal{F} \) is the Fourier transform acting on the spatial variable \( x \). Furthermore, it is shown that the Riesz fractional derivative \([24]\) can also be defined as
\[
\frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t) = -\frac{1}{2\cos \frac{\pi\alpha}{4}} \left[ -\infty \ D^\alpha_x u(x,t) + x \ D^\alpha_{-\infty} u(x,t) \right],
\]
where \( -\infty \ D^\alpha_x u(x,t) \) and \( x \ D^\alpha_{-\infty} u(x,t) \) are the left and right Riemann-Liouville derivatives, respectively. When \( \alpha = 2 \), the system (1.1) is reduced to the classical CNLS equations, which describe a wide class of physical nonlinear phenomena, such as the hydrodynamics, the nonlinear optics and the dynamics of the two-component Bose-Einstein condensate.

Generally, closed-form analytical solutions of the space fractional CNLS equations are not available. Consequently, the numerical methods become important and powerful which are few until now. Recently, based on the fractional centered difference formula, in \([21–23]\) the authors proposed an implicit conservative difference scheme to discretize the space fractional CNLS equations which is unconditionally stable. The coefficient matrix of the discretized linear system is equal to the sum of the complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and the symmetric Toeplitz-plus-diagonal matrix. As the coefficient matrix is non-Hermitian Toeplitz-like, we can employ Krylov subspace methods, such as BiCGSTAB, to solve the discretized linear system. Using the fast Fourier transform (FFT), the Toeplitz matrix-vector multiplication can be done in \( O(M \log M) \) operations, where \( M \) is the number of grid points. Nevertheless, the resulting system in general is ill-conditioned and the convergence rates of Krylov subspace methods tend to be considerably worse. In order to improve their performance and reliability, preconditioning is a common technique.

Circulant preconditioners for Toeplitz matrices have been theoretically and numerically studied with numerous applications for over twenty years; see \([7, 8, 15]\). However, circulant preconditioners do not work for such Toeplitz-plus-diagonal systems. Chan and Ng \([9]\) considered banded preconditioners for Toeplitz-plus-band systems. The main drawback of this method is that the generating function should be known in order to construct effective banded preconditioners. In general, the generating function of the corresponding Toeplitz matrix is unknown. In \([17]\), multigrid methods are studied for Toeplitz-plus-diagonal linear systems arising from Sinc-Galerkin methods. As the generating functions are known, the proposed multigrid method is to incorporate the diagonal matrix into the interpolating process. Lately, Ng and Pan \([16]\) proposed approximate inverse circulant-plus-diagonal preconditioners for solving Hermitian positive definite Toeplitz-plus-diagonal systems. Their
idea is to use circulant matrices to approximate the inverses of Toeplitz matrices and then combine these circulant matrices together. Motivated by this idea of construction of preconditioners, Lei and Sun [14] and Pan et al. [19] presented circulant preconditioners and approximate inverse preconditioners for diagonal-times-Toeplitz matrices in fractional diffusion equations.

In this paper, we firstly employ the approximate inverse circulant-plus-diagonal preconditioners, and then propose and develop new preconditioners based on Hermitian and skew-Hermitian splitting (HSS) for solving the Toeplitz-like system arising from the discretization of the space fractional CNLS equations. The outline of the rest of the paper is as follows. In Section 2, the discretized system is presented. In Section 3, we employ the approximate inverse circulant-plus-diagonal preconditioners and construct new preconditioners based on HSS for the discretized system. Theoretically, we show that the spectra of the resulting preconditioned matrices lie in the interior of the disk of radius 1 centered at the point (1, 0). In Section 4, numerical examples are given to show the performances of these preconditioners. Finally, concluding remarks are presented in Section 5.

2. Discretization of the Space Fractional CNLS Equations

Let $\tau = T/N$ and $h = (b - a)/(M + 1)$ be the sizes of time step and spatial grid, respectively, where $N$ and $M$ are positive integers. We define a temporal and spatial partition $t_n = n\tau$ for $n = 0, 1, \cdots, N$ and $x_j = a + jh$ for $j = 0, 1, \cdots, M + 1$. Let $u_j^n \approx u(x_j, t_n)$ and $v_j^n \approx v(x_j, t_n)$ denote the corresponding numerical solutions. By the fractional centered difference formula [18], we can discrete the fractional Laplacian $(-\Delta)^{\alpha/2}$ in the truncated bounded domain as

$$(-\Delta)^{\alpha/2}u(x_j) = -\frac{\partial^\alpha}{\partial |x|^\alpha}u(x_j) = \frac{1}{h^\alpha} \sum_{k=1}^{M} c_{j-k}u_k + O(h^2),$$

where the coefficients

$$c_k = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\alpha/2 - k + 1) \Gamma(\alpha/2 + k + 1)}$$

with $\Gamma(\cdot)$ being the gamma function. Moreover, the coefficients $c_k$ satisfy the following properties:

$$c_0 \geq 0; \quad c_k = c_{-k} \leq 0, \quad k = 1, 2, \cdots; \quad \sum_{k=0}^{\infty} |c_k| = c_0.$$

They proposed the following implicit difference scheme for the space fractional CNLS equations (1.1):

$$
\begin{cases}
\frac{i}{2\tau} (u_j^{n+1} - u_j^n) + \frac{\mu}{h^2} \sum_{k=1}^{M} c_{j-k} (u_j^{n+1} + u_j^{n+1}) + \rho |u_j^n|^2 + \beta |v_j^n|^2 \frac{u_j^{n+1} + u_j^{n-1}}{2} = 0, \\
\frac{i}{2\tau} (v_j^{n+1} - v_j^n) + \frac{\mu}{h^2} \sum_{k=1}^{M} c_{j-k} (v_j^{n+1} + v_j^{n+1}) + \rho |v_j^n|^2 + \beta |u_j^n|^2 \frac{v_j^{n+1} + v_j^{n-1}}{2} = 0,
\end{cases}
$$

(2.1)
where \( j = 1, 2, \cdots, M, n = 1, 2, \cdots, N - 1 \), and proved that the scheme is unconditionally stable [21–23]. By the initial boundary value conditions, we have \( u^n_0 = u_0(x_j), v^n_0 = v_0(x_j) \), \( u^n_0 = u^n_{M+1} = 0, v^n_0 = v^n_{M+1} = 0 \). In addition, the first step can be obtained by some second or higher order time integrators. The structure of the first difference equation in (2.1) is the same as the second one. Denote

\[
\begin{align*}
\mathbf{u}^{n+1} &= \left[ u^{n+1}_1, \cdots, u^{n+1}_M \right]^T, \\
\mathbf{b}^{n+1} &= \left[ b^{n+1}_1, \cdots, b^{n+1}_M \right]^T,
\end{align*}
\]

where

\[
\begin{align*}
\mu &= \frac{\gamma \tau}{h^2}, \\
d_j^{n+1} &= \rho \tau (|u_j^n|^2 + \beta |v_j^n|^2),
\end{align*}
\]

Then we can rewrite the first difference scheme in (2.1) into the following matrix vector form:

\[
A^{n+1} \mathbf{u}^{n+1} = \mathbf{b}^{n+1}, \quad n = 1, 2, \cdots, N - 1,
\]

where the coefficient matrix \( A^{n+1} \) is of the form

\[
A^{n+1} = D^{n+1} + T + iI.
\]

Here \( D^{n+1} \) is the diagonal matrix defined by \( D^{n+1} = \text{diag}(d_1^{n+1}, d_2^{n+1}, \cdots, d_M^{n+1}) \), \( I \) is the identity matrix and \( T \) is the Toeplitz matrix

\[
T = \mu \begin{pmatrix}
c_0 & c_{-1} & \cdots & c_{M-2} & c_{M-1} \\
c_1 & c_0 & \cdots & c_{M-3} & c_{M-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{M-2} & c_{M-3} & \cdots & c_0 & c_{-1} \\
c_{M-1} & c_{M-2} & \cdots & c_1 & c_0
\end{pmatrix}.
\]

From the fact \( \gamma > 0, \rho > 0, \beta \geq 0 \) and the properties of the coefficients \( c_k \) we see that the Toeplitz matrix \( T \) is symmetric strictly diagonally dominant, then symmetric positive definite, and \( D^{n+1} \) is nonnegative diagonal matrix. Thus, the matrix \( D^{n+1} + T \) is symmetric positive definite. Based on these facts, the coefficient matrix \( A^{n+1} \) is non-Hermitian positive definite.

### 3. Construction of the Preconditioners

In the following, we consider the iterative solution of the system of linear equations

\[
Au = b, \quad A \in \mathbb{C}^{M \times M} \text{ nonsingular}, \quad \text{and} \quad u, b \in \mathbb{C}^M,
\]

where \( A \) is a complex symmetric matrix of the form

\[
A = D + T + iI
\]
and \( i = \sqrt{-1} \) is the imaginary unit, \( I \) is the identity matrix, \( D = \text{diag}(d_1, \cdots, d_M) \) is the nonnegative diagonal matrix with \( d_j \geq 0, j = 1, \cdots, M \), \( T \) is the symmetric positive definite Toeplitz matrix defined in (2.3). Since \( A \) is non-Hermitian, we can use Krylov subspace methods, such as BiCGSTAB, to solve the linear system (3.1). In order to speed up its convergence rate, preconditioning technique is often employed. It is widely recognized that preconditioning is the most critical ingredient in the development of efficient solvers for challenging problems in scientific computations. In the following, we consider preconditioners for the matrix \( A \).

The approximate inverse circulant-plus-diagonal preconditioners proposed in [16] for solving non-Hermitian Toeplitz-plus-diagonal system (3.1) are constructed as follows. Define \( K_j = d_jI + T + il, j = 1, 2, \cdots, M \). Clearly, all \( K_j \) are Toeplitz matrices. According to the fact that \( e_j^\top A = e_j^\top K_j \), the first preconditioner is based on the following approximation \( e_j^\top A^{-1} = e_j^\top K_j^{-1} \), where \( e_j \) denotes the \( j \)-th column of the identity matrix. This means that the \( j \)-th row of the inverse of \( A \) is approximated by the \( j \)-th row of the inverse of \( K_j \). Therefore, the following preconditioner \( P_T \) whose inverse is defined by

\[
P_T^{-1} = \sum_{j=1}^{M} e_j e_j^\top K_j^{-1}.
\]

To construct \( P_T^{-1} \), we need to compute the inverse of \( K_j \) \( (j = 1, 2, \cdots, M) \), which is impractical. However, as \( K_j \) is a Toeplitz matrix, we can approximate \( K_j \) by a circulant matrix. Let \( C \) be the Strang circulant approximation [10] of the Toeplitz matrix \( T \), that is, the first column of the circulant matrix \( C \) is given by

\[
\begin{cases}
  \mu[c_0, c_1, \cdots, c_{M-1}, 0, c_{-M+1}, \cdots, c_{-1}]^\top, & \text{if } M \text{ is even}, \\
  \mu[c_0, c_1, \cdots, c_{M-1}, c_{-M+1}, \cdots, c_{-1}]^\top, & \text{if } M \text{ is odd}.
\end{cases}
\]

(3.3)

Let \( C_j = d_jI + C + il, j = 1, 2, \cdots, M \). Then the approximate inverse circulant-plus-diagonal preconditioner \( P_C \) is

\[
P_C^{-1} = \sum_{j=1}^{M} e_j e_j^\top C_j^{-1}.
\]

It is well known that circulant matrices can be diagonalized in \( \mathcal{O}(M \log M) \) operations by making use of FFT. Hence the product \( C_j^{-1} r \) for any vector \( r \) can be computed by FFT in \( \mathcal{O}(M \log M) \) operations.

Based on the HSS preconditioning technique, we construct the following new preconditioners. The Hermitian and skew-Hermitian parts of \( A \) in (3.2) are given by \( H = D + T \) and \( S = ilI \), respectively. Obviously, the Hermitian part \( H \) is symmetric positive definite and the skew-Hermitian part \( S \) is diagonal. The HSS iteration method [4] and its relevant iteration methods [1–3, 5, 6] were proposed. The HSS iteration method for solving the non-Hermitian positive definite linear system (3.1) is as follows:

\[
\begin{align*}
\left( \tilde{a}I + D + T \right) u^{(k+\frac{1}{2})} &= (\tilde{a}I - il) u^{(k)} + b, \\
\left( \tilde{a}I + ilI \right) u^{(k+1)} &= (\tilde{a}I - D - T) u^{(k+\frac{1}{2})} + b,
\end{align*}
\]
where \( \bar{a} \) is a given positive constant and \( I \) is the identity matrix. Induced from the HSS iteration method, the HSS preconditioner is

\[
P_H = \frac{1}{2\bar{a}}(\bar{a}I + D + T)(\bar{a}I + iI) \equiv \frac{\bar{a} + i}{2\bar{a}}(\bar{a}I + D + T).
\]

Since the coefficient matrix \( A \) is non-Hermitian positive definite, we know that the HSS iteration method converges to the unique solution of the system of linear equations (3.1) for any positive constant \( \bar{a} \). Then, all the eigenvalues of the HSS preconditioned matrix \( P_H^{-1}A \) lie in the interior of the disk of radius 1 centered at the point \((1,0)\). In particular, the preconditioned matrix is positive stable. As is well known, these are desirable properties to fast convergent rate of the Krylov subspace methods like BiCGSTAB. Moreover, if \( \lambda_{\min} \) and \( \lambda_{\max} \) are the minimum and the maximum eigenvalues of the symmetric positive definite matrix \( D + T \), respectively, then the optimal parameter \( \bar{a}^* \) involved in the preconditioner \( P_H \) is \( \bar{a}^* = \sqrt{\lambda_{\min} \lambda_{\max}} \). We emphasize that, the optimal parameter \( \bar{a}^* \) minimizes the spectral radius of the iteration matrix of the HSS iteration itself and thus is really the optimal parameter in theory. For any given vector \( r \), the residual equation \( P_H w = r \) in the preconditioned BiCGSTAB iteration method can be solved either exactly by a Cholesky factorization or inexacty by some conjugate gradient or multigrid scheme. In order to reduce computation complexity, we can further approximate \( \bar{a}I + D + T \) involved in the HSS preconditioner \( P_H \) by employing its unsymmetric Gauss-Seidel (UGS) splitting, and we can obtain the following two-stage preconditioner:

\[
P_{H-UGS} = \frac{\bar{a} + i}{2\bar{a}}(\bar{D} + \bar{L})\bar{D}^{-1}(\bar{D} + \bar{U}),
\]

where \( \bar{D}, \bar{L} \) and \( \bar{U} \) being the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix \( \bar{a}I + D + T \), respectively.

Obviously, at each step of the HSS iteration for the linear system (3.1) we need to solve the linear sub-system with its coefficient matrix being the symmetric positive definite one \( \bar{a}I + D + T \). Note that the coefficient matrix \( \bar{a}I + D + T \) is the Toeplitz-like matrix which is equal to the diagonal-plus-Toeplitz matrix. Unfortunately, unlike the case of the Toeplitz systems, no fast direct solvers have been developed for solving diagonal-plus-Toeplitz systems. In a word, the first-half step of the HSS iteration which finds solution \( u^{(k+\frac{1}{2})} \) with matrix \( \bar{a}I + D + T \) being large and full, although being symmetric positive definite, is very costly and impractical in actual implementations. In order to overcome the disadvantage of the HSS iteration, the HSS-like iteration method [20] is proposed:

\[
\begin{align*}
(\bar{a}I + T)u^{(k+\frac{1}{2})} &= (\bar{a}I - D - iI)u^{(k)} + b, \\
(\bar{a}I + D + iI)u^{(k+1)} &= (\bar{a}I - T)u^{(k+\frac{1}{2})} + b.
\end{align*}
\]

Concerning the convergence of the HSS-like iteration method, we have the following theorem.

**Theorem 3.1.** Let \( A \in \mathbb{C}^{M \times M} \) be defined in (3.2), \( D + T \) and \( il \) be its Hermitian and skew-Hermitian parts, respectively, and \( \bar{a} \) be a positive constant. Then the iteration matrix \( M(\bar{a}) \) of
Thus, the HSS-like iteration converges to the unique solution of the system of linear equations

\[ M(\tilde{a}) = (\tilde{a}I + D + iI)^{-1}(\tilde{a}I - T)(\tilde{a}I + T)^{-1}(\tilde{a}I - D - iI), \]

and its spectral radius \( \rho(M(\tilde{a})) \) is bounded by

\[
\sigma(\tilde{a}) \equiv \max_{\lambda_j \in \lambda(T)} \frac{\tilde{a} - \lambda_j}{\tilde{a} + \lambda_j} \cdot \max_{1 \leq j \leq M} \sqrt{\frac{(\tilde{a} - d_j)^2 + 1}{(\tilde{a} + d_j)^2 + 1}},
\]

where \( \lambda(T) \) is the spectral set of the matrix \( T \). Therefore, it holds that

\[
\rho(M(\tilde{a})) \leq \sigma(\tilde{a}) < 1, \quad \forall \tilde{a} > 0.
\]

Thus, the HSS-like iteration converges to the unique solution of the system of linear equations (3.1).

Induced from the HSS-like method, the HSS-like preconditioner \( P_{HL} \) is

\[
P_{HL} = \frac{1}{2\tilde{a}}(\tilde{a}I + T)(\tilde{a}I + D + iI).
\]

Obviously, the HSS-like preconditioner \( P_{HL} \) is the Toeplitz-times-diagonal matrix. Thus, for any given vector \( r \), the residual equation \( P_{HL}w = r \) in the preconditioned BiCGSTAB iteration method can be solved by the fast direct methods. The same as the HSS preconditioned matrix, all eigenvalues of the HSS-like preconditioned matrix \( P_{HL}^{-1}A \) lie in the interior of the disk of radius 1 centered at the point \((1,0)\). Then, HSS-like preconditioner \( P_{HL} \) is a good preconditioner for the matrix \( A \). From Theorem 3.1, we have

\[
\sigma(\tilde{a}) \equiv \max_{\lambda_j \in \lambda(T)} \frac{\tilde{a} - \lambda_j}{\tilde{a} + \lambda_j} \cdot \max_{1 \leq j \leq M} \sqrt{\frac{(\tilde{a} - d_j)^2 + 1}{(\tilde{a} + d_j)^2 + 1}}
\]

\[
\leq \left\{ \begin{array}{l}
\max_{\lambda_j \in \lambda(T)} \frac{\tilde{a} - \lambda_j}{\tilde{a} + \lambda_j} = \sigma_1(\tilde{a}), \\
\max_{\lambda_j \in \lambda(T)} \sqrt{\frac{(\tilde{a} - d_j)^2 + 1}{(\tilde{a} + d_j)^2 + 1}} \cdot \max_{1 \leq j \leq M} \sqrt{\frac{(\tilde{a} - d_j)^2 + 1}{(\tilde{a} + d_j)^2 + 1}} = \sigma_2(\tilde{a}).
\end{array} \right.
\]

Thus, the spectral radius \( \rho(M(\tilde{a})) \) are also bounded by \( \sigma_1(\tilde{a}) \) and \( \sigma_2(\tilde{a}) \). We remark that if the minimum and the maximum eigenvalues of the symmetric positive definite Toeplitz matrix \( T \) are known, then the optimal parameters for the upper bound \( \sigma_1(\tilde{a}) \) and \( \sigma_2(\tilde{a}) \) can be obtained, respectively. These facts are precisely stated as the following theorems.

**Theorem 3.2.** Let \( A \in \mathbb{C}^{M \times M} \) be defined in (3.2), let \( \lambda_{\min} \) and \( \lambda_{\max} \) be the minimum and the maximum eigenvalues of the symmetric positive definite matrix \( T \), respectively, and let \( \bar{a} \) be a positive constant, then

\[
\tilde{a}_* \equiv \arg \min_{\tilde{a}} \left\{ \max_{\lambda_{\min} \leq \lambda \leq \lambda_{\max}} \left| \frac{\tilde{a} - \lambda}{\tilde{a} + \lambda} \right| = \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} \right\} = \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}}.
\]
and the corresponding upper bound is

\[ \sigma_1(\tilde{\alpha}_*) = \frac{\sqrt{\lambda_{\text{max}}^2 + \sqrt{\lambda_{\text{min}}^2}}}{\sqrt{\lambda_{\text{max}}^2 - \sqrt{\lambda_{\text{min}}^2}}} = \frac{\kappa(T) - 1}{\kappa(T) + 1}, \]

where \( \kappa(T) \) is the spectral condition number of \( T \).

**Theorem 3.3.** Let \( A \in \mathbb{C}^{M \times M} \) be defined in (3.2), let \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) be the minimum and the maximum eigenvalues of the symmetric positive definite matrix \( T \), respectively, and \( \delta_{\text{min}} = \min\{\lambda_{\text{min}}, d_1, d_2, \ldots, d_M\} \), \( \delta_{\text{max}} = \max\{\lambda_{\text{max}}, d_1, d_2, \ldots, d_M\} \), and let \( \tilde{\alpha} \) be a positive constant, then

\[
\tilde{\alpha}_* = \arg \min_{\tilde{\alpha}} \left\{ \frac{\max_{\delta_{\text{min}} \leq \delta \leq \delta_{\text{max}}} (\tilde{\alpha} - \delta)^2 + 1}{\tilde{\alpha} + \delta} \right\} = \begin{cases} \sqrt{\delta_{\text{min}}^2 \delta_{\text{max}}^2 - 1}, & \text{for } 1 < \sqrt{\delta_{\text{min}}^2 \delta_{\text{max}}^2}, \\ \frac{\sqrt{\delta_{\text{min}}^2 + 1 - \delta_{\text{min}}}}{\sqrt{\delta_{\text{min}}^2 + 1 + \delta_{\text{min}}}}, & \text{for } 1 \geq \sqrt{\delta_{\text{min}}^2 \delta_{\text{max}}^2}. \end{cases}
\]

and the corresponding upper bound is

\[
\sigma_2(\tilde{\alpha}_*) = \begin{cases} \frac{\delta_{\text{min}}^2 + \delta_{\text{max}}^2 - 2 \sqrt{\delta_{\text{min}}^2 \delta_{\text{max}}^2 - 1}}{\delta_{\text{min}}^2 + \delta_{\text{max}}^2 + 2 \sqrt{\delta_{\text{min}}^2 \delta_{\text{max}}^2 - 1}}, & \text{for } 1 < \sqrt{\delta_{\text{min}}^2 \delta_{\text{max}}^2}, \\ \frac{\sqrt{\delta_{\text{min}}^2 + 1 - \delta_{\text{min}}}}{\sqrt{\delta_{\text{min}}^2 + 1 + \delta_{\text{min}}}}, & \text{for } 1 \geq \sqrt{\delta_{\text{min}}^2 \delta_{\text{max}}^2}. \end{cases}
\]

The proofs of Theorem 3.2 and Theorem 3.3 can be found in [4]. We emphasize that the optimal parameters \( \tilde{\alpha} \) and \( \tilde{\alpha}_* \) minimize only the upper bounds \( \sigma_1(\tilde{\alpha}) \) and \( \sigma_2(\tilde{\alpha}) \) of the spectral radius of the iteration matrix but does not minimize the spectral radius itself.

Note that \( \tilde{\alpha} I + T \) and \( \tilde{\alpha} I + D + i I \) are the Toeplitz matrix and the diagonal matrix, respectively. We can further approximate the Toeplitz matrix \( \tilde{\alpha} I + T \) by the circulant matrix \( \tilde{\alpha} I + C \), where \( C \) is the Strang circulant approximation of the Toeplitz matrix \( T \) defined in (3.3). Then we obtain another new approximate preconditioner \( P_{\text{HL-C}} \) with

\[
P_{\text{HL-C}} = \frac{1}{2\tilde{\alpha}} (\tilde{\alpha} I + C)(\tilde{\alpha} I + D + i I).
\]

Thus, for any given vector \( r \), the residual equation \( P_{\text{HL-C}}w = r \) in the preconditioned BiCGSTAB iteration method can be very efficiently solved by making use of IFFT in \( \mathcal{O}(M \log M) \) operations.

### 4. Numerical Examples

In this section, we carry out numerical experiments to study the performance of the proposed preconditioners \( P_C, P_H, P_{\text{HL-UGS}}, P_{\text{HL}}, \) and \( P_{\text{HL-C}} \). We employ the preconditioned BiCGSTAB methods to solve the linear system (3.1). All parameters are set to be the experimentally optimal ones. All numerical experiments are started from the zero vector,
Table 1: Numerical results for Example 4.1 when $\alpha = 1.3$.

<table>
<thead>
<tr>
<th>$M$</th>
<th>Iter</th>
<th>CPU</th>
<th>Iter</th>
<th>CPU</th>
<th>Iter</th>
<th>CPU</th>
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<td>0.17</td>
<td>5</td>
<td>0.08</td>
<td>4</td>
<td>0.06</td>
<td>4</td>
<td>0.08</td>
<td>4</td>
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Table 2: Numerical results for Example 4.1 when $\alpha = 1.6$.

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<th>CPU</th>
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<tr>
<td>1600</td>
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<td>0.86</td>
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<tr>
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<td>9.66</td>
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<td>2.16</td>
<td>5</td>
<td>3.49</td>
<td>5</td>
<td>2.08</td>
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performed in MATLAB with machine precision $10^{-16}$, and terminated when the current iterate satisfies

$$\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6},$$

where $r^{(k)}$ is the residual vector of the $k$th iteration and $r^{(0)}$ is the initial residual vector.

**Example 4.1.** Let $\gamma = 1$, $\rho = 2$, $\beta = 0$, $1 < \alpha < 2$. Then the system (1.1) is decoupled and becomes

$$iu_t + (-\Delta)^{\frac{\alpha}{2}}u + 2|u|^2u = 0, \quad -20 \leq x \leq 20, \quad 0 < t \leq 2,$$

subjected to the initial boundary value conditions $u(x, 0) = \text{sech}(x) \cdot \exp(2ix)$, $u(-20, t) = u(20, t) = 0$.

**Example 4.2.** For the following coupled system with $\gamma = 1$, $\rho = 2$, $\beta = 1$, $1 < \alpha < 2$,

$$\begin{cases} 
i u_t + (-\Delta)^{\frac{\alpha}{2}}u + 2(|u|^2 + |v|^2)u = 0, & -20 \leq x \leq 20, \quad 0 < t \leq 2, \\
i v_t + (-\Delta)^{\frac{\alpha}{2}}v + 2(|v|^2 + |u|^2)v = 0, & -20 \leq x \leq 20, \quad 0 < t \leq 2, \end{cases}$$

we take the initial boundary value conditions in the form

$$\begin{cases} 
u(x, 0) = \text{sech}(x + 1) \cdot \exp(2ix), & v(x, 0) = \text{sech}(x - 1) \cdot \exp(-2ix), \\
u(-20, t) = v(20, t) = 0, & v(-20, t) = v(20, t) = 0. \end{cases}$$

The numerical results are listed in Tables 1–4, where “$M$” denotes the number of spatial grid points, “Iter” denotes the iteration steps of all the iteration methods, “CPU” denotes
the total CPU time in seconds for solving the discretized system, and “$\alpha$” denotes the order of the space fractional CNLS equations.

In Tables 1–2 and Tables 3–4 we list the iteration steps and the CPU time in seconds for the BiCGSTAB method with or without using a preconditioner for the Examples 4.1 and 4.2 when $\alpha = 1.3$ and $\alpha = 1.6$ with respect to different sizes of spatial grid. The time step sizes for all numerical experiments are set to be 0.01. In these tables, the symbol “I” indicates that no preconditioner is used when solving the system of linear equations (3.1), while “$P_C$”, “$P_H$”, “$P_{H-UGS}$”, “$P_{HL}$” and “$P_{HL-C}$” denote the BiCGSTAB method with the preconditioners $P_C$, $P_H$, $P_{H-UGS}$, $P_{HL}$ and $P_{HL-C}$, respectively.

From these tables we see that the iteration step and the CPU time of the vanilla BiCGSTAB method are very large and are monotonically increasing with respect to $M$, but those of the preconditioned BiCGSTAB methods are considerably small which keep almost a constant, especially when the spatial grid becomes fine. To achieve the prescribed convergence criterion, the $P_C$-preconditioned BiCGSTAB method requires less number of iteration steps but more computing time than preconditioned BiCGSTAB methods with preconditioners $P_H$, $P_{H-UGS}$, $P_{HL}$, $P_{HL-C}$ based on HSS. The number of iteration steps of the $P_H$, $P_{H-UGS}$, $P_{HL}$, and $P_{HL-C}$-preconditioned BiCGSTAB methods are almost same when the spatial grid becomes fine. Among all the iterations, the $P_{HL-C}$-preconditioned BiCGSTAB method is the most effective in terms of CPU time, and the $P_{H-UGS}$-preconditioned BiCGSTAB method is the second most effective in terms of CPU time. We emphasize that the preconditioners $P_H$, $P_{H-UGS}$, $P_{HL}$, $P_{HL-C}$ based on HSS outperform the approximate inverse circulant-plus-diagonal preconditioner $P_C$ in terms of the CPU time.

<table>
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<tr>
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<th>Iter</th>
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<th>Iter</th>
<th>CPU</th>
<th>Iter</th>
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5. Concluding Remarks

In this paper, we have considered discretized linear systems arising from the space fractional CNLS equations. The coefficient matrix of the discretized linear system is equal to the sum of a complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and a symmetric Toeplitz-plus-diagonal matrix. The main contribution of this paper is to develop matrix splitting and approximate preconditioners based on HSS for these Toeplitz-like matrices so that BiCGSTAB method for solving these preconditioned systems can converge very quickly. We have presented numerical examples and have shown that the proposed preconditioners are very effective and efficient.

Acknowledgments

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References


