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Higher-Order Rogue Wave and Rational Soliton Solutions of Discrete Complex mKdV Equations

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Abstract. The generalised perturbation $(n, N-n)$-fold Darboux transformation is used to derive new higher-order rogue wave and rational soliton solutions of the discrete complex mKdV equations. The structure of such waves and details of their evolution are investigated via numerical simulations, showing that the strong interaction yields weak oscillation and stability whereas the weak interaction is associated with strong oscillation and instability. A small noise has a weak influence on the wave propagation for the strong interaction, but substantially changes the wave behaviour in the weak interaction case.

AMS subject classifications: 35Q51, 35Q53, 37K05, 37K10

Key words: Discrete complex mKdV equation, modulational instability, generalised perturbation $(n, N-n)$-fold Darboux transformation, higher-order rogue wave solutions, higher-order rational soliton solutions.

1. Introduction

Nonlinear partial differential equations (NPDEs) frequently arise in scientific fields such as fluid dynamics, plasma physics or optics [31–36, 39], and their discrete analogues in areas such as population dynamics, plasma physics, real ionic fluids, and the evolution of slowly varying envelopes of electric fields [1, 2, 4, 5, 16–18, 20–23, 27, 29, 30]. Physicists and mathematicians have found rogue wave solutions of NPDEs in nonlinear optics, plasma physics, Bose-Einstein condensates, and financial markets [10, 11, 13, 19, 47, 48]. For continuous NPDEs, the rogue wave solutions have been constructed using the Darboux transformation and its generalisations [10,11,13,15]. Discrete rogue wave solutions of the single-component Ablowitz-Ladik and discrete Hirota equations [6–9], the focusing and defocusing Ablowitz-Ladik equations [26] and a discrete nonlocal nonlinear Schrödinger equation [46] have also been considered. Nevertheless, the discrete rogue wave solutions of discrete integrable NPDEs remain little studied.

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Let us consider the discrete complex mKdV equation

$$u_{n,t} = (1+|u_n|^2)(u_{n+1}-u_{n-1}), \tag{1.1}$$

where $u_n = u_n(t) = u(n,t)$ is the complex field of discrete and time variables n and t respectively, $u_{n,t}$ denotes the derivative du_n/dt, and $|u_n|^2 := u_n u_n^*$ where the asterisk denotes complex conjugation. Eq. (1.1) has been discussed in Ref. [8], and may be viewed as a discrete version of the continuous complex mKdV equation

$$iu_t + \frac{1}{2}u_{xxx} + 3|u|^2 u_x = 0$$

with many applications — e.g. in electrodynamics, electromagnetics, density stratification, elastic media and traffic flow [25,49,50].

Let $E : f(n,t) \to f(n+1,t)$ and $E^{-1} : f(n,t) \to f(n-1,t)$, $n \in \mathbb{Z}, t \in \mathbb{R}$ be the shift operators, where sometimes the variable t is repressed so $f(n,t)$ is abbreviated as $f(n)$. According to Refs. [3,40], the Lax pairs associated with Eq. (1.1) are

$$E\varphi_n = U_n\varphi_n = \begin{pmatrix} \lambda^2 & \lambda u_n \\ -\lambda u_n^* & 1 \end{pmatrix}\varphi_n\,, \tag{1.2}$$

$$\varphi_{n,t} = V_n\varphi_n = \begin{pmatrix} \dfrac{\lambda^2}{2} - \dfrac{1}{2\lambda^2} + u_n u_{n-1}^* & \lambda u_n + \dfrac{u_{n-1}}{\lambda} \\ -\lambda u_{n-1}^* - \dfrac{u_n^*}{\lambda} & -\dfrac{\lambda^2}{2} + \dfrac{1}{2\lambda^2} + u_n^* u_{n-1} \end{pmatrix}\varphi_n\,, \tag{1.3}$$

where the eigenvalue parameter λ does not depend on n and t, $\varphi_n := (\varphi_{1,n}, \varphi_{2,n})^T$ is the vector eigenfunction and T denotes the transposition. It is easily seen that the compatibility condition $\varphi_{n,t} = \varphi_{t,n}$ for Eqs. (1.2) and (1.3) yields Eq. (1.1). The approach adopted in Ref. [8] to obtain the two lowest-order rogue wave solutions of the discrete Hirota equation can also be used to produce low-order rogue wave solutions of Eq. (1.1).

In Section 2, a discrete integrable hierarchy associated with Eq. (1.1) is presented and new discrete integrable NPDEs are obtained. Section 3 deals with the modulation instability of the general plane-wave states of Eq. (1.1). Section 4 is devoted to a discrete version of the generalised perturbation $(n, N-n)$-fold Darboux transformation of Eq. (1.1), used to study integrable continuous NPDEs [41,42,44,45] and the discrete coupled Ablowitz-Ladik equation [43]. These ideas are applied to Eq. (1.1) in Section 5, and a higher-order discrete rogue wave solution that differs from the rogue wave solutions of discrete coupled Ablowitz-Ladik equations in Ref. [43] is obtained. Moreover, we also derive higher-order discrete rational soliton solutions. To the best of the author's knowledge, such solutions for discrete NPDEs have not been reported in literature before. Concluding remarks are in Section 6.

2. Discrete Integrable Hierarchy

Let us now consider the discrete matrix spectral problem

$$E\varphi_n = U_n(u,\lambda)\varphi_n, U_n(u,\lambda) = \begin{pmatrix} \lambda^2 & \lambda u_n \\ \lambda v_n & \beta \end{pmatrix}, \tag{2.1}$$