East Asian Journal on Applied Mathematics Vol. **8**, No. 1, pp. 139-150
doi: 10.4208/eajam.100916.061117a February 2018

A Modified Newton Method for Nonlinear Eigenvalue Problems

Xiao-Ping Chen[1,2] and Hua Dai[1,*]

[1] *Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China.*
[2] *Department of Mathematics, Taizhou University, Taizhou 225300, China.*

Received 10 September 2016; Accepted (in revised version) 6 November 2017.

Abstract. A modification to the Newton method for nonlinear eigenvalue problems is proposed and locally quadratic convergence of this algorithm is established. Numerical examples show the efficiency of the method and reduced computational cost.

AMS subject classifications: 65F15

Key words: Nonlinear eigenvalue problem, smallest singular value, Newton method, quadratic convergence.

1. Introduction

In this paper we consider the nonlinear eigenvalue problem (NEP)

$$T(\lambda)x = 0, \tag{1.1}$$

where $T(\lambda)$ is an analytic $n \times n$ matrix-valued function of complex variable λ. The problem (1.1) includes the classical eigenvalue problem ($T(\lambda) = \lambda I - A$), the generalised eigenvalue problem ($T(\lambda) = \lambda A - B$), the quadratic eigenvalue problem ($T(\lambda) = \lambda^2 A + \lambda B + C$) — [16, 27], the polynomial eigenvalue problem ($T(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + \lambda A_1 + A_0$) — [6, 16] and the delay eigenvalue problem ($T(\lambda) = \lambda I - A - \sum_{i=1}^{m} A_i e^{-\tau_i \lambda}$) — [10,11]. Nonlinear eigenvalue problems arise in various applications, including nonlinear ordinary differential equations [26], acoustic surface waves [32], photonic band structures [24], vibration of viscoelastic structures [1, 5] and fluid-solid structures [3, 30], simulation of quantum dots [8, 31], the stability of time-delay systems [20] and so on. For more information about possible applications of the NEP (1.1) and numerical methods for its solution the reader can consult Ref. [19].

It is well known that λ is an eigenvalue of the problem (1.1) if and only if it satisfies the characteristic equation

$$\det\big(T(\lambda)\big) = 0. \tag{1.2}$$

*Corresponding author. *Email addresses:* cxpnuaa@163.com (X.-P. Chen), hdai@nuaa.edu.cn (H. Dai)

The root of the equation (1.2) can be derived by the Newton method

$$\lambda_k = \lambda_{k-1} - \frac{1}{\mathrm{tr}\,(T(\lambda_{k-1})^{-1}T'(\lambda_{k-1}))}, \quad k = 1, 2, \cdots,$$

where $T'(\lambda_{k-1})$ is the derivative of $T(\lambda)$ at the point $\lambda = \lambda_{k-1}$ and $\mathrm{tr}\,(A)$ denotes the trace of the matrix A — cf. Ref. [13,16]. If an eigenvalue of the matrix T is known, the associated eigenvector can be constructed by the inverse iteration [21].

Let $T(\lambda)P(\lambda) = Q(\lambda)R(\lambda)$ be a QR decomposition with column pivoting of the matrix $T(\lambda)$ — i.e. $P(\lambda)$ and $Q(\lambda)$ are, respectively, permutation and unitary matrices, and the matrix $R(\lambda) = (r_{ij}(\lambda))$ is upper-triangular with the diagonal entries ordered as

$$\left|r_{11}(\lambda)\right| \geq \left|r_{22}(\lambda)\right| \geq \cdots \geq \left|r_{nn}(\lambda)\right|.$$

Kublanovskaya [15] used the Newton method for the equation $r_{nn}(\lambda) = 0$ to find an eigenvalue of the NEP (1.1). Jian and Singhal [9] improved the approach of [15] and established the quadratic convergence of the method. Li [17,18] presented sufficient conditions for a smooth QR decomposition and developed numerical methods for NEP (1.1). A similar method, but based on a smooth LU decomposition of $T(\lambda)$, has been studied in [4,32]. Nevertheless, for large matrices, the above methods are not as efficient as required.

If $\sigma_{min}(T(\lambda))$ is the smallest singular value of the matrix $T(\lambda)$, then λ is an eigenvalue of the NEP (1.1) if and only if it satisfies the equation

$$\sigma_{min}\big(T(\lambda)\big) = 0.$$

Hence, the singular value decomposition (SVD) of the matrix $T(\lambda)$ can be used to determine the solution of the NEP (1.1) and such an approach, combined with the Newton method, has been employed by Guo *et al.* — cf. Ref. [7]. On the other hand, the necessity to compute SVD of the corresponding matrix at each iteration step presents additional challenges in the method implementation. Therefore, the inverse iteration method is used to approximate the smallest singular value and the associated left and right singular vectors of $T(\lambda)$.

The paper is organized as follows. In Section 2, we consider the solution of the NEP (1.1) by the Newton method based on singular value decomposition and propose a modification of this approach. Section 3 deals with the convergence of the method. In particular, it is shown that convergence is locally quadratic. Numerical examples are discussed in Section 4, and our concluding remarks are in Section 5.

2. Modified Newton Method for NEP

Let I denote the identity matrix, $\mathrm{diag}\,(a_1, a_2, \cdots, a_n)$ the diagonal matrix with the diagonal entries $a_1, a_2, \cdots, a_n$, $T^H(\lambda)$ the conjugate transpose to the matrix $T(\lambda)$, and $\|A\|$ the Euclidean norm of a matrix (a vector) A. Any matrix A can be written in the form $A = [\alpha_1, \alpha_2, \cdots, \alpha_n]$, with the column-vectors α_j. This notation is used in what follows.