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On a New SSOR-Like Method with Four Parameters for the Augmented Systems

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Abstract. In this paper, we propose a new SSOR-like method with four parameters to solve the augmented system. And we analyze the convergence of the method and get the optimal convergence factor under suitable conditions. It is proved that the optimal convergence factor is the same as the GMPSD method [M.A. Louka and N.M. Missirlis, A comparison of the extrapolated successive overrelaxation and the preconditioned simultaneous displacement methods for augmented systems, Numer. Math. 131(2015) 517-540] with five parameters under the same assumption.

AMS subject classifications: 65F10

Key words: Convergence, SSOR-like method, convergence factor.

1. Introduction

Consider the saddle point problem or the augmented system in the following form:

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b \\ q \end{pmatrix}, \tag{1.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ with $rank(B) = n$ and $m \geq n$, $b \in R^m$ and $q \in R^n$ are two given vectors. Denote B^T as the transpose of the matrix B. Under these restrictions, the system (1.1) has a unique solution.

The system (1.1) appears in many scientific and engineering applications such as the mixed finite element approximation of elliptic partial differential equations [1], optimal control [2], constrained optimization [19], and so on [3, 4, 21, 22, 24, 25].

For the system (1.1), there are many efficient iterative methods constructed such as the Uzawa-type methods [8, 20, 26, 27, 33, 36, 50, 52] and the Krylov subspace methods [6, 9, 30, 34, 43], the HSS-type methods [5, 7, 11–15, 29, 37, 42, 47, 54] and the SOR-like methods [10, 16, 23, 28, 32, 35, 44], the AOR-like methods [31, 39] and SSOR-like methods [17, 18, 38, 40, 41, 45, 46, 49, 51, 53].

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According to the number of parameters, the existing SSOR-like methods can be separated into the ones with one parameter in [17,46,49] and two parameters in [45,51,53], three parameters in [18,40] and four parameters in [41] and five parameters in [38], respectively.

In this paper, we present a new SSOR-like method with four parameters. It has the simplest form than other SSOR-like methods [17,18,38,40,41,45,46,49,51,53]. We discuss the convergence of the new method and get the optimal convergence factor. It is proved that the new method has the same optimal convergence factor as the one of the generalized modified preconditioned simultaneous displacement(GMPSD) method with five parameters in [38], so the former with simpler form is at least as good as the latter.

The paper is organized as follows. In Section 2, the new SSOR-like method is proposed. The convergence analysis is done in Section 3, and the optimal convergence factor is estimated under a certain condition in Section 4. In Section 5, the optimal convergent parameters and the optimal convergence factors of the method applied to two frequently used examples are listed.

Throughout the paper, denote $N(B^T)$ and $\rho(B)$ as the null space of the matrix B^T and the spectral radius of the matrix B, respectively.

2. The New SSOR-Like Method

Let $Q \in R^{n\times n}$ be nonsingular and symmetric, and denote the initial vector by $x_0 \in R^m$, $y_0 \in R^n$. The new SSOR-like method with four given parameters ω, δ, γ, υ is defined by

$$\begin{cases} y_{k+1} = y_k + Q^{-1}B^T(\upsilon x_k - \delta A^{-1}By_k + \delta A^{-1}b) - (\delta+\upsilon)Q^{-1}q, \\ x_{k+1} = (1-\omega)x_k - A^{-1}\big\{B\left[(\omega-\gamma)y_k + \gamma y_{k+1}\right] - \omega b\big\}, \end{cases} \quad k \ge 0, \tag{2.1}$$

where parameters satisfy

$$\omega(\delta+\upsilon) \neq 0. \tag{2.2}$$

Denote $I_m \in R^{m\times m}$ and $I_n \in R^{n\times n}$ as identity matrices, respectively. The new SSOR-like method (2.1) can be rewritten as

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = H_{\omega,\delta,\gamma,\nu} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + b_{\omega,\delta,\gamma,\nu}, \quad k \ge 0, \tag{2.3}$$

where

$$b_{\omega,\delta,\gamma,\nu} = \begin{pmatrix} \omega A^{-1}b - \gamma\delta A^{-1}BQ^{-1}B^TA^{-1}b + \gamma(\delta+\nu)A^{-1}BQ^{-1}q \\ \delta Q^{-1}B^TA^{-1}b - (\delta+\nu)Q^{-1}q \end{pmatrix}$$

and

$$H_{\omega,\delta,\gamma,\nu} = \begin{pmatrix} (1-\omega)I_m - \gamma\upsilon A^{-1}BQ^{-1}B^T & -\omega A^{-1}B + \gamma\delta A^{-1}BQ^{-1}B^TA^{-1}B \\ \nu Q^{-1}B^T & I_n - \delta Q^{-1}B^TA^{-1}B \end{pmatrix}. \tag{2.4}$$

Especially, when $\gamma(\gamma-\omega)(2\gamma-\omega)(\delta+\upsilon+4) \neq 0$, (2.1) as well as (2.3) can be splitted as

$$\begin{pmatrix} x_{k+\frac{1}{2}} \\ y_{k+\frac{1}{2}} \end{pmatrix} = M_{\omega,\delta,\gamma,\upsilon} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + (D-\Omega L)^{-1} \begin{pmatrix} b \\ -q \end{pmatrix}, \quad k \ge 0, \tag{2.5}$$

and

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = N_{\omega,\delta,\gamma,\upsilon} \begin{pmatrix} x_{k+\frac{1}{2}} \\ y_{k+\frac{1}{2}} \end{pmatrix} + (D - \Omega U)^{-1} \begin{pmatrix} b \\ -q \end{pmatrix}, \quad k \geq 0, \tag{2.6}$$

where

$$\Omega = \begin{pmatrix} \left(2 - \frac{\omega}{\gamma}\right) I_m & 0 \\ 0 & \frac{2(\delta+\upsilon)}{\delta+\upsilon+4} I_n \end{pmatrix},$$

$$N_{\omega,\delta,\gamma,\upsilon} = (D - \Omega U)^{-1} \left[(I - \Omega) D + \Omega L \right] = \begin{pmatrix} (1-\gamma) I_m - \frac{\gamma}{2} A^{-1} B Q^{-1} B^T & -\gamma A^{-1} B \\ \frac{\delta+\upsilon}{2} Q^{-1} B^T & I_n \end{pmatrix},$$

and

$$\begin{aligned} M_{\omega,\delta,\gamma,\upsilon} =& (D - \Omega L)^{-1} \left[(I - \Omega) D + \Omega U \right] \\ =& \begin{pmatrix} \frac{\omega}{2\gamma} I_m & \left(\frac{\omega}{2\gamma} - 1\right) A^{-1} B \\ \frac{\delta+\upsilon}{2} \left(1 - \frac{(\gamma-\omega)(\delta+\upsilon+4)}{\gamma-1}\right) Q^{-1} B^T & I_n - \frac{(\delta+\upsilon)(\gamma-\omega)(\delta+\upsilon+4)}{2(\gamma-1)} Q^{-1} B^T A^{-1} B \end{pmatrix}, \end{aligned}$$

with

$$D = \begin{pmatrix} \left(1 + \frac{\gamma-1}{\gamma-\omega} - \frac{1}{\gamma}\right) A & 0 \\ 0 & Q \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 \\ B^T & \frac{1}{2} Q \end{pmatrix}$$

and

$$U = \begin{pmatrix} \left(\frac{\gamma-1}{\gamma-\omega} - \frac{1}{\gamma}\right) A & -B \\ 0 & \frac{1}{2} Q \end{pmatrix},$$

satisfying

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = D - L - U.$$

So (2.1) can also be regarded as the generalized SSOR-like method.

3. The Convergence Analysis

In this section, we discuss the convergence of the SSOR-like method (2.1) for the system (1.1). In the following theorem, the assumption that Q is nonsingular and symmetric follows from that in Section 2 in [38].

Theorem 3.1. *Let Q be a nonsingular and symmetric matrix and four parameters ω, δ, γ, υ satisfy* (2.2). *Suppose all the eigenvalues of $Q^{-1}B^TA^{-1}B$ are real, and denote ρ and μ as the maximum and the minimum eigenvalues of $Q^{-1}B^TA^{-1}B$ in modulus respectively. Then*
(i) for the case $\upsilon(\gamma-1) \neq 0$ and $m = n$, the method (2.1), *with the ω, δ, γ, υ given, converges to the unique solution of* (1.1) *if and only if all the eigenvalues of $Q^{-1}B^TA^{-1}B$ have the same sign,*

$$f_{low\_b}(\omega,\gamma,\upsilon) < sign(\mu)\delta < f_{up\_b}(\omega,\gamma,\upsilon), \tag{3.1}$$

where

$$f_{low\_b}(\omega,\gamma,\upsilon)=\begin{cases} sign(\mu)L(\omega,\gamma,\upsilon), & \omega<0,\\ \max\{-sign(\mu)\upsilon, sign(\mu)L(\omega,\gamma,\upsilon)\}, & 0<\omega<1,\\ -sign(\mu)\upsilon,\ \ with\,|\upsilon(\gamma-1)|<sign(\mu)\frac{1}{\rho}, & \omega=1,\\ \max\{-sign(\mu)\upsilon, sign(\mu)F(\omega,\gamma,\upsilon)\}, & 1<\omega<2,\\ \max\{-sign(\mu)\upsilon, sign(\mu)F(\omega,\gamma,\upsilon)\}, with\ sign(\mu)\upsilon(1-\gamma)>0, & \omega=2,\\ \max\{-sign(\mu)\upsilon, sign(\mu)F(\omega,\gamma,\upsilon), sign(\mu)G(\omega,\gamma,\upsilon)\}, & \omega>2, \end{cases}$$

and

$$f_{up\_b}(\omega,\gamma,\upsilon)=\begin{cases} \min\{-sign(\mu)\upsilon, sign(\mu)F(\omega,\gamma,\upsilon), sign(\mu)G(\omega,\gamma,\upsilon)\}, & \omega<0,\\ \min\{sign(\mu)F(\omega,\gamma,\upsilon), sign(\mu)G(\omega,\gamma,\upsilon)\}, & 0<\omega<1,\\ sign(\mu)G(\omega,\gamma,\upsilon), with\ |\upsilon(\gamma-1)|<sign(\mu)\frac{1}{\rho}, & \omega=1,\\ \min\{sign(\mu)L(\omega,\gamma,\upsilon), sign(\mu)G(\omega,\gamma,\upsilon)\}, & 1<\omega<2,\\ sign(\mu)L(\omega,\gamma,\upsilon),\ with\ sign(\mu)\upsilon(1-\gamma)>0, & \omega=2,\\ sign(\mu)L(\omega,\gamma,\upsilon), & \omega>2, \end{cases}$$

with

$$F(\omega,\gamma,\upsilon)=\begin{cases} \frac{\omega-2}{(\omega-1)\mu}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega\geq 2,\\ \frac{\omega-2}{(\omega-1)\rho}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega\neq 0,1,\ \omega<2,\\ -\upsilon,\ \ with\ \ |\upsilon(\gamma-1)|<sign(\mu)\frac{1}{\rho}, & \omega=1, \end{cases}$$

$$G(\omega,\gamma,\upsilon)=\begin{cases} \frac{2}{\mu}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}, & \omega>2,\\ -\upsilon,\ \ with\ \ sign(\mu)\upsilon(1-\gamma)>0, & \omega=2,\\ \frac{2}{\rho}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}, & \omega\neq 0, \omega<2, \end{cases}$$

and

$$L(\omega,\gamma,\upsilon)=\begin{cases} \frac{\omega}{(\omega-1)\mu}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega<0,\\ \frac{\omega}{(\omega-1)\rho}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega\neq 1, \omega>0,\\ -\upsilon,\ \ with\ |\upsilon(\gamma-1)|<sign(\mu)\frac{1}{\rho}, & \omega=1. \end{cases}$$

(ii) *for other cases, the method* (2.1)*, with the* ω*,* δ*,* γ*,* υ *given, converges to the unique solution of* (1.1) *if and only if all the eigenvalues of* $Q^{-1}B^TA^{-1}B$ *have the same sign,*

$$0<\omega<2 \tag{3.2}$$

and (3.1) *hold.*

When Q is symmetric positive definite or symmetric negative definite, the description of Theorem 3.1 becomes more clear.

Corollary 3.1. *Suppose that* Q *is a symmetric positive definite matrix and four parameters* ω*,* δ*,* γ*,* υ *satisfy* (2.2)*. Denote* μ_{max} *and* μ_{min} *as the maximum and the minimum eigenvalues*

of $Q^{-1}B^TA^{-1}B$ respectively. Then
(i) for the case $\upsilon(\gamma-1)\neq 0$ and $m=n$, the method (2.1) *converges to the unique solution of* (1.1) *if and only if*

$$h_{low\_b}(\omega,\gamma,\upsilon)<\delta<h_{up\_b}(\omega,\gamma,\upsilon), \tag{3.3}$$

where

$$h_{low\_b}(\omega,\gamma,\upsilon)=\begin{cases}\frac{\omega}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega<0,\\ \max\left\{-\upsilon,\frac{\omega}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}\right\}, & 0<\omega<1,\\ -\upsilon,\ \ with\ |\upsilon(\gamma-1)|<\frac{1}{\mu_{\max}}, & \omega=1,\\ \max\left\{-\upsilon,\frac{\omega-2}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}\right\}, & 1<\omega<2,\\ \max\{-\upsilon,\upsilon(\gamma-2)\},\ \ with\ \upsilon(1-\gamma)>0, & \omega=2,\\ \max\left\{-\upsilon,\frac{\omega-2}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\min}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & \omega>2,\end{cases}$$

and

$$h_{up\_b}(\omega,\gamma,\upsilon)=\begin{cases}\min\left\{-\upsilon,\frac{\omega-2}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\max}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & \omega<0,\\ \min\left\{\frac{\omega-2}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\max}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & 0<\omega<1,\\ \frac{2}{\mu_{\max}}+\upsilon(1-2\gamma),\ \ with\ |\upsilon(\gamma-1)|<\frac{1}{\mu_{\max}}, & \omega=1,\\ \min\left\{\frac{\omega}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\max}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & 1<\omega<2,\\ \frac{2}{\mu_{\max}}+\upsilon(\gamma-2),\ \ with\ \upsilon(1-\gamma)>0, & \omega=2,\\ \frac{\omega}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega>2.\end{cases}$$

(ii) for other cases, the method (2.1)*, with the ω, δ, γ, υ given, converges to the unique solution of* (1.1) *if and only if*

$$0<\omega<2 \tag{3.4}$$

and (3.3) *hold.*

Corollary 3.2. *Suppose that Q is a symmetric negative definite matrix and four parameters ω, δ, γ, υ satisfy* (2.2)*. Denote μ_{max} and μ_{min} as the maximum and the minimum eigenvalues of $Q^{-1}B^TA^{-1}B$ respectively. Then*
(i) for the case $\upsilon(\gamma-1)\neq 0$ and $m=n$, the method (2.1) *converges to the unique solution of* (1.1) *if and only if*

$$g_{low\_b}(\omega,\gamma,\upsilon)<\delta<g_{up\_b}(\omega,\gamma,\upsilon), \tag{3.5}$$

where

$$g_{low\_b}(\omega,\gamma,\upsilon)=\begin{cases}\max\left\{-\upsilon,\frac{\omega-2}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\min}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & \omega<0,\\ \max\left\{\frac{\omega-2}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\min}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & 0<\omega<1,\\ \frac{2}{\mu_{\min}}+\upsilon(1-2\gamma),\ \ with\ |\upsilon(\gamma-1)|<-\frac{1}{\mu_{\min}}, & \omega=1,\\ \max\left\{\frac{\omega}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\min}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & 1<\omega<2,\\ \frac{2}{\mu_{\min}}+\upsilon(\gamma-2),\ \ with\ \upsilon(1-\gamma)<0, & \omega=2,\\ \frac{\omega}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega>2,\end{cases}$$

and

$$g_{up\_b}(\omega,\gamma,\upsilon)=\begin{cases}\frac{\omega}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}, & \omega<0,\\ \min\left\{-\upsilon,\frac{\omega}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}\right\}, & 0<\omega<1,\\ -\upsilon,\quad \text{with } |\upsilon(\gamma-1)|<-\frac{1}{\mu_{\min}}, & \omega=1,\\ \min\left\{-\upsilon,\frac{\omega-2}{(\omega-1)\mu_{\min}}+\frac{\upsilon(\gamma-\omega)}{\omega-1}\right\}, & 1<\omega<2,\\ \min\{-\upsilon,\upsilon(\gamma-2)\},\quad \text{with } \upsilon(1-\gamma)<0, & \omega=2,\\ \min\left\{-\upsilon,\frac{\omega-2}{(\omega-1)\mu_{\max}}+\frac{\upsilon(\gamma-\omega)}{\omega-1},\frac{2}{\mu_{\max}}+\frac{\upsilon(\omega-2\gamma)}{2-\omega}\right\}, & \omega>2.\end{cases}$$

(ii) *for other cases, the method* (2.1)*, with the* ω*,* δ*,* γ*,* υ *given, converges to the unique solution of* (1.1) *if and only if*

$$0<\omega<2 \tag{3.6}$$

and (3.5) *hold.*

To prove Theorem 3.1, we first introduce three useful lemmas.

Lemma 3.1. *If* $\upsilon(\gamma-1)\neq 0$ *and* $m=n$*, then* $\lambda=1-\omega$ *is not an eigenvalue of the matrix* $H_{\omega,\delta,\gamma,\upsilon}$*. Otherwise,* $\lambda=1-\omega$ *is an eigenvalue of the matrix* $H_{\omega,\delta,\gamma,\upsilon}$*.*

Proof. What we need to do is to discuss when

$$H_{\omega,\delta,\gamma,\upsilon}\begin{pmatrix}x\\ y\end{pmatrix}=(1-\omega)\begin{pmatrix}x\\ y\end{pmatrix} \tag{3.7}$$

has a nonzero solution $(x^T,y^T)^T\in R^{m+n}$. By (2.4), (3.7) is equivalent to

$$\begin{cases}\omega(\gamma-1)A^{-1}By=0,\\ \upsilon Q^{-1}B^Tx=\left(\delta Q^{-1}B^TA^{-1}B-\omega I_n\right)y.\end{cases} \tag{3.8}$$

We separate the following analysis into three cases: $\upsilon(\gamma-1)\neq 0$; $\upsilon\neq 0$ and $\gamma=1$; $\upsilon=0$.

For the first case that $\upsilon(\gamma-1)\neq 0$, (3.8) is equivalent to

$$\begin{cases}By=0,\\ \upsilon Q^{-1}B^Tx=\left(\delta Q^{-1}B^TA^{-1}B-\omega I_n\right)y,\end{cases} \tag{3.9}$$

by $\omega\neq 0$ in (2.2).

When $m=n$, B and B^T are nonsingular by the condition that $rank(B)=n$. Then (3.9) only has zero solution with $x=0\in R^m$ and $y=0\in R^n$ by $\upsilon\neq 0$. So does (3.7). That is to say, $\lambda=1-\omega$ is not an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$.

When $m>n$, $rank(B^T)=n<m$, so there exists a nonzero vector $x\in N(B^T)$ such that $(x^T,0)\in R^{m+n}$ is a nonzero solution of (3.8), or, equivalently of (3.7). So $\lambda=1-\omega$ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$.

For the second case that $\upsilon\neq 0$ and $\gamma=1$, (3.8) is equivalent to

$$\upsilon Q^{-1}B^Tx=\left(\delta Q^{-1}B^TA^{-1}B-\omega I_n\right)y. \tag{3.10}$$

If $rank(\delta Q^{-1}B^TA^{-1}B-\omega I_n)<n$, there exists a nonzero vector $y\in N(\delta Q^{-1}B^TA^{-1}B-\omega I_n)$. Then $(0,y^T)^T\in R^{m+n}$ is nonzero and satisfies (3.10). If $rank(\delta Q^{-1}B^TA^{-1}B-\omega I_n)=n$, then

$$y=\upsilon(\delta Q^{-1}B^TA^{-1}B-\omega I_n)^{-1}Q^{-1}B^Tx \tag{3.11}$$

holds by (3.10). For any nonzero vector $x\in R^m$ given and $y\in R^n$ computed by (3.11), $(x^T,y^T)^T\in R^{m+n}$ is nonzero and satisfies (3.10). Anyway, (3.10) or (3.7) always has nonzero solutions, so $\lambda=1-\omega$ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$ in this case.

For the last case that $\upsilon=0$, for any nonzero vector $x\in R^n$ given, it can be easily checked that $(x^T,0)^T\in R^{m+n}$ is nonzero and satisfies (3.8) or (3.7). So $\lambda=1-\omega$ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$.

In summary, if $\upsilon(\gamma-1)\neq 0$ and $m=n$, $\lambda=1-\omega$ is not an eigenvalue of the matrix $H_{\omega,\delta,\gamma,\upsilon}$. Otherwise, $\lambda=1-\omega$ is an eigenvalue of the matrix $H_{\omega,\delta,\gamma,\upsilon}$. □

Lemma 3.2. *If*

$$\lambda\neq 1-\omega,\quad (\lambda-1)(\upsilon\gamma+\delta)+\omega(\delta+\upsilon)=0, \tag{3.12}$$

then λ is not an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$.

Proof. We can conclude from (3.12) that $\lambda\neq 1$ by (2.2). If λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$, there exists a nonzero eigenvector $(x^T,y^T)^T\in R^{m+n}$ such that

$$H_{\omega,\delta,\gamma,\upsilon}\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix},$$

or, equivalently,

$$\begin{cases}(1-\omega)x-\gamma\upsilon A^{-1}BQ^{-1}B^Tx-\omega A^{-1}By+\gamma\delta A^{-1}BQ^{-1}B^TA^{-1}By=\lambda x,\\ \upsilon Q^{-1}B^Tx=\delta Q^{-1}B^TA^{-1}By+(\lambda-1)y.\end{cases} \tag{3.13}$$

Substituting $\upsilon Q^{-1}B^Tx$ in the first equality of (3.13) by the second one yields

$$(1-\omega-\lambda)x=[\omega+\gamma(\lambda-1)]A^{-1}By.$$

Since $\lambda\neq 1-\omega$, we have

$$x=\frac{\omega+\gamma(\lambda-1)}{1-\omega-\lambda}A^{-1}By. \tag{3.14}$$

Putting (3.14) into the second equality of (3.13) yields

$$\frac{(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)}{1-\omega-\lambda}Q^{-1}B^TA^{-1}By=(\lambda-1)y. \tag{3.15}$$

By the assumption that $(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)=0$, it holds

$$(\lambda-1)y=0.$$

So $y=0$ by $\lambda\neq 1$, and then $x=0$ by (3.14), i.e. $(x^T,y^T)^T=0\in R^{m+n}$. It is a contradiction that $(x^T,y^T)^T\in R^{m+n}$ is a corresponding eigenvector, which completes the proof. □

Lemma 3.3. *Suppose $\lambda \neq 1-\omega$, then λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$ if and only if there exists an eigenvalue of $Q^{-1}B^TA^{-1}B$, denoted by μ, such that*

$$\mu\big[(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)\big]=(\lambda-1)(1-\omega-\lambda) \tag{3.16}$$

is satisfied.

Proof. For any $\lambda \neq 1-\omega$ given, if λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$, then

$$(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)\neq 0 \tag{3.17}$$

holds. Otherwise, λ is not an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$ by Lemma 3.2.
Then there exists a nonzero vector $(x^T,y^T)^T\in R^{m+n}$ such that

$$H_{\omega,\delta,\gamma,\upsilon}\begin{pmatrix} x \\ y \end{pmatrix}=\lambda\begin{pmatrix} x \\ y \end{pmatrix}.$$

By the same deduction in the proof for Lemma 3.2, we have that (3.13)-(3.15) still hold here. So

$$Q^{-1}B^TA^{-1}By=\frac{(\lambda-1)(1-\omega-\lambda)}{(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)}y \tag{3.18}$$

by the assumption that $(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)\neq 0$. We can conclude that $y\neq 0$. Otherwise, $x=0$ will follow from the equality (3.14). So $(x^T,y^T)^T=0$ which is a contradiction. Let

$$\mu=\frac{(\lambda-1)(1-\omega-\lambda)}{(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)}, \tag{3.19}$$

then μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$ with the corresponding eigenvector $y\in R^n$ by (3.18), and (3.16) follows.

Conversely, for any $\lambda\neq 1-\omega$ given, if λ is a root of (3.16) for an eigenvalue μ of $Q^{-1}B^TA^{-1}B$, then (3.17) still holds. In fact, if

$$(\upsilon\gamma+\delta)(\lambda-1)+\omega(\delta+\upsilon)=0, \tag{3.20}$$

it holds that $\lambda=1$ by (3.16). Then $\omega(\delta+\upsilon)=0$ follows, which is a contradiction with (2.2).

Suppose $y\in R^n$ is an eigenvector of $Q^{-1}B^TA^{-1}B$ belonging to μ, we have that (3.19) and (3.18) hold by (3.17). Since (3.18) is equivalent to

$$\frac{\upsilon[\gamma(\lambda-1)+\omega]}{1-\omega-\lambda}Q^{-1}B^TA^{-1}By-\delta Q^{-1}B^TA^{-1}By=(\lambda-1)y \tag{3.21}$$

by the assumption $\lambda\neq 1-\omega$. For such a nonzero vector y, let

$$x=\frac{\omega+\gamma(\lambda-1)}{1-\omega-\lambda}A^{-1}By,$$

then it follows from (3.21) that

$$\begin{cases} x = \frac{\gamma(\lambda-1)+\omega}{1-\omega-\lambda} A^{-1}By, \\ \upsilon Q^{-1}B^T x - \delta Q^{-1}B^T A^{-1}By = (\lambda-1)y, \end{cases} \quad y \neq 0,$$

which is equivalent to

$$H_{\omega,\delta,\gamma,\upsilon}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \quad y \neq 0.$$

That is to say, λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$. The proof is completed. □

Lemma 3.4 ([48]). *Both roots of the real quadratic equation $\lambda^2 - b\lambda + c = 0$ are less than unity in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Proof of Theorem 3.1

Proof. Let $\mu_i,\ i = 1, 2, \cdots, n$, be all the real eigenvalues of $Q^{-1}B^TA^{-1}B$, and $\mu_{\max}$ and $\mu_{\min}$ be the maximum and the minimum eigenvalues of them. By Lemma 3.3, we have the following conclusion:

If $\lambda \neq 1-\omega$, then λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$ if and only if there exists $1 \leq i \leq n$ such that λ is a root of the equation:

$$\lambda^2 + \big[\mu_i(\upsilon\gamma+\delta) - (2-\omega)\big]\lambda + \mu_i\big[\upsilon(\omega-\gamma) - (1-\omega)\delta\big] + 1 - \omega = 0, \tag{3.22}$$

or, equivalently,

$$\mu_i\big[(\upsilon\gamma+\delta)(\lambda-1) + \omega(\delta+\upsilon)\big] = (\lambda-1)(1-\omega-\lambda), \tag{3.23}$$

i.e., (3.16) with μ substituted by μ_i for some $1 \leq i \leq n$.

Then, we turn to consider the convergence of the method.

Since $\lambda = 1-\omega$ is not an eigenvalue of the matrix $H_{\omega,\delta,\gamma,\upsilon}$ for the case that $\upsilon(\gamma-1) \neq 0$ and $m = n$ and it is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$ for other cases, we analyze the convergence separately.

For the case that $\upsilon(\gamma-1) \neq 0$ and $m = n$, if λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$, then $\lambda \neq 1-\omega$ is true by Lemma 3.1. So, by the conclusion proved above, λ is a root of (3.22) for some $1 \leq i \leq n$.

Conversely, if λ is a root of (3.22) as well as (3.23) for some $1 \leq i \leq n$, then $\lambda \neq 1-\omega$ holds. Otherwise, by (3.23),

$$\mu_i\omega\upsilon(\gamma-1) = 0. \tag{3.24}$$

Since $Q^{-1}B^TA^{-1}B$ is nonsingular by the supposition that A is symmetric positive definite and that B is of full column rank, $\mu_i \neq 0$ holds. It follows from (3.24) and (2.2) that $\upsilon(\gamma-1) = 0$, which is in contradiction with the supposition $\upsilon(\gamma-1) \neq 0$. So, by the conclusion proved, λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$.

In a word, λ is an eigenvalue of $H_{\omega,\delta,\gamma,\upsilon}$ if and only if λ is a root of (3.22) in this case.

Therefore, by Lemma 3.4 and (3.22), $\rho(H_{\omega,\delta,\gamma,\upsilon})<1$ if and only if

$$\begin{cases} \left|\mu_i[\upsilon(\omega-\gamma)-(1-\omega)\delta]+1-\omega\right|<1, \\ \left|\mu_i(\upsilon\gamma+\delta)-(2-\omega)\right|<\mu_i[\upsilon(\omega-\gamma)-(1-\omega)\delta]+2-\omega, \end{cases} \quad i=1,2,\cdots,n, \tag{3.25}$$

or,

$$\begin{cases} \omega-2<\mu_i[\upsilon(\omega-\gamma)-(1-\omega)\delta]<\omega, \\ \mu_i\omega(\delta+\upsilon)>0, \\ \mu_i[2(\upsilon\gamma+\delta)-\omega(\delta+\upsilon)]<2(2-\omega), \end{cases} \quad i=1,2,\cdots,n. \tag{3.26}$$

(3.26) is equivalent to $\mu_1,\cdots,\mu_n$ have the same sign,

$$\begin{cases} \omega(\delta+\upsilon)>0, \\ \max\left\{\frac{\omega-2}{\mu_{\min}},\frac{\omega-2}{\mu_{\max}}\right\}+\upsilon(\gamma-\omega)<(\omega-1)\delta<\min\left\{\frac{\omega}{\mu_{\min}},\frac{\omega}{\mu_{\max}}\right\}+\upsilon(\gamma-\omega), \quad if \quad \mu_{\min}>0 \\ (2-\omega)\delta<\upsilon(\omega-2\gamma)+\min\left\{\frac{2(2-\omega)}{\mu_{\min}},\frac{2(2-\omega)}{\mu_{\max}}\right\}, \end{cases} \tag{3.27}$$

and

$$\begin{cases} \omega(\delta+\upsilon)<0, \\ \max\left\{\frac{\omega}{\mu_{\min}},\frac{\omega}{\mu_{\max}}\right\}+\upsilon(\gamma-\omega)<(\omega-1)\delta<\min\left\{\frac{\omega-2}{\mu_{\min}},\frac{\omega-2}{\mu_{\max}}\right\}+\upsilon(\gamma-\omega), \quad if \quad \mu_{\max}<0 \\ (2-\omega)\delta>\upsilon(\omega-2\gamma)+\max\left\{\frac{2(2-\omega)}{\mu_{\min}},\frac{2(2-\omega)}{\mu_{\max}}\right\}, \end{cases} \tag{3.28}$$

hold.

When $\mu_i>0,\ i=1,2,\cdots,n$, (3.27) can be written, for different cases of ω, as

$$\begin{cases} \delta+\upsilon<0, \\ \frac{\omega-2}{\mu_{\max}}+\upsilon(\gamma-\omega)<(\omega-1)\delta<\frac{\omega}{\mu_{\min}}+\upsilon(\gamma-\omega), \\ (2-\omega)\delta<\upsilon(\omega-2\gamma)+\frac{2(2-\omega)}{\mu_{\max}}, \end{cases} \quad \omega<0, \tag{3.29a}$$

$$\begin{cases} \delta+\upsilon>0, \\ \frac{\omega-2}{\mu_{\max}}+\upsilon(\gamma-\omega)<(\omega-1)\delta<\frac{\omega}{\mu_{\max}}+\upsilon(\gamma-\omega), \\ (2-\omega)\delta<\upsilon(\omega-2\gamma)+\frac{2(2-\omega)}{\mu_{\max}}, \end{cases} \quad 0<\omega<2,\ \omega\neq 1, \tag{3.29b}$$

$$\begin{cases} \delta+\upsilon>0, \\ -\frac{1}{\mu_{\max}}<\upsilon(1-\gamma)<\frac{1}{\mu_{\max}}, \\ \delta<\upsilon(1-2\gamma)+\frac{2}{\mu_{\max}}, \end{cases} \quad \omega=1, \tag{3.29c}$$

$$\begin{cases} \delta+\upsilon>0, \\ \upsilon(\gamma-2)<\delta<\frac{2}{\mu_{\max}}+\upsilon(\gamma-2), \\ \upsilon(1-\gamma)>0, \end{cases} \quad \omega=2, \tag{3.29d}$$

$$\begin{cases} \delta+\upsilon>0, \\ \frac{\omega-2}{\mu_{\min}}+\upsilon(\gamma-\omega)<(\omega-1)\delta<\frac{\omega}{\mu_{\max}}+\upsilon(\gamma-\omega), \\ (2-\omega)\delta<\upsilon(\omega-2\gamma)+\frac{2(2-\omega)}{\mu_{\min}}, \end{cases} \quad \omega>2, \tag{3.29e}$$

(3.29a)-(3.29e) imply that (3.1) holds for the case $\mu_i>0,\ i=1,2,\cdots,n$.

Similarly, it can be shown that (3.1) also holds for the case $\mu_i < 0,\ i = 1, 2, \cdots, n$, by (3.28). So $\rho(H_{\omega,\delta,\gamma,v}) < 1$ if and only if all the eigenvalues of $Q^{-1}B^TA^{-1}B$ have the same sign and (3.1) hold. (i) is proved.

For any other cases, $\lambda = 1-\omega$ is an eigenvalue of $H_{\omega,\delta,\gamma,v}$ by Lemma 3.1. By the conclusion proved, we have that λ is an eigenvalue of $H_{\omega,\delta,\gamma,v}$ if and only if $\lambda = 1-\omega$ or $\lambda \neq 1-\omega$ is a root of (3.22) for some $1 \leq i \leq n$.

Then $\rho(H_{\omega,\delta,\gamma,v}) < 1$ if and only if

$$|1-\omega| < 1 \tag{3.30}$$

and (3.25) hold. So based on the discussion above, $\rho(H_{\omega,\delta,\gamma,v}) < 1$ if and only if all the eigenvalues of $Q^{-1}B^TA^{-1}B$ have the same sign, (3.1) with $1 < \omega < 2$ hold. (ii) is proved. □

4. The Optimal Convergence Factor

In this section, we discuss the optimal convergence factor of the SSOR-like method (2.1). We first introduce GMPSD method presented in [38] and the known corresponding convergence results.

For a nonsingular and symmetric matrix $Q \in R^{n\times n}$, the initial vectors $x_0 \in R^m$, $y_0 \in R^n$ and parameters $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5$, satisfying

$$\alpha_1\alpha_2\left(1-\alpha_4\alpha_5\right)\left[1-(1-\alpha_5)\,\alpha_4\right] \neq 0,$$

the GMPSD method is

$$\begin{cases} y_{k+1} = y_k + \frac{1}{[1-(1-\alpha_5)\alpha_4](1-\alpha_4\alpha_5)}Q^{-1} \\ \qquad\qquad \times \left\{B^T[(\alpha_2-\alpha_1\alpha_4)x_k + \alpha_1\alpha_4A^{-1}(b-By_k)] - \alpha_2 q\right\}, \quad k \geq 0. \\ x_{k+1} = (1-\alpha_1)x_k + A^{-1}\{B\left[(\alpha_3-\alpha_1)y_k - \alpha_3 y_{k+1}\right] + \alpha_1 b\}, \end{cases} \tag{4.1}$$

The iterative matrix of the GMPSD method (4.1) is

$$\tilde{H}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5} = \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{pmatrix} \tag{4.2}$$

where

$$\begin{cases} \tilde{H}_{11} = (1-\alpha_1)I_m - \frac{\alpha_3(\alpha_2-\alpha_1\alpha_4)}{(1-\alpha_4\alpha_5)[1-(1-\alpha_5)\alpha_4]}A^{-1}BQ^{-1}B^T, \\ \tilde{H}_{12} = -\alpha_1A^{-1}B + \frac{\alpha_1\alpha_3\alpha_4}{(1-\alpha_4\alpha_5)[1-(1-\alpha_5)\alpha_4]}A^{-1}BQ^{-1}B^TA^{-1}B, \\ \tilde{H}_{21} = \frac{\alpha_2-\alpha_1\alpha_4}{(1-\alpha_4\alpha_5)[1-(1-\alpha_5)\alpha_4]}Q^{-1}B^T, \\ \tilde{H}_{22} = I_n - \frac{\alpha_1\alpha_4}{(1-\alpha_4\alpha_5)[1-(1-\alpha_5)\alpha_4]}Q^{-1}B^TA^{-1}B. \end{cases}$$

Lemma 4.1 (Theorem 3.3 in [38]). *Let A and Q be symmetric positive definite matrices and $B \in R^{m\times n}$ be of full column rank. Denote the minimum and the maximum eigenvalues of $Q^{-1}B^TA^{-1}B$ by $\mu_{\min}$ and $\mu_{\max}$ respectively. The spectral radius of the GMPSD method is minimized for any $\alpha_4 \neq \alpha_{2_{opt}}/\alpha_{1_{opt}}$ at*

$$\begin{cases} \alpha_{3_{opt}} = \frac{\alpha_{1_{opt}}(\alpha_{2_{opt}}-\alpha_4)}{\alpha_{2_{opt}}-\alpha_{1_{opt}}\alpha_4}, \\ \alpha_{1_{opt}} = \frac{4\sqrt{\mu_{\min}\mu_{\max}}}{(\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}})^2}, \quad \alpha_{2_{opt}} = \frac{(1-\alpha_4\alpha_5)[1-(1-\alpha_5)\alpha_4]}{\sqrt{\mu_{\min}\mu_{\max}}}, \end{cases} \tag{4.3}$$

and its corresponding value is

$$\rho(\tilde{H}_{\alpha_{1_{opt}},\alpha_{2_{opt}},\alpha_{3_{opt}},\alpha_4,\alpha_5}) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}}}.$$

Based on Lemma 4.1, we have

Theorem 4.1. *Under the assumption in Lemma* 4.1*, the optimal convergence factor of the SSOR-like method* (2.1) *with the four parameters ω and δ, γ and υ, satisfying* (2.2)*, is*

$$\rho_{opt} = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}}}, \tag{4.4}$$

and the corresponding optimal convergent parameters are

$$\begin{cases} \omega^* = \frac{4\sqrt{\mu_{\min}\mu_{\max}}}{\left(\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}}\right)^2}, & \delta^* = c, \\ \gamma^* = \frac{\frac{4}{(\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}})^2}-c}{\frac{1}{\sqrt{\mu_{\min}\mu_{\max}}}-c}, & \upsilon^* = \frac{1}{\sqrt{\mu_{\min}\mu_{\max}}}-c, \end{cases} \tag{4.5}$$

where c is real and satisfies $c \neq 1/\sqrt{\mu_{\min}\mu_{\max}}$.

Proof. Denote

$$Iter\_Matrix_{new} = \left\{H_{\omega,\delta,\gamma,\upsilon} \mid \omega(\delta+\upsilon)\neq 0\right\}$$

and

$$Iter\_Matrix_{GMPSD} = \left\{\tilde{H}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5} \mid \alpha_1\alpha_2\left(1-\alpha_4\alpha_5\right)\left[1-(1-\alpha_5)\alpha_4\right]\neq 0\right\},$$

where $H_{\omega,\delta,\gamma,\upsilon}$ and $\tilde{H}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$ are defined by (2.4) and (4.2) respectively.

For any $\tilde{H}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5} \in Iter\_Matrix_{GMPSD}$, with parameters α_1, α_2, α_3, α_4, α_5, it holds that

$$\alpha_1\alpha_2\left(1-\alpha_4\alpha_5\right)\left[1-(1-\alpha_5)\alpha_4\right]\neq 0.$$

Let

$$\begin{aligned} \omega &= \alpha_1, \quad \delta = \frac{\alpha_1\alpha_4}{\left(1-\alpha_4\alpha_5\right)\left[1-(1-\alpha_5)\alpha_4\right]}, \\ \gamma &= \alpha_3, \quad \upsilon = \frac{\alpha_2-\alpha_1\alpha_4}{\left(1-\alpha_4\alpha_5\right)\left[1-(1-\alpha_5)\alpha_4\right]}. \end{aligned} \tag{4.6}$$

Then $\omega(\delta+\upsilon)\neq 0$ and

$$\tilde{H}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}=H_{\omega,\delta,\gamma,\upsilon}\in Iter\_Matrix_{new}.$$

So

$$Iter\_Matrix_{GMPSD}\subseteq Iter\_Matrix_{new} \tag{4.7}$$

is true by the arbitrariness of $H_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}$.

Conversely, for any $H_{\omega,\delta,\gamma,\upsilon}\in Iter\_Matrix_{new}$, with parameters ω, δ, γ, υ, it holds that $\omega(\delta+\upsilon)\neq 0$. Let

$$\begin{cases} \alpha_1=\omega,\ \alpha_2=\delta+\upsilon,\ \alpha_3=\gamma,\ \alpha_4=0,\ \alpha_5=\frac{1}{2}, & if\ \delta=0,\\ \alpha_1=\omega,\ \alpha_2=-(\delta+\upsilon),\ \alpha_3=\gamma,\ \alpha_4=-\frac{\delta}{\omega},\ \alpha_5=\frac{1}{2}+\sqrt{\left(\frac{1}{2}+\frac{\omega}{\delta}\right)^2+\frac{\omega^2}{\delta^2}}, & if\ \delta\neq 0. \end{cases}$$

Then

$$\tilde{H}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}=H_{\omega,\delta,\gamma,\upsilon}\in Iter\_Matrix_{GMPSD}.$$

It follows from the arbitrariness of $H_{\omega,\delta,\gamma,\upsilon}$ that

$$Iter\_Matrix_{new}\subseteq Iter\_Matrix_{GMPSD}. \tag{4.8}$$

(4.7) together with (4.8) implies

$$Iter\_Matrix_{new}=Iter\_Matrix_{GMPSD}.$$

So,

$$\min_{\substack{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\\ \alpha_1\alpha_2(1-\alpha_4\alpha_5)[1-(1-\alpha_5)\alpha_4]\neq 0}}\rho\left(\tilde{H}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}\right)=\min_{\substack{\omega,\delta,\gamma,\upsilon\\ \omega(\delta+\upsilon)\neq 0}}\rho\left(H_{\omega,\delta,\gamma,\upsilon}\right)$$

follows. Therefore, the SSOR-like method defined by (2.1) and the GMPSD method own the same optimal convergence factors. By Lemma 4.1, the optimal convergence factor is ρ_{opt} defined by (4.4). The optimal convergence factor can be reached at all possible parameters ω^*, δ^*, γ^*, υ^* chosen by (4.5) due to

$$H_{\omega^*,\delta^*,\gamma^*,\upsilon^*}=\tilde{H}_{\alpha_{1_{opt}},\alpha_{2_{opt}},\alpha_{3_{opt}},\alpha_4,\alpha_5},$$

where $\alpha_{1_{opt}}$, $\alpha_{2_{opt}}$, $\alpha_{3_{opt}}$ are defined by (4.3), and α_4, α_5, satisfying

$$\alpha_4\neq\frac{\alpha_{2_{opt}}}{\alpha_{1_{opt}}},\qquad \left(1-\alpha_4\alpha_5\right)\left[1-(1-\alpha_5)\alpha_4\right]\neq 0,$$

are any suitable real numbers.

Table 1: The choice of Q for Example 5.1.

Case No.	Matrix Q	Description	Case No.	Matrix Q
I	$B^T\hat{A}^{-1}B$	$\hat{A} = diag(A)$	IV	$tridiag\left(B^TA^{-1}B\right)$
II	$B^T\hat{A}^{-1}B$	$\hat{A} = tridiag(A)$	V	$Q = \sqrt{\lambda_{\min}\lambda_{\max}}B^TB$
III	$tridiag\left(B^T\hat{A}^{-1}B\right)$	$\hat{A} = tridiag(A)$		

* where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and the maximum eigenvalues of A respectively.

5. Numerical Examples

In the following, we list the optimal convergent parameters and the optimal convergence factor of the new SSOR-like method (2.1) when it is applied to two frequently used examples. As is listed in Table 1 and Table 3, Q is chosen to be symmetric positive definite.

Example 5.1 ([10, 17, 18, 35, 39, 45, 51, 53]). Let

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in R^{2p^2 \times 2p^2}, \qquad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2p^2 \times p^2}$$

with

$$T = \frac{1}{h^2} tridiag(-1, 2, -1) \in R^{p \times p}, \qquad F = \frac{1}{h} tridiag(-1, 1, 0) \in R^{p \times p},$$

where $\otimes$ is the Kronecker product symbol and $h = 1/(p+1)$.

Let $m = 2p^2$, $n = p^2$ and Q be chosen in Table 1 as is listed in references. Then the optimal parameters and the optimal convergence factors are shown in Table 2 for different cases of m and n as is down in [10]. From Table 2, it can be seen that the optimal convergence factor is increasing gradually when m increases, and that $Q = tridiag(B^TA^{-1}B)$ may be the best choice for the method (2.1) from the view of the optimal convergence factor. By Theorem 4.1, the new method and the GMPSD method not only have the same optimal convergence factor but also have the same iteration matrix at the corresponding optimal parameters, so they have the same CPU time and iteration numbers. And then we can refer to the CPU time and iteration numbers of GMPSD method in Table 4 in [38].

Example 5.2 ([10, 18, 26, 39, 41, 44]). Let $A = (a_{ij})_{m\times m}$, $B = (b_{ij})_{m\times n}$ with

$$a_{ij} = \begin{cases} i+1, & i = j, \\ 1, & |i-j| = 1, \quad i, j = 1, 2, \cdots, m, \\ 0, & otherwise, \end{cases}$$

$$b_{ij} = \begin{cases} j, & i = j + m - n, \quad i = 1, 2, \cdots, m, \\ 0, & otherwise, \quad j = 1, 2, \cdots, n. \end{cases}$$

Let $m = 2p^2$, $n = p^2$ and Q be chosen in Table 3 as is listed in references, the optimal parameters and the optimal convergence factors are listed in Table 4 for four cases of m

Table 2: The optimal parameters and the optimal convergence factors for Example 5.1.

	m	n	$\rho_{opt} \approx$	$(\omega^*, \delta^*, \gamma^*, \upsilon^*)^T \approx$
Case I	128	64	0.675550	$\left(0.543632, c, \frac{0.203911-c}{0.375090-c}, 0.375090-c\right)^T$
	512	256	0.811229	$\left(0.341907, c, \frac{0.070648-c}{0.206629-c}, 0.206629-c\right)^T$
	1152	576	0.866671	$\left(0.248881, c, \frac{0.035411-c}{0.142280-c}, 0.142280-c\right)^T$
	2048	1024	0.896909	$\left(0.195554, c, \frac{0.021207-c}{0.108445-c}, 0.108445-c\right)^T$
Case II	128	64	0.580251	$\left(0.663309, c, \frac{0.331240-c}{0.499375-c}, 0.499375-c\right)^T$
	512	256	0.746384	$\left(0.442911, c, \frac{0.126417-c}{0.285422-c}, 0.285422-c\right)^T$
	1152	576	0.818124	$\left(0.330674, c, \frac{0.065628-c}{0.198468-c}, 0.198468-c\right)^T$
	2048	1024	0.858206	$\left(0.263483, c, \frac{0.040027-c}{0.151914-c}, 0.151914-c\right)^T$
Case III	128	64	0.492171	$\left(0.757767, c, \frac{1.478271-c}{1.950825-c}, 1.950825-c\right)^T$
	512	256	0.607108	$\left(0.631420, c, \frac{1.597456-c}{2.529944-c}, 2.529944-c\right)^T$
	1152	576	0.664441	$\left(0.558518, c, \frac{1.661205-c}{2.974309-c}, 2.974309-c\right)^T$
	2048	1024	0.700924	$\left(0.508706, c, \frac{1.705639-c}{3.352898-c}, 3.352898-c\right)^T$
Case IV	128	64	0.447748	$\left(0.799522, c, \frac{1.675696-c}{2.095872-c}, 2.095872-c\right)^T$
	512	256	0.560710	$\left(0.685604, c, \frac{1.949608-c}{2.843637-c}, 2.843637-c\right)^T$
	1152	576	0.619599	$\left(0.616097, c, \frac{2.098967-c}{3.406877-c}, 3.406877-c\right)^T$
	2048	1024	0.658135	$\left(0.566858, c, \frac{2.199517-c}{3.880193-c}, 3.880193-c\right)^T$
Case V	128	64	0.675550	$\left(0.543632, c, \frac{7321.192939-c}{13467.184744-c}, 13467.184744-c\right)^T$
	512	256	0.811229	$\left(0.341907, c, \frac{17347.719184-c}{50738.090075-c}, 50738.090075-c\right)^T$
	1152	576	0.866671	$\left(0.248881, c, \frac{27738.310313-c}{111452.280909-c}, 111452.280909-c\right)^T$
	2048	1024	0.896909	$\left(0.195554, c, \frac{38250.046189-c}{195597.917434-c}, 195597.917434-c\right)^T$

* where c is any real number such that $\upsilon^* \neq 0$.

Table 3: The choice of Q for Example 5.2.

Case No.	Matrix Q	Description	Case No.	Matrix Q
I	$B^T\hat{A}^{-1}B$	$\hat{A} = diag(A)$	II	B^TB

Table 4: The optimal parameters and the optimal convergence factors for Example 5.2.

	m	n	$\rho_{opt} \approx$	$(\omega^*, \delta^*, \gamma^*, \upsilon^*)^T \approx$
Case I	128	64	0.013754	$\left(0.999811, c, \frac{0.999427-c}{0.999616-c}, 0.999616-c\right)^T$
	512	256	0.003718	$\left(0.999986, c, \frac{0.999958-c}{0.999972-c}, 0.999972-c\right)^T$
	1152	576	0.001688	$\left(0.999997, c, \frac{0.999991-c}{0.999994-c}, 0.999994-c\right)^T$
	2048	1024	0.000958	$\left(0.999999, c, \frac{0.999997-c}{0.999998-c}, 0.999998-c\right)^T$
Case II	128	64	0.170187	$\left(0.971063, c, \frac{89.341790-c}{92.006648-c}, 92.006648-c\right)^T$
	512	256	0.171216	$\left(0.970685, c, \frac{352.883722-c}{363.540858-c}, 363.540858-c\right)^T$
	1152	576	0.171413	$\left(0.970618, c, \frac{792.111409-c}{816.090175-c}, 816.090175-c\right)^T$
	2048	1024	0.171483	$\left(0.970594, c, \frac{1407.028989-c}{1449.658194-c}, 1449.658194-c\right)^T$

* where c is any real number such that $\upsilon^* \neq 0$.

Table 5: The numerical results of Example 5.2 at optimal parameters.

		m	128	512	1152
		n	64	256	576
Case I	c=-1	IT	6	5	5
		CPU	9.1902e-004	0.0383	0.1868
		RES	3.8556e-010	1.6085e-010	6.8094e-012
	c=0	IT	6	5	5
		CPU	0.0015	0.0391	0.1855
		RES	1.9428e-010	8.0538e-011	3.4070e-012
	c=0.5	IT	6	5	4
		CPU	0.0010	0.0395	0.1494
		RES	9.8842e-011	4.0385e-011	6.8527e-010
	c=100	IT	7	6	5
		CPU	0.0012	0.0471	0.1859
		RES	1.9314e-010	2.1774e-011	3.3673e-010
Case II	c=-1	IT	13	13	13
		CPU	0.0019	0.0988	0.4833
		RES	9.6794e-010	6.6326e-010	5.8908e-010
	c=0	IT	13	13	13
		CPU	0.0021	0.0960	0.4794
		RES	9.6012e-010	6.6191e-010	5.8854e-010
	c=0.5	IT	13	13	13
		CPU	0.0025	0.0948	0.4748
		RES	9.5621e-010	6.6123e-010	5.8828e-010
	c=100	IT	12	13	13
		CPU	0.0019	0.0959	0.4874
		RES	5.9981e-010	5.2926e-010	5.3558e-010

and n as is down in [10]. It can be seen that $Q = B^T \hat{A}^{-1} B$ with $\hat{A} = tridiag(A)$ may be the best choice for the method (2.1) from the view of the optimal convergence factor.

In the following, the number of iterations (denoted by IT), CPU time and the RES at the optimal parameters are shown in Table 5 for two cases listed in Table 3 respectively, where CPU time is the average of cpu times in ten times and RES is defined by

$$RES = \frac{\sqrt{\|b - Ax_k - By_k\|^2 + \|q - B^T x_k\|^2}}{\sqrt{\|b - Ax_0 - By_0\|^2 + \|q - B^T x_0\|^2}}.$$

The initial guess was always a zero vector, the vector $(b^T, q^T)^T \in R^{m+n}$ was chosen such that $(1, 1, \cdots, 1)^T \in R^{m+n}$ was the exact solution of (1.1) and the numerical experiments stopped when the current iteration satisfies $RES < 10^{-9}$. It can been that the IT, CPU time and RES may be different if c is chosen to be different. All runs are performed on a PC with a 2.60 GHz 32-bit processor and 4.00 GB memory via MATLAB7.11.

References

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