East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.190716.051116b

Vol. **7**, No. 1, pp. 70-81
February 2017

On Preconditioners Based on HSS for the Space Fractional CNLS Equations

Yu-Hong Ran[1,*], Jun-Gang Wang[2] and Dong-Ling Wang[1]

[1] *Center for Nonlinear Studies, School of Mathematics, Northwest University, Xi'an, Shaanxi 710127, China.*
[2] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China.*

Received 19 July 2016; Accepted (in revised version) 5 November 2016.

Abstract. The space fractional coupled nonlinear Schrödinger (CNLS) equations are discretized by an implicit conservative difference scheme with the fractional centered difference formula, which is unconditionally stable. The coefficient matrix of the discretized linear system is equal to the sum of a complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and a symmetric Toeplitz-plus-diagonal matrix. In this paper, we present new preconditioners based on Hermitian and skew-Hermitian splitting (HSS) for such Toeplitz-like matrix. Theoretically, we show that all the eigenvalues of the resulting preconditioned matrices lie in the interior of the disk of radius 1 centered at the point $(1,0)$. Thus Krylov subspace methods with the proposed preconditioners converge very fast. Numerical examples are given to illustrate the effectiveness of the proposed preconditioners.

AMS subject classifications: 65F10, 65F15

Key words: The space fractional Schrödinger equations, Toeplitz matrix, Hermitian and skew-Hermitian splitting, preconditioner, Krylov subspace methods.

1. Introduction

The classical Schrödinger equations describe the evolution of microscopic particles, and they can be derived from the path integral over the Brownian motion. Laskin [12] generalized the path integral method from the Brownian motion to the Lévy-α process to obtain the space fractional Schrödinger equations [13].

In this paper, we consider the space fractional coupled nonlinear Schrödinger (CNLS) equations

$$\begin{cases} iu_t + \gamma(-\Delta)^{\frac{\alpha}{2}}u + \rho\big(|u|^2 + \beta|v|^2\big)u = 0, \\ iv_t + \gamma(-\Delta)^{\frac{\alpha}{2}}v + \rho\big(|v|^2 + \beta|u|^2\big)v = 0, \end{cases} \quad a \le x \le b, \quad 0 < t \le T, \tag{1.1}$$

*Corresponding author. *Email address:* `ranyh@nwu.edu.cn` (Y.-H. Ran)

with the initial boundary value conditions

$$\begin{cases} u(x,0) = u_0(x), & v(x,0) = v_0(x), & a \le x \le b, \\ u(a,t) = u(b,t) = 0, & v(a,t) = v(b,t) = 0, & 0 \le t \le T, \end{cases}$$

where $i = \sqrt{-1}$, $1 < \alpha < 2$ and the parameters $\gamma > 0,\ \rho > 0,\ \ \beta \ge 0$ are constants. The fractional Laplacian [11] can be characterized as

$$(-\Delta)^{\frac{\alpha}{2}} u(x,t) = \mathscr{F}^{-1}\Big(|\xi|^{\alpha}\mathscr{F}\big(u(x,t)\big)\Big),$$

where $\mathscr{F}$ is the Fourier transform acting on the spatial variable x. Furthermore, it is shown that the Riesz fractional derivative [24] can also be defined as

$$\frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t) = -\frac{1}{2\cos\frac{\pi\alpha}{2}}\Big[{}_{-\infty}D_x^{\alpha} u(x,t) + {}_x D_{+\infty}^{\alpha} u(x,t)\Big],$$

where ${}_{-\infty}D_x^{\alpha}u(x,t)$ and ${}_xD_{+\infty}^{\alpha}u(x,t)$ are the left and right Riemann-Liouville derivatives, respectively. when $\alpha = 2$, the system (1.1) is reduced to the classical CNLS equations, which describe a wide class of physical nonlinear phenomena, such as the hydrodynamics, the nonlinear optics and the dynamics of the two-component Bose-Einstein condensate.

Generally, closed-form analytical solutions of the space fractional CNLS equations are not available. Consequently, the numerical methods become important and powerful which are very few until now. Recently, based on the fractional centered difference formula, in [21–23] the authors proposed an implicit conservative difference scheme to discretize the space fractional CNLS equations which is unconditionally stable. The coefficient matrix of the discretized linear system is equal to the sum of the complex scaled identity matrix which can be written as the imaginary unit times the identity matrix and the symmetric Toeplitz-plus-diagonal matrix. As the coefficient matrix is non-Hermitian Toeplitz-like, we can employ Krylov subspace methods, such as BiCGSTAB, to solve the discretized linear system. Using the fast Fourier transform (FFT), the Toeplitz matrix-vector multiplication can be done in $\mathscr{O}(M\log M)$ operations, where M is the number of grid points. Nevertheless, the resulting system in general is ill-conditioned and the convergence rates of Krylov subspace methods tend to be considerably worse. In order to improve their performance and reliability, preconditioning is a common technique.

Circulant preconditioners for Toeplitz matrices have been theoretically and numerically studied with numerous applications for over twenty years; see [7, 8, 15]. However, circulant preconditioners do not work for such Toeplitz-plus-diagonal systems. Chan and Ng [9] considered banded preconditioners for Toeplitz-plus-band systems. The main drawback of this method is that the generating function should be known in order to construct effective banded preconditioners. In general, the generating function of the corresponding Toeplitz matrix is unknown. In [17], multigrid methods are studied for Toeplitz-plus-diagonal linear systems arising from Sinc-Galerkin methods. As the generating functions are known, the proposed multigrid method is to incorporate the diagonal matrix into the interpolating process. Lately, Ng and Pan [16] proposed approximate inverse circulant-plus-diagonal preconditioners for solving Hermitian positive definite Toeplitz-plus-diagonal systems. Their