East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.190716.051116b

Vol. 7, No. 1, pp. 70-81
February 2017

On Preconditioners Based on HSS for the Space Fractional CNLS Equations

Yu-Hong Ran[1,*], Jun-Gang Wang[2] and Dong-Ling Wang[1]

[1] *Center for Nonlinear Studies, School of Mathematics, Northwest University, Xi'an, Shaanxi 710127, China.*
[2] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China.*

Received 19 July 2016; Accepted (in revised version) 5 November 2016.

Abstract. The space fractional coupled nonlinear Schrödinger (CNLS) equations are discretized by an implicit conservative difference scheme with the fractional centered difference formula, which is unconditionally stable. The coefficient matrix of the discretized linear system is equal to the sum of a complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and a symmetric Toeplitz-plus-diagonal matrix. In this paper, we present new preconditioners based on Hermitian and skew-Hermitian splitting (HSS) for such Toeplitz-like matrix. Theoretically, we show that all the eigenvalues of the resulting preconditioned matrices lie in the interior of the disk of radius 1 centered at the point $(1,0)$. Thus Krylov subspace methods with the proposed preconditioners converge very fast. Numerical examples are given to illustrate the effectiveness of the proposed preconditioners.

AMS subject classifications: 65F10, 65F15

Key words: The space fractional Schrödinger equations, Toeplitz matrix, Hermitian and skew-Hermitian splitting, preconditioner, Krylov subspace methods.

1. Introduction

The classical Schrödinger equations describe the evolution of microscopic particles, and they can be derived from the path integral over the Brownian motion. Laskin [12] generalized the path integral method from the Brownian motion to the Lévy-α process to obtain the space fractional Schrödinger equations [13].

In this paper, we consider the space fractional coupled nonlinear Schrödinger (CNLS) equations

$$\begin{cases} iu_t+\gamma(-\Delta)^{\frac{\alpha}{2}}u+\rho\big(|u|^2+\beta|v|^2\big)u=0, \\ iv_t+\gamma(-\Delta)^{\frac{\alpha}{2}}v+\rho\big(|v|^2+\beta|u|^2\big)v=0, \end{cases} \quad a\le x\le b, \quad 0<t\le T, \tag{1.1}$$

*Corresponding author. *Email address:* ranyh@nwu.edu.cn (Y.-H. Ran)

with the initial boundary value conditions

$$\begin{cases} u(x,0)=u_0(x), & v(x,0)=v_0(x), & a\le x\le b,\\ u(a,t)=u(b,t)=0, & v(a,t)=v(b,t)=0, & 0\le t\le T, \end{cases}$$

where $i=\sqrt{-1}$, $1<\alpha<2$ and the parameters $\gamma>0$, $\rho>0$, $\beta\ge 0$ are constants. The fractional Laplacian [11] can be characterized as

$$(-\Delta)^{\frac{\alpha}{2}}u(x,t)=\mathscr{F}^{-1}\Big(|\xi|^{\alpha}\mathscr{F}\big(u(x,t)\big)\Big),$$

where $\mathscr{F}$ is the Fourier transform acting on the spatial variable x. Furthermore, it is shown that the Riesz fractional derivative [24] can also be defined as

$$\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,t)=-(-\Delta)^{\frac{\alpha}{2}}u(x,t)=-\frac{1}{2\cos\frac{\pi\alpha}{2}}\Big[{}_{-\infty}D_x^{\alpha}u(x,t)+{}_xD_{+\infty}^{\alpha}u(x,t)\Big],$$

where ${}_{-\infty}D_x^{\alpha}u(x,t)$ and ${}_xD_{+\infty}^{\alpha}u(x,t)$ are the left and right Riemann-Liouville derivatives, respectively. when $\alpha=2$, the system (1.1) is reduced to the classical CNLS equations, which describe a wide class of physical nonlinear phenomena, such as the hydrodynamics, the nonlinear optics and the dynamics of the two-component Bose-Einstein condensate.

Generally, closed-form analytical solutions of the space fractional CNLS equations are not available. Consequently, the numerical methods become important and powerful which are very few until now. Recently, based on the fractional centered difference formula, in [21–23] the authors proposed an implicit conservative difference scheme to discretize the space fractional CNLS equations which is unconditionally stable. The coefficient matrix of the discretized linear system is equal to the sum of the complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and the symmetric Toeplitz-plus-diagonal matrix. As the coefficient matrix is non-Hermitian Toeplitz-like, we can employ Krylov subspace methods, such as BiCGSTAB, to solve the discretized linear system. Using the fast Fourier transform (FFT), the Toeplitz matrix-vector multiplication can be done in $\mathcal{O}(M\log M)$ operations, where M is the number of grid points. Nevertheless, the resulting system in general is ill-conditioned and the convergence rates of Krylov subspace methods tend to be considerably worse. In order to improve their performance and reliability, preconditioning is a common technique.

Circulant preconditioners for Toeplitz matrices have been theoretically and numerically studied with numerous applications for over twenty years; see [7, 8, 15]. However, circulant preconditioners do not work for such Toeplitz-plus-diagonal systems. Chan and Ng [9] considered banded preconditioners for Toeplitz-plus-band systems. The main drawback of this method is that the generating function should be known in order to construct effective banded preconditioners. In general, the generating function of the corresponding Toeplitz matrix is unknown. In [17], multigrid methods are studied for Toeplitz-plus-diagonal linear systems arising from Sinc-Galerkin methods. As the generating functions are known, the proposed multigrid method is to incorporate the diagonal matrix into the interpolating process. Lately, Ng and Pan [16] proposed approximate inverse circulant-plus-diagonal preconditioners for solving Hermitian positive definite Toeplitz-plus-diagonal systems. Their

idea is to use circulant matrices to approximate the inverses of Toeplitz matrices and then combine these circulant matrices together. Motivated by this idea of construction of preconditioners, Lei and Sun [14] and Pan et al. [19] presented circulant preconditioners and approximate inverse preconditioners for diagonal-times-Toeplitz matrices in fractional diffusion equations.

In this paper, we firstly employ the approximate inverse circulant-plus-diagonal preconditioners, and then propose and develop new preconditioners based on Hermitian and skew-Hermitian splitting (HSS) for solving the Toeplitz-like system arising from the discretization of the space fractional CNLS equations. The outline of the rest of the paper is as follows. In Section 2, the discretized system is presented. In Section 3, we employ the approximate inverse circulant-plus-diagonal preconditioners and construct new preconditioners based on HSS for the discretized system. Theoretically, we show that the spectra of the resulting preconditioned matrices lie in the interior of the disk of radius 1 centered at the point $(1,0)$. In Section 4, numerical examples are given to show the performances of these preconditioners. Finally, concluding remarks are presented in Section 5.

2. Discretization of the Space Fractional CNLS Equations

Let $\tau = T/N$ and $h = (b-a)/(M+1)$ be the sizes of time step and spatial grid, respectively, where N and M are positive integers. We define a temporal and spatial partition $t_n = n\tau$ for $n = 0,1,\cdots,N$ and $x_j = a + jh$ for $j = 0,1,\cdots,M+1$. Let $u_j^n \approx u(x_j,t_n)$ and $v_j^n \approx v(x_j,t_n)$ denote the corresponding numerical solutions. By the fractional centered difference formula [18], we can discrete the fractional Laplacian $(-\Delta)^{\alpha/2}$ in the truncated bounded domain as

$$(-\Delta)^{\frac{\alpha}{2}}u(x_j) = -\frac{\partial^\alpha}{\partial|x|^\alpha}u(x_j) = \frac{1}{h^\alpha}\sum_{k=1}^{M} c_{j-k}u_k + \mathcal{O}(h^2),$$

where the coefficients

$$c_k = \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}$$

with $\Gamma(\cdot)$ being the gamma function. Moreover, the coefficients c_k satisfy the following properties:

$$c_0 \geq 0; \quad c_k = c_{-k} \leq 0, \ \ k = 1,2,\cdots; \quad \sum_{k=-\infty,k\neq 0}^{+\infty} |c_k| = c_0.$$

They proposed the following implicit difference scheme for the space fractional CNLS equations (1.1):

$$\begin{cases} i\frac{u_j^{n+1}-u_j^{n-1}}{2\tau} + \frac{\gamma}{h^\alpha}\sum\limits_{k=1}^{M} c_{j-k}\frac{u_k^{n+1}+u_k^{n-1}}{2} + \rho\big(|u_j^n|^2+\beta|v_j^n|^2\big)\frac{u_j^{n+1}+u_j^{n-1}}{2} = 0, \\ i\frac{v_j^{n+1}-v_j^{n-1}}{2\tau} + \frac{\gamma}{h^\alpha}\sum\limits_{k=1}^{M} c_{j-k}\frac{v_k^{n+1}+v_k^{n-1}}{2} + \rho\big(|v_j^n|^2+\beta|u_j^n|^2\big)\frac{v_j^{n+1}+v_j^{n-1}}{2} = 0, \end{cases} \tag{2.1}$$

where $j = 1, 2, \cdots, M$, $n = 1, 2, \cdots, N-1$, and proved that the scheme is unconditionally stable [21–23]. By the initial boundary value conditions, we have $u_j^0 = u_0(x_j)$, $v_j^0 = v_0(x_j)$, $u_0^n = u_{M+1}^n = 0$, $v_0^n = v_{M+1}^n = 0$. In addition, the first step can be obtained by some second or higher order time integrators. The structure of the first difference equation in (2.1) is the same as the second one. Denote

$$u^{n+1} = \left[u_1^{n+1}, \cdots, u_M^{n+1}\right]^{\mathrm{T}}, \qquad b^{n+1} = \left[b_1^{n+1}, \cdots, b_M^{n+1}\right]^{\mathrm{T}},$$
$$\mu = \frac{\gamma\tau}{h^\alpha}, \qquad d_j^{n+1} = \rho\tau\left(|u_j^n|^2 + \beta|v_j^n|^2\right),$$

where

$$b_j^{n+1} = iu_j^{n-1} - \mu\sum_{k=1}^{M} c_{j-k}u_k^{n-1} - d_j^{n+1}u_j^{n-1}, \quad j = 1, 2, \cdots, M.$$

Then we can rewrite the first difference scheme in (2.1) into the following matrix vector form:

$$A^{n+1}u^{n+1} = b^{n+1}, \quad n = 1, 2, \cdots, N-1, \tag{2.2}$$

where the coefficient matrix A^{n+1} is of the form

$$A^{n+1} = D^{n+1} + T + iI.$$

Here D^{n+1} is the diagonal matrix defined by $D^{n+1} = \mathrm{diag}(d_1^{n+1}, d_2^{n+1}, \cdots, d_M^{n+1})$, I is the identity matrix and T is the Toeplitz matrix

$$T = \mu\begin{pmatrix} c_0 & c_{-1} & \cdots & c_{2-M} & c_{1-M} \\ c_1 & c_0 & \cdots & c_{3-M} & c_{2-M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{M-2} & c_{M-3} & \cdots & c_0 & c_{-1} \\ c_{M-1} & c_{M-2} & \cdots & c_1 & c_0 \end{pmatrix}. \tag{2.3}$$

From the fact $\gamma > 0$, $\rho > 0$, $\beta \geq 0$ and the properties of the coefficients c_k we see that the Toeplitz matrix T is symmetric strictly diagonally dominant, then symmetric positive definite, and D^{n+1} is nonnegative diagonal matrix. Thus, the matrix $D^{n+1} + T$ is symmetric positive definite. Based on these facts, the coefficient matrix A^{n+1} is non-Hermitian positive definite.

3. Construction of the Preconditioners

In the following, we consider the iterative solution of the system of linear equations

$$Au = b, \quad A \in \mathbb{C}^{M\times M} \quad \text{nonsingular,} \quad \text{and} \quad u, b \in \mathbb{C}^M, \tag{3.1}$$

where A is a complex symmetric matrix of the form

$$A = D + T + iI \tag{3.2}$$

and $i=\sqrt{-1}$ is the imaginary unit, I is the identity matrix, $D=\mathrm{diag}(d_1,\cdots,d_M)$ is the nonnegative diagonal matrix with $d_j\geq 0, j=1,\cdots,M$, T is the symmetric positive definite Toeplitz matrix defined in (2.3). Since A is non-Hermitian, we can use Krylov subspace methods, such as BiCGSTAB, to solve the linear system (3.1). In order to speed up its convergence rate, preconditioning technique is often employed. It is widely recognized that preconditioning is the most critical ingredient in the development of efficient solvers for challenging problems in scientific computations. In the following, we consider preconditioners for the matrix A.

The approximate inverse circulant-plus-diagonal preconditioners proposed in [16] for solving non-Hermitian Toeplitz-plus-diagonal system (3.1) are constructed as follows. Define $K_j=d_jI+T+iI, j=1,2,\cdots,M$. Clearly, all K_j are Toeplitz matrices. According to the fact that $e_j^{\intercal}A=e_j^{\intercal}K_j$, the first preconditioner is based on the following approximation $e_j^{\intercal}A^{-1}=e_j^{\intercal}K_j^{-1}$, where e_j denotes the j-th column of the identity matrix. This means that the j-th row of the inverse of A is approximated by the j-th row of the inverse of K_j. Therefore, the following preconditioner P_{T} whose inverse is defined by

$$P_{\mathrm{T}}^{-1}=\sum_{j=1}^{M}e_je_j^{\intercal}K_j^{-1}.$$

To construct P_{T}^{-1}, we need to compute the inverse of K_j $(j=1,2,\cdots,M)$, which is impractical. However, as K_j is a Toeplitz matrix, we can approximate K_j by a circulant matrix. Let C be the Strang circulant approximation [10] of the Toeplitz matrix T, that is, the first column of the circulant matrix C is given by

$$\begin{cases}\mu\big[c_0,c_1,\cdots,c_{\frac{M}{2}-1},0,c_{-\frac{M}{2}+1},\cdots,c_{-1}\big]^{\mathrm{T}}, & \text{if } M \text{ is even},\\ \mu\big[c_0,c_1,\cdots,c_{\frac{M-1}{2}},c_{-\frac{M-1}{2}},\cdots,c_{-1}\big]^{\mathrm{T}}, & \text{if } M \text{ is odd}.\end{cases}\tag{3.3}$$

Let $C_j=d_jI+C+iI$, $j=1,2,\cdots,M$. Then the approximate inverse circulant-plus-diagonal preconditioner P_{C} is

$$P_{\mathrm{C}}^{-1}=\sum_{j=1}^{M}e_je_j^{\intercal}C_j^{-1}.$$

It is well known that circulant matrices can be diagonalized in $\mathcal{O}(M\log M)$ operations by making use of FFT. Hence the product $C_j^{-1}r$ for any vector r can be computed by FFT in $\mathcal{O}(M\log M)$ operations.

Based on the HSS preconditioning technique, we construct the following new preconditioners. The Hermitian and skew-Hermitian parts of A in (3.2) are given by $H=D+T$ and $S=iI$, respectively. Obviously, the Hermitian part H is symmetric positive definite and the skew-Hermitian part S is diagonal. The HSS iteration method [4] and its relevant iteration methods [1–3, 5, 6] were proposed. The HSS iteration method for solving the non-Hermitian positive definite linear system (3.1) is as follows:

$$\begin{cases}\big(\tilde{\alpha}I+D+T\big)u^{(k+\frac{1}{2})}=\big(\tilde{\alpha}I-iI\big)u^{(k)}+b,\\ \big(\tilde{\alpha}I+iI\big)u^{(k+1)}=\big(\tilde{\alpha}I-D-T\big)u^{(k+\frac{1}{2})}+b,\end{cases}$$

where $\tilde{\alpha}$ is a given positive constant and I is the identity matrix. Induced from the HSS iteration method, the HSS preconditioner is

$$P_{\mathrm{H}} = \frac{1}{2\tilde{\alpha}}(\tilde{\alpha}I + D + T)(\tilde{\alpha}I + iI) \equiv \frac{\tilde{\alpha}+i}{2\tilde{\alpha}}(\tilde{\alpha}I + D + T).$$

Since the coefficient matrix A is non-Hermitian positive definite, we know that the HSS iteration method converges to the unique solution of the system of linear equations (3.1) for any positive constant $\tilde{\alpha}$. Then, all the eigenvalues of the HSS preconditioned matrix $P_{\mathrm{H}}^{-1}A$ lie in the interior of the disk of radius 1 centered at the point $(1,0)$. In particular, the preconditioned matrix is positive stable. As is well known, these are desirable properties to fast convergent rate of the Krylov subspace methods like BiCGSTAB. Moreover, if $\tilde{\lambda}_{\min}$ and $\tilde{\lambda}_{\max}$ are the minimum and the maximum eigenvalues of the symmetric positive definite matrix $D+T$, respectively, then the optimal parameter $\tilde{\alpha}^*$ involved in the preconditioner P_{H} is $\tilde{\alpha}^* = \sqrt{\tilde{\lambda}_{\min}\tilde{\lambda}_{\max}}$. We emphasize that, the optimal parameter $\tilde{\alpha}^*$ minimizes the spectral radius of the iteration matrix of the HSS iteration itself and thus is really the optimal parameter in theory. For any given vector r, the residual equation $P_{\mathrm{H}}w = r$ in the preconditioned BiCGSTAB iteration method can be solved either exactly by a Cholesky factorization or inexactly by some conjugate gradient or multigrid scheme. In order to reduce computation complexity, we can further approximate $\tilde{\alpha}I + D + T$ involved in the HSS preconditioner P_{H} by employing its unsymmetric Gauss-Seidel (UGS) splitting, and we can obtain the following two-stage preconditioner:

$$P_{\mathrm{H-UGS}} = \frac{\tilde{\alpha}+i}{2\tilde{\alpha}}(\widetilde{D} + \widetilde{L})\widetilde{D}^{-1}(\widetilde{D} + \widetilde{U}),$$

where $\widetilde{D}, \widetilde{L}$ and $\widetilde{U}$ being the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix $\tilde{\alpha}I + D + T$, respectively.

Obviously, at each step of the HSS iteration for the linear system (3.1) we need to solve the linear sub-system with its coefficient matrix being the symmetric positive definite one $\tilde{\alpha}I + D + T$. Note that the coefficient matrix $\tilde{\alpha}I + D + T$ is the Toeplitz-like matrix which is equal to the diagonal-plus-Toeplitz matrix. Unfortunately, unlike the case of the Toeplitz systems, no fast direct solvers have been developed for solving diagonal-plus-Toeplitz systems. In a word, the first-half step of the HSS iteration which finds solution $u^{(k+\frac{1}{2})}$ with matrix $\tilde{\alpha}I + D + T$ being large and full, although being symmetric positive definite, is very costly and impractical in actual implementations. In order to overcome the disadvantage of the HSS iteration, the HSS-like iteration method [20] is proposed:

$$\begin{cases} \big(\tilde{\alpha}I + T\big)u^{(k+\frac{1}{2})} = \big(\tilde{\alpha}I - D - iI\big)u^{(k)} + b, \\ \big(\tilde{\alpha}I + D + iI\big)u^{(k+1)} = \big(\tilde{\alpha}I - T\big)u^{(k+\frac{1}{2})} + b. \end{cases}$$

Concerning the convergence of the HSS-like iteration method, we have the following theorem.

Theorem 3.1. *Let $A \in \mathbb{C}^{M\times M}$ be defined in* (3.2)*, $D+T$ and iI be its Hermitian and skew-Hermitian parts, respectively, and $\tilde{\alpha}$ be a positive constant. Then the iteration matrix $M(\tilde{\alpha})$ of*

the HSS-like iteration is given by

$$M(\tilde{\alpha}) = \big(\tilde{\alpha}I + D + iI\big)^{-1}\big(\tilde{\alpha}I - T\big)\big(\tilde{\alpha}I + T\big)^{-1}\big(\tilde{\alpha}I - D - iI\big),$$

and its spectral radius $\rho(M(\tilde{\alpha}))$ is bounded by

$$\sigma(\tilde{\alpha}) \equiv \max_{\lambda_j \in \lambda(T)} \left| \frac{\tilde{\alpha} - \lambda_j}{\tilde{\alpha} + \lambda_j} \right| \cdot \max_{1 \le j \le M} \sqrt{\frac{(\tilde{\alpha} - d_j)^2 + 1}{(\tilde{\alpha} + d_j)^2 + 1}},$$

where $\lambda(T)$ is the spectral set of the matrix T. Therefore, it holds that

$$\rho(M(\tilde{\alpha})) \le \sigma(\tilde{\alpha}) < 1, \quad \forall \tilde{\alpha} > 0.$$

Thus, the HSS-like iteration converges to the unique solution of the system of linear equations (3.1).

Induced from the HSS-like method, the HSS-like preconditioner P_{HL} is

$$P_{\mathrm{HL}} = \frac{1}{2\tilde{\alpha}}(\tilde{\alpha}I + T)(\tilde{\alpha}I + D + iI).$$

Obviously, the HSS-like preconditioner P_{HL} is the Toeplitz-times-diagonal matrix. Thus, for any given vector r, the residual equation $P_{\mathrm{HL}}w = r$ in the preconditioned BiCGSTAB iteration method can be solved by the fast direct methods. The same as the HSS preconditioned matrix, all eigenvalues of the HSS-like preconditioned matrix $P_{\mathrm{HL}}^{-1}A$ lie in the interior of the disk of radius 1 centered at the point $(1,0)$. Then, HSS-like preconditioner P_{HL} is a good preconditioner for the matrix A. From Theorem 3.1, we have

$$\begin{aligned}
\sigma(\tilde{\alpha}) &\equiv \max_{\lambda_j \in \lambda(T)} \left| \frac{\tilde{\alpha} - \lambda_j}{\tilde{\alpha} + \lambda_j} \right| \cdot \max_{1 \le j \le M} \sqrt{\frac{(\tilde{\alpha} - d_j)^2 + 1}{(\tilde{\alpha} + d_j)^2 + 1}} \\
&\le \begin{cases} \max\limits_{\lambda_j \in \lambda(T)} \left| \frac{\tilde{\alpha} - \lambda_j}{\tilde{\alpha} + \lambda_j} \right| \equiv \sigma_1(\tilde{\alpha}), \\ \max\limits_{\lambda_j \in \lambda(T)} \sqrt{\frac{(\tilde{\alpha} - \lambda_j)^2 + 1}{(\tilde{\alpha} + \lambda_j)^2 + 1}} \cdot \max\limits_{1 \le j \le M} \sqrt{\frac{(\tilde{\alpha} - d_j)^2 + 1}{(\tilde{\alpha} + d_j)^2 + 1}} \equiv \sigma_2(\tilde{\alpha}). \end{cases}
\end{aligned}$$

Thus, the spectral radius $\rho(M(\tilde{\alpha}))$ are also bounded by $\sigma_1(\tilde{\alpha})$ and $\sigma_2(\tilde{\alpha})$. We remark that if the minimum and the maximum eigenvalues of the symmetric positive definite Toeplitz matrix T are known, then the optimal parameters for the upper bound $\sigma_1(\tilde{\alpha})$ and $\sigma_2(\tilde{\alpha})$ can be obtained, respectively. These facts are precisely stated as the following theorems.

Theorem 3.2. *Let $A \in \mathbb{C}^{M \times M}$ be defined in* (3.2)*, let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the maximum eigenvalues of the symmetric positive definite matrix T, respectively, and let $\tilde{\alpha}$ be a positive constant, then*

$$\tilde{\alpha}_* \equiv \arg \min_{\tilde{\alpha}} \left\{ \max_{\lambda_{\min} \le \lambda \le \lambda_{\max}} \left| \frac{\tilde{\alpha} - \lambda}{\tilde{\alpha} + \lambda} \right| \right\} = \sqrt{\lambda_{\min}\lambda_{\max}},$$

and the corresponding upper bound is

$$\sigma_1(\tilde{\alpha}_*) = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}} = \frac{\sqrt{\kappa(T)} - 1}{\sqrt{\kappa(T)} + 1},$$

where $\kappa(T)$ is the spectral condition number of T.

Theorem 3.3. *Let $A \in \mathbb{C}^{M\times M}$ be defined in* (3.2)*, let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the maximum eigenvalues of the symmetric positive definite matrix T, respectively, and $\delta_{\min} = \min\{\lambda_{\min}, d_1, d_2, \cdots, d_M\}$, $\delta_{\max} = \max\{\lambda_{\max}, d_1, d_2, \cdots, d_M\}$, and let $\tilde{\alpha}$ be a positive constant, then*

$$\tilde{\alpha}_{**} \equiv \arg \min_{\tilde{\alpha}} \left\{ \max_{\delta_{\min}\le\delta\le\delta_{\max}} \frac{(\tilde{\alpha}-\delta)^2+1}{(\tilde{\alpha}+\delta)^2+1} \right\} = \begin{cases} \sqrt{\delta_{\min}\delta_{\max}} - 1, & \text{for} \quad 1 < \sqrt{\delta_{\min}\delta_{\max}}, \\ \sqrt{\delta_{\min}^2 + 1}, & \text{for} \quad 1 \ge \sqrt{\delta_{\min}\delta_{\max}}, \end{cases}$$

and the corresponding upper bound is

$$\sigma_2(\tilde{\alpha}_{**}) = \begin{cases} \dfrac{\delta_{\min}+\delta_{\max}-2\sqrt{\delta_{\min}\delta_{\max}-1}}{\delta_{\min}+\delta_{\max}+2\sqrt{\delta_{\min}\delta_{\max}-1}}, & \text{for} \quad 1 < \sqrt{\delta_{\min}\delta_{\max}}, \\ \dfrac{\sqrt{\delta_{\min}^2+1}-\delta_{\min}}{\sqrt{\delta_{\min}^2+1}+\delta_{\min}}, & \text{for} \quad 1 \ge \sqrt{\delta_{\min}\delta_{\max}}. \end{cases}$$

The proofs of Theorem 3.2 and Theorem 3.3 can be found in [4]. We emphasize that the optimal parameters $\tilde{\alpha}_*$ and $\tilde{\alpha}_{**}$ minimize only the upper bounds $\sigma_1(\tilde{\alpha})$ and $\sigma_2(\tilde{\alpha})$ of the spectral radius of the iteration matrix but does not minimize the spectral radius itself.

Note that $\tilde{\alpha}I + T$ and $\tilde{\alpha}I + D + iI$ are the Toeplitz matrix and the diagonal matrix, respectively. We can further approximate the Toeplitz matrix $\tilde{\alpha}I + T$ by the circulant matrix $\tilde{\alpha}I + C$, where C is the Strang circulant approximation of the Toeplitz matrix T defined in (3.3). Then we obtain another new approximate preconditioner $P_{\text{HL–C}}$ with

$$P_{\text{HL–C}} = \frac{1}{2\tilde{\alpha}}(\tilde{\alpha}I + C)(\tilde{\alpha}I + D + iI).$$

Thus, for any given vector r, the residual equation $P_{\text{HL–C}}w = r$ in the preconditioned BiCGSTAB iteration method can be very efficiently solved by making use of IFFT in $\mathcal{O}(M \log M)$ operations.

4. Numerical Examples

In this section, we carry out numerical experiments to study the performance of the proposed preconditioners $P_{\text{C}}, P_{\text{H}}$, $P_{\text{H–UGS}}$, P_{HL} and $P_{\text{HL–C}}$. We employ the preconditioned BiCGSTAB methods to solve the linear system (3.1). All parameters are set to be the experimentally optimal ones. All numerical experiments are started from the zero vector,

Table 1: Numerical results for Example 4.1 when $\alpha = 1.3$.

	I		P_C		P_H		P_{H-UGS}		P_{HL}		P_{HL-C}	
M	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
800	40	0.23	3	0.17	5	0.08	4	0.06	4	0.08	4	0.04
1600	63	1.58	3	0.55	5	0.31	4	0.24	4	0.29	4	0.17
3200	139	14.16	3	2.91	5	1.36	3	0.73	4	1.21	4	0.67
5000	179	43.87	3	8.50	5	3.69	3	2.09	4	3.44	4	1.67

Table 2: Numerical results for Example 4.1 when $\alpha = 1.6$.

	I		P_C		P_H		P_{H-UGS}		P_{HL}		P_{HL-C}	
M	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
800	42	0.25	2	0.17	5	0.07	4	0.07	5	0.08	5	0.05
1600	86	3.54	2	0.54	5	0.31	3	0.19	5	0.27	5	0.20
3200	172	17.56	2	2.65	5	1.44	3	0.75	5	1.40	5	0.86
5000	221	51.86	2	9.66	5	3.81	3	2.16	5	3.49	5	2.08

performed in MATLAB with machine precision 10^{-16}, and terminated when the current iterate satisfies

$$\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6},$$

where $r^{(k)}$ is the residual vector of the kth iteration and $r^{(0)}$ is the initial residual vector.

Example 4.1. Let $\gamma = 1$, $\rho = 2$, $\beta = 0$, $1 < \alpha < 2$. Then the system (1.1) is decoupled and becomes

$$iu_t + (-\Delta)^{\frac{\alpha}{2}}u + 2|u|^2u = 0, \quad -20 \le x \le 20, \ \ 0 < t \le 2,$$

subjected to the initial boundary value conditions $u(x,0) = \operatorname{sech}(x)\cdot\exp(2ix)$, $u(-20,t) = u(20,t) = 0$.

Example 4.2. For the following coupled system with $\gamma = 1$, $\rho = 2$, $\beta = 1$, $1 < \alpha < 2$,

$$\begin{cases} iu_t + (-\Delta)^{\frac{\alpha}{2}}u + 2(|u|^2 + |v|^2)u = 0, \\ iv_t + (-\Delta)^{\frac{\alpha}{2}}v + 2(|v|^2 + |u|^2)v = 0, \end{cases} \quad -20 \le x \le 20, \ \ 0 < t \le 2,$$

we take the initial boundary value conditions in the form

$$\begin{cases} u(x,0) = \operatorname{sech}(x+1)\cdot\exp(2ix), & v(x,0) = \operatorname{sech}(x-1)\cdot\exp(-2ix), \\ u(-20,t) = u(20,t) = 0, & v(-20,t) = v(20,t) = 0. \end{cases}$$

The numerical results are listed in Tables 1–4, where "M" denotes the number of spatial grid points, "Iter" denotes the iteration steps of all the iteration methods, "CPU" denotes

Table 3: Numerical results for Example 4.2 when $\alpha = 1.3$.

	I		P_C		P_H		P_{H-UGS}		P_{HL}		P_{HL-C}	
M	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
800	38	0.22	2	0.16	4	0.07	4	0.06	4	0.07	4	0.04
1600	67	1.69	2	0.51	4	0.26	4	0.24	4	0.23	4	0.17
3200	89	9.34	2	3.21	5	1.45	4	0.93	4	1.39	4	0.69
5000	218	51.25	2	10.71	5	3.95	4	2.45	4	3.82	4	1.69

Table 4: Numerical results for Example 4.2 when $\alpha = 1.6$

	I		P_C		P_H		P_{H-UGS}		P_{HL}		P_{HL-C}	
M	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
800	43	0.36	2	0.16	5	0.07	4	0.06	5	0.08	5	0.06
1600	72	1.82	2	0.52	5	0.34	3	0.24	5	0.28	5	0.20
3200	112	11.90	2	4.60	5	1.91	3	0.73	5	1.80	5	0.95
5000	237	57.69	2	11.86	5	4.83	3	2.32	5	4.63	6	2.20

the total CPU time in seconds for solving the discretized system, and "α" denotes the order of the space fractional CNLS equations.

In Tables 1—2 and Tables 3—4 we list the iteration steps and the CPU time in seconds for the BiCGSTAB method with or without using a preconditioner for the Examples 4.1 and 4.2 when $\alpha = 1.3$ and $\alpha = 1.6$ with respect to different sizes of spatial grid. The time step sizes for all numerical experiments are set to be 0.01. In these tables, the symbol "I" indicates that no preconditioner is used when solving the system of linear equations (3.1), while "P_C", "P_H", "P_{H-UGS}", "P_{HL}" and "P_{HL-C}" denote the BiCGSTAB method with the preconditioners P_C, P_H, P_{H-UGS}, P_{HL} and P_{HL-C}, respectively.

From these tables we see that the iteration step and the CPU time of the vanilla BiCGSTAB method are very large and are monotonically increasing with respect to M, but those of the preconditioned BiCGSTAB methods are considerably small which keep almost a constant, especially when the spatial grid becomes fine. To achieve the prescribed convergence criterion, the P_C-preconditioned BiCGSTAB method requires less number of iteration steps but more computing time than preconditioned BiCGSTAB methods with preconditioners P_H, P_{H-UGS}, P_{HL}, P_{HL-C} based on HSS. The number of iteration steps of the P_H-, P_{H-UGS}-, P_{HL}-, and P_{HL-C}-preconditioned BiCGSTAB methods are almost same when the spatial grid becomes fine. Among all the iterations, the P_{HL-C}-preconditioned BiCGSTAB method is the most effective in terms of CPU time, and the P_{H-UGS}-preconditioned BiCGSTAB method is the second most effective in terms of CPU time. We emphasize that the preconditioners P_H, P_{H-UGS}, P_{HL}, P_{HL-C} based on HSS outperform the approximate inverse circulant-plus-diagonal preconditioner P_C in terms of the CPU time.

5. Concluding Remarks

In this paper, we have considered discretized linear systems arising from the space fractional CNLS equations. The coefficient matrix of the discretized linear system is equal to the sum of a complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and a symmetric Toeplitz-plus-diagonal matrix. The main contribution of this paper is to develop matrix splitting and approximate preconditioners based on HSS for these Toeplitz-like matrices so that BiCGSTAB method for solving these preconditioned systems can converge very quickly. We have presented numerical examples and have shown that the proposed preconditioners are very effective and efficient.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Nos. 11601425, 11601432, 11501447) and the Shaanxi Provincial Education Department (Program No. 16JK1760).

References

[1] Z.-Z. Bai, M. Benzi, F. Chen, Modified HSS iteration methods for a class of complex symmetric linear systems, *Computing*, 87(2010), pp. 93-111.
[2] Z.-Z. Bai, M. Benzi, F. Chen, On preconditioned MHSS iteration methods for complex symmetric linear systems, *Numerical Algorithms*, 56(2011), pp. 297-317.
[3] Z.-Z. Bai, G.H. Golub and C.-K. Li, Convergence properties of preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite matrices, *Mathematics of Computation*, 76(2007), pp. 287-298.
[4] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *SIAM Journal on Matrix Analysis and Applications*, 24(2003), pp. 603-626.
[5] Z.-Z. Bai, G.H. Golub, M.K. Ng, On inexact Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *Linear Algebra and its Applications*, 428(2008), pp. 413-440.
[6] Z.-Z. Bai, G.H. Golub, J.-Y. Pan, Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems, *Numerische Mathematik*, 98(2004), pp. 1-32.
[7] R.H. Chan, X.-Q. Jin, An Introduction to Iterative Toeplitz Solvers, *SIAM, Philadelphia*, 2007.
[8] R.H. Chan, M.K. Ng, Conjugate gradient methods for Toeplitz systems, *SIAM Review*, 38(1996), pp. 427-482.
[9] R.H. Chan, K.P. Ng, Fast iterative solvers for Toeplitz-plus-band systems, *SIAM Journal on Scientific Computing*, 14(1993), pp. 1013-1019.
[10] R.H. Chan, G. Strang, Toeplitz equations by conjugate gradients with circulant preconditioner, *SIAM Journal on Scientific and Statistical Computing*, 10(1989), pp. 104-119.
[11] F. Demengel, G. Demengel, Fractional Sobolev Spaces, *Springer, London*, 2012.
[12] N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Physics Letters A*, 268(2000), pp. 298-305.
[13] N. Laskin, Fractional Schrödinger equation, *Physical Review E*, 66(2002), pp. 56-108.

[14] S.-L. Lei, H.-W. Sun, A circulant preconditioner for fractional diffusion equations, *Journal of Computational Physics*, 242(2013), pp. 715-725.
[15] M.K. Ng, Iterative Methods for Toeplitz Systems, *Numerical Mathematics and Scientific Computation, Oxford University Press, New York*, 2004.
[16] M.K. Ng, J.-Y. Pan, Approximate inverse circulant-plus-diagonal preconditioners for Toeplitz-plus-diagonal matrices, *SIAM Journal on Scientific Computing*, 32(2010), pp. 1442-1464.
[17] M.K. Ng, S. Serra-Capizzano, C. Tablino-Possio, Multigrid methods for symmetric Sinc-Galerkin systems, *Linear Algebra and its Applications*, 12(2005), pp. 261-269.
[18] M.D. Ortigueira, Riesz potential opeators and inverses via fractional centred derivatives, *International Journal of Mathematics and Mathematical Sciences*, 2006(2006), pp. 1-12.
[19] J.-Y. Pan, R.-H. Ke, M. K. Ng, H.-W. Sun, Preconditioning techniques for diagonal-times-Toeplitz matrices in fractional diffusion equations, *SIAM Journal on Scientific Computing*, 36(2014), pp. A2698-A2719.
[20] Y.-H. Ran, J.-G. Wang, D.-L. Wang, On HSS-like iteration method for the space fractional coupled nonlinear Schrödinger equations, *Applied Mathematics and Computation*, 271(2015), pp. 482-488.
[21] D.-L. Wang, A.-G. Xiao, W. Yang, Crank-Nicolson difference scheme for the coupled nonlinear Schrödinger equations with the Riesz space fractional derivative, *Journal of Computational Physics*, 242(2013), pp. 670-681.
[22] D.-L. Wang, A.-G. Xiao, W. Yang, A linearly implicit conservative difference scheme for the space fractional coupled nonlinear Schrödinger equations, *Journal of Computational Physics*, 272(2014), pp. 644-655.
[23] D.-L. Wang, A.-G. Xiao, W. Yang, Maximum-norm error analysis of a difference scheme for the space fractional CNLS, *Applied Mathematics and Computation*, 257(2015), pp. 241-251.
[24] Q. Yang, F. Liu, I. Turner, Numerical methods for fractional partial differential equations with Riesz space fractional derivatives, *Applied Mathmatical Modelling*, 34(2010), pp. 200-218.