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On a New SSOR-Like Method with Four Parameters for the Augmented Systems

Hui-Di Wang and Zheng-Da Huang*

School of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, China.

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Abstract. In this paper, we propose a new SSOR-like method with four parameters to solve the augmented system. And we analyze the convergence of the method and get the optimal convergence factor under suitable conditions. It is proved that the optimal convergence factor is the same as the GMPSD method [M.A. Louka and N.M. Missirlis, A comparison of the extrapolated successive overrelaxation and the preconditioned simultaneous displacement methods for augmented systems, Numer. Math. 131(2015) 517-540] with five parameters under the same assumption.

AMS subject classifications: 65F10

Key words: Convergence, SSOR-like method, convergence factor.

1. Introduction

Consider the saddle point problem or the augmented system in the following form:

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b \\ q \end{pmatrix}, \tag{1.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ with $rank(B) = n$ and $m \geq n$, $b \in R^m$ and $q \in R^n$ are two given vectors. Denote B^T as the transpose of the matrix B. Under these restrictions, the system (1.1) has a unique solution.

The system (1.1) appears in many scientific and engineering applications such as the mixed finite element approximation of elliptic partial differential equations [1], optimal control [2], constrained optimization [19], and so on [3,4,21,22,24,25].

For the system (1.1), there are many efficient iterative methods constructed such as the Uzawa-type methods [8, 20, 26, 27, 33, 36, 50, 52] and the Krylov subspace methods [6, 9, 30, 34, 43], the HSS-type methods [5, 7, 11–15, 29, 37, 42, 47, 54] and the SOR-like methods [10,16,23,28,32,35,44], the AOR-like methods [31,39] and SSOR-like methods [17,18,38,40,41,45,46,49,51,53].

*Corresponding author. *Email addresses:* `hdwang@zju.edu.cn` (H.-D. Wang), `zdhuang@zju.edu.cn` (Z.-D. Huang)

According to the number of parameters, the existing SSOR-like methods can be separated into the ones with one parameter in [17, 46, 49] and two parameters in [45, 51, 53], three parameters in [18, 40] and four parameters in [41] and five parameters in [38], respectively.

In this paper, we present a new SSOR-like method with four parameters. It has the simplest form than other SSOR-like methods [17,18,38,40,41,45,46,49,51,53]. We discuss the convergence of the new method and get the optimal convergence factor. It is proved that the new method has the same optimal convergence factor as the one of the generalized modified preconditioned simultaneous displacement(GMPSD) method with five parameters in [38], so the former with simpler form is at least as good as the latter.

The paper is organized as follows. In Section 2, the new SSOR-like method is proposed. The convergence analysis is done in Section 3, and the optimal convergence factor is estimated under a certain condition in Section 4. In Section 5, the optimal convergent parameters and the optimal convergence factors of the method applied to two frequently used examples are listed.

Throughout the paper, denote $N(B^T)$ and $\rho(B)$ as the null space of the matrix B^T and the spectral radius of the matrix B, respectively.

2. The New SSOR-Like Method

Let $Q \in R^{n\times n}$ be nonsingular and symmetric, and denote the initial vector by $x_0 \in R^m$, $y_0 \in R^n$. The new SSOR-like method with four given parameters ω, δ, γ, υ is defined by

$$\begin{cases} y_{k+1} = y_k + Q^{-1}B^T(\upsilon x_k - \delta A^{-1}By_k + \delta A^{-1}b) - (\delta+\upsilon)Q^{-1}q, \\ x_{k+1} = (1-\omega)x_k - A^{-1}\big\{B\,[(\omega-\gamma)y_k + \gamma y_{k+1}] - \omega b\big\}, \end{cases} \quad k \geq 0, \tag{2.1}$$

where parameters satisfy

$$\omega(\delta+\upsilon) \neq 0. \tag{2.2}$$

Denote $I_m \in R^{m\times m}$ and $I_n \in R^{n\times n}$ as identity matrices, respectively. The new SSOR-like method (2.1) can be rewritten as

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = H_{\omega,\delta,\gamma,\nu}\begin{pmatrix} x_k \\ y_k \end{pmatrix} + b_{\omega,\delta,\gamma,\nu}, \quad k \geq 0, \tag{2.3}$$

where

$$b_{\omega,\delta,\gamma,\nu} = \begin{pmatrix} \omega A^{-1}b - \gamma\delta A^{-1}BQ^{-1}B^TA^{-1}b + \gamma(\delta+\nu)A^{-1}BQ^{-1}q \\ \delta Q^{-1}B^TA^{-1}b - (\delta+\nu)Q^{-1}q \end{pmatrix}$$

and

$$H_{\omega,\delta,\gamma,\nu} = \begin{pmatrix} (1-\omega)I_m - \gamma\upsilon A^{-1}BQ^{-1}B^T & -\omega A^{-1}B + \gamma\delta A^{-1}BQ^{-1}B^TA^{-1}B \\ \nu Q^{-1}B^T & I_n - \delta Q^{-1}B^TA^{-1}B \end{pmatrix}. \tag{2.4}$$

Especially, when $\gamma(\gamma-\omega)(2\gamma-\omega)(\delta+\upsilon+4) \neq 0$, (2.1) as well as (2.3) can be splitted as

$$\begin{pmatrix} x_{k+\frac{1}{2}} \\ y_{k+\frac{1}{2}} \end{pmatrix} = M_{\omega,\delta,\gamma,\upsilon}\begin{pmatrix} x_k \\ y_k \end{pmatrix} + (D-\Omega L)^{-1}\begin{pmatrix} b \\ -q \end{pmatrix}, \quad k \geq 0, \tag{2.5}$$