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A Weak Galerkin Finite Element Method for Multi-Term Time-Fractional Diffusion Equations

Jun Zhou[1,*], Da Xu[1,*] and Hongbin Chen[2]

[1] *Key Laboratory of High Performance Computing and Stochastic Information Processing, College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, P. R. China.*
[2] *College of Science, Central South University of Forestry and Technology, Changsha 410004, Hunan, China.*

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Abstract. The stability and convergence of a weak Galerkin finite element method for multi-term time-fractional diffusion equations with one-dimensional space variable are proved. Numerical experiments are consistent with theoretical analysis.

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Key words: Multi-term time-fractional diffusion equation, weak Galerkin finite element method, stability.

1. Introduction

Let $\tilde{P}_{\alpha,\alpha_1,\cdots,\alpha_n}(D_t)$ be the differential operator,

$$\tilde{P}_{\alpha,\alpha_1,\cdots,\alpha_m}(D_t) = D_t^{\alpha} + \sum_{j=1}^{m} d_j D_t^{\alpha_j}, \tag{1.1}$$

where d_j, $j = 1, \cdots, m$ are positive numbers, $0 < \alpha_m \leq \cdots \leq \alpha_1 < \alpha < 1$,

$$D_t^r u(t) := \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} u'(s) ds, \quad 0 < r < 1, \tag{1.2}$$

is the Caputo fractional derivative of order r with respect to variable t — cf. Ref. [17], and Γ the Γ-function.

Fractional differential equations often arise in applications [3,12,18]. It is not always possible to find an analytic solution of such equations, hence numerical methods have to

*Corresponding author. *Email addresses:* gfd25562@163.com (J. Zhou), daxu@hunnu.edu.cn (D. Xu), hongbinchen@csuft.edu.cn (H. Chen)

be used. In this work, we apply a weak Galerkin finite element method to initial boundary value problem for multi-term time-fractional diffusion equation with one-dimensional space variable

$$\begin{aligned}&\tilde{P}_{\alpha,\alpha_1,\dots,\alpha_m}(D_t)u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t),\\&u(0,t)=0,\quad u(L,t)=0,\\&u(x,0)=\psi(x),\\&x\in(0,L),\quad t\in(0,T],\end{aligned}\tag{1.3}$$

where $f(x,t)$ is a sufficiently smooth function.

Time fractional diffusion equations are studied in [8,13]. For numerical solution various methods have been proposed — e.g. Liu *et al.* [8] employed a finite difference method and developed a fractional predictor-corrector method, Zhao *et al.* [22] constructed a finite element method in space and finite difference method in time, Lopez-Marcos [5] and Lubich [7] investigated the spectral and finite element methods.

Weak Galerkin finite element method was initially introduced by Wang and Ye [19] to solve the second order elliptic problems. The main idea behind this method consists in the replacement of classical derivatives in standard variational equations by weak ones. This allows using of totally discontinuous finite elements with inferior values not related to the boundaries — cf. Ref. [16]. Nowadays, the method is widely recognised — e.g. Zhang *et al.* [21] applied it to elliptic problems with one-dimensional space variable, Chen and Zhang [1] to one-dimensional Burgers' equation, Li and Wang to parabolic equations [6], Mu *et al.* to Stokes [14] and Maxwell equations [15]. However, to the best of our knowledge, so far the weak Galerkin finite element method has not been employed in multi-term time-fractional diffusion equations. Here, we use the weak Galerkin finite element method in space and the backward Euler method in time. The corresponding integral terms are discretised by first-order convolution quadratures. We also prove the stability of the method, its convergence in L^2-norm, and derive error estimates.

The paper is organized as follows. In Section 2, we describe the weak Galerkin finite element method and write down a fully discrete weak Galerkin finite element equations. Section 3 is devoted to the stability and convergence of the method. In Section 4, we present results of numerical experiments and discuss their correlation with theoretical analysis. Our conclusion is in Section 5.

2. Weak Galerkin Finite Element Method

Let I refer to the interval $(0,L)$ and let $H^s(I)$ and $L^2(I)$ be, respectively, the usual Sobolev space and the space of square summable functions on I. The norms in $H^s(I)$ and $L^2(I)$ are denoted by $\|\cdot\|_s=\|\cdot\|_{H^s(I)}$ and $\|\cdot\|$, while $(\cdot,\cdot)$ is the inner product in $L^2(I)$. Moreover, we also consider the space $H_0^1(I)=\{v: v\in H^1(I), v(0)=0\}$.

Let us rewrite the problem (1.3) in the weak variational form — cf. Theorem 2.2 and Theorem 2.3 in Ref. [5]. Multiplying the equation (1.3) by $v\in H_0^1(I)$ and integrating the products arising, we obtain

$$\begin{aligned}&\big(\tilde{P}_{\alpha,\alpha_1,\cdots,\alpha_m}(D_t)u,v\big)+(u_x,v_x)=(f,v),\quad \forall v\in H_0^1(I),\quad 0<t\le T,\\&u(x,0)=\psi(x),\quad x\in I.\end{aligned}\tag{2.1}$$

By $\tilde{I}_a$ we denote a closed interval $[x_a,x_b]$ and by $I_a=(x_a,x_b)$ the interior of $\tilde{I}_a$. Weak functions v on $\tilde{I}_a$ are defined as $v=\{v_0,v_a,v_b\}$, where $v_0:=v|_{I_a}$, $v_a:=v(x_a)$ and $v_b:=v(x_b)$. Note that, in general, the values of v_0 at x_a and x_b are not connected with v_a and v_b. The space of all such functions is denoted by $W(I_a)$ — i.e.

$$W(I_a):=\big\{v=\{v_0,v_a,v_b\}:v_0\in L^2(I_a),\quad |v_a|+|v_b|<\infty\big\}.$$

Moreover, let r be a non-negative integer and $P_r(I_a)$ the set of all polynomials on I_a of degree at most r.

Proposition 2.1 (cf. Zhang & Tang [21]). *For any fixed element $v\in W(I_a)$ and for any $q\in P_r(I_a)$ there exists a unique solution $u\in P_r(I_a)$ of the equation*

$$\int_{I_a} uvq\,dx=-\int_{I_a} v_0q'dx+v_bq(x_b)-v_aq(x_a),\tag{2.2}$$

which does not depend on the choice of q.

This unique solution u of equation (2.2) is denoted by $d_{w,r}$ and called the discrete weak derivative of v. Thus Proposition 2.1 means that if $v\in H^1(I_a)$, then $d_{w,r}v$ is the L^2 projection of v_x onto subspace $P_r(I_a)$.

Now we can introduce a weak Galerkin finite element method for the problem (1.3). For $M\in\mathbb{N}$, we set $h=L/M$ and consider the partition $I_h=\{x_j|x_j=(j-1)h,1\le j\le M+1\}$ of the interval $[0,L]$ into the subintervals $I_i=(x_i,x_{i+1})$, $i=1,2,\cdots,M$. For any $k\in\mathbb{N}$, the discrete weak function space $W(I_h,k)$ on I_h is defined by

$$W(I_h,k):=\big\{v:v|_{I_i}\in W(I_i,k),\ i=1,2,\cdots,M\big\},\tag{2.3}$$

where

$$W(I_i,k):=\big\{v=\{v_0,v_i,v_{i+1}\}:v_0\in P_k(I_i),\quad |v_i|+|v_{i+1}|<\infty\big\}.\tag{2.4}$$

According to the definition, for weak functions $v=\{v_0,v_i,v_{i+1}\}\in W(I_i,k)$, the one-sided limits of the function v_0 at the points x_i and x_{i+1} not necessarily coincide with $v_i=v(x_i)$ and $v_{i+1}=v(x_{i+1}$. Let $\tilde{I}_{i,-}:=[x_{i-1},x_i]$ and $\tilde{I}_{i,+}:=[x_i,x_{i+1}]$. For a function $v=\{v_0,v_i,v_{i+1}\}\in W(I_i,k)$, let $v|_{\tilde{I}_{i,-}}=\{(v_0)_-,(v_{i-1})_-,(v_i)_-\}$ and $v|_{\tilde{I}_{i,+}}=\{(v_0)_+,(v_i)_+,(v_{i+1})_+\}$, then the jump of $v\in W(I_h,k)$ at the point x_i is

$$[v]_{x_i}=(v_i)_- -(v_i)_+.$$

Thus if $[v]_{x_i}=0$ then v is continuous at the point x_i. Now we consider the weak finite element spaces

$$S_h:=\big\{v:v\in W(I_h,k),\ [v]_{x_i}=0,\ i=2,\cdots,M\big\},\tag{2.5}$$

and

$$S_h^0 := \{v : v \in S_h, \quad v_1 = 0, \quad v_{M+1} = 0\}. \tag{2.6}$$

The discrete inner L^2-product on the space $W(I_h, k)$ is defined by

$$(u,v)_h = \sum_{i=1}^{M}(u,v)_{I_i} = \sum_{i=1}^{M}\int_{I_i} uv dx,$$

so that the corresponding norm of $u \in W(I_h, k)$ is $\|u\|_h^2 = (u,u)_h$. The semi-discrete weak Galerkin finite element scheme for problem (1.3) is based on the variational formulation of the problem (2.1) and consists in finding an element $u_h(t) \in S_h^0$ satisfying the equations

$$\big(\tilde{P}_{\alpha,\alpha_1,\ldots,\alpha_m}(D_t)u_{h,0}, v_0\big) + (d_{w,r}u_h, d_{w,r}v) = (f, v_0) \quad \text{for all} \quad v \in S_h^0, \tag{2.7}$$

$$u_h(x,0) = E_h\psi(x), \quad x \in I, \tag{2.8}$$

with an operator E_h introduced in the next section. Recall that for $\alpha \in (0,1)$, the Riemann-Liouville integral $I^{(\alpha)}$ is defined by

$$I^{(\alpha)}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \tag{2.9}$$

so that

$$D_t^\alpha u(t) = I^{(1-\alpha)}u_t(t). \tag{2.10}$$

Let $N \in \mathbb{N}$. Setting $\tau := T/N$ and $\Omega_\tau := \{(t_{n-1}, t_n)|t_n = n\tau, 0 \le n \le N\}$, we consider the time discretisation of the problem (1.3) at the time moment $t = t_n$. If $u_h^n = u_h^n(x) \in S_h$ denotes an approximation for $u(x, t_n)$ at the point $t = t_n$, $n = 0, 1, \cdots, N$, then the derivative u_t of u at the point t_n is obtained by the backward Euler method — viz.

$$(u_h)_t(t_n) \approx \delta_t u_h^n = \frac{1}{\tau}(u_h^n - u_h^{n-1}), \quad x \in [0, L], \quad n \ge 1. \tag{2.11}$$

Moreover, the integral $I^{(\alpha)}\varphi(t_n)$ is approximated according to the Lubich [9–11, 20] first order convolution quadrature formula

$$H_n^{(\alpha)}(\varphi) = \tau^\alpha \sum_{p=1}^{n} c_{n-p}^{(\alpha)}\varphi^p = \tau^\alpha \sum_{p=0}^{n-1} c_p^{(\alpha)}\varphi^{n-p}, \tag{2.12}$$

where the weights $c_p^{(\alpha)}$ are determined by the generating function

$$(1-z)^{-\alpha} = \sum_{p=0}^{\infty} c_p^{(\alpha)} z^p. \tag{2.13}$$

The fully discrete weak Galerkin finite element method for problem (1.3) consists in finding an element $u_h^n \in S_h$ ($u_h^n = \{u_{h,0}^n, u_{h,i}^n, u_{h,i+1}^n\}$ on each I_i, $i = 1, 2, \cdots, M$) which satisfies the equations

$$\sum_{p=0}^{n-1} \tilde{c}_p (\delta_t u_{h,0}^{n-p}, v_0) + (d_{w,r} u_h^n, d_{w,r} v)_h = (f^n, v_0), \quad \forall v \in S_h^0, \tag{2.14}$$

$$u_h^n(0) = u_h^n(L) = 0, \quad 1 \le n \le N, \tag{2.15}$$

$$u_h^0 = E_h u^0(x), \quad x \in [0, L], \tag{2.16}$$

where

$$\tilde{c}_p = \tau^{1-\alpha} c_p^{(1-\alpha)} + \sum_{j=1}^{m} d_j \tau^{1-\alpha_j} c_p^{(1-\alpha_j)}, \tag{2.17}$$

and the operator E_h is defined in next section.

3. Analysis of Weak Galerkin Finite Element Scheme

3.1. Stability

Let us start with auxiliary results needed in stability analysis.

Lemma 3.1. *If $\{c_p^{(\alpha)}\}_{p=0}^{\infty}$ is the sequence defined by* (2.13)*, then*

$$\left| \sum_{p=1}^{n-1} \frac{c_{p-1}^{(\alpha)}}{p} \right| \le C, \tag{3.1}$$

where the constant C does not depend on n.

The proof directly follows from the relation

$$c_p^{(\alpha)} = \frac{p^{\alpha-1}}{\Gamma(\alpha)} + O(p^{\alpha-2}), \quad p \to \infty, \tag{3.2}$$

cf. formula (4.6) in Ref. [9].

Lemma 3.2. *If*

$$0 < \alpha_m \le \cdots \le \alpha_1 < \alpha < 1, \quad m \in \mathbb{N}, \tag{3.3}$$

then the coefficients $\tilde{c}_p$ of (2.17) *satisfy the inequality*

$$\tilde{c}_n \le \tilde{c}_{n-1} \le \cdots \le \tilde{c}_1 \le \tilde{c}_0, \quad n = 1, 2, \cdots. \tag{3.4}$$

Proof. Indeed, if $\alpha \in (0,1)$, then $c_0^{(\alpha)} = 1$ and for any $p \ge 1$,

$$c_p^{(\alpha)} = (-1)^p \binom{-\alpha}{p} = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+p-1)}{p!}. \tag{3.5}$$

Therefore,

$$c_p^{(\alpha)} = \left(1 - \frac{1-\alpha}{p}\right) c_{p-1}^{(\alpha)}, \quad p \geq 1, \tag{3.6}$$

and taking into account the positivity of the coefficients d_j, we obtain

$$\begin{aligned}
\tilde{c}_{p-1} - \tilde{c}_p &= \tau^{1-\alpha}\left(c_{p-1}^{(1-\alpha)} - c_p^{(1-\alpha)}\right) + \sum_{j=1}^{m} d_j \tau^{1-\alpha_j}\left(c_{p-1}^{(1-\alpha_j)} - c_p^{(1-\alpha_j)}\right) \\
&= \tau^{1-\alpha}\alpha \frac{c_{p-1}^{(1-\alpha)}}{p} + \sum_{j=1}^{m} d_j \tau^{1-\alpha_j}\alpha_j \frac{c_{p-1}^{(1-\alpha_j)}}{p} \\
&\leq \alpha\left(\tau^{1-\alpha} + \sum_{j=1}^{m} d_j \tau^{1-\alpha_j}\right)\frac{c_{p-1}^{(1-\alpha_m)}}{p} = \alpha \tilde{c}_0 \frac{c_{p-1}^{(1-\alpha_m)}}{p},
\end{aligned} \tag{3.7}$$

so that the inequalities (3.4) follow. □

Theorem 3.1. *Any solution u_h^n of the equations* (2.14)-(2.16) *satisfies the inequality*

$$\|u_{h,0}^n\| \leq C\left(\|u_{h,0}^0\| + \frac{\tau}{\tilde{c}_0}\|f^n\|\right), \tag{3.8}$$

where here, and in what follows, C means a constant independent of h and n.

Proof. Substitution of $v = u_h^n$ into (2.14) yields

$$\sum_{p=0}^{n-1} \tilde{c}_p(\delta_t u_{h,0}^{n-p}, u_{h,0}^n) + \|d_{w,r} u_h^n\|_h^2 = (f^n, u_{h,0}^n). \tag{3.9}$$

Using the relation (2.11), we rewrite the first term in the left-hand side of (3.9) as

$$\begin{aligned}
\sum_{p=0}^{n-1} \tilde{c}_p(\delta_t u_{h,0}^{n-p}, u_{h,0}^n) =& \frac{1}{\tau}\sum_{p=0}^{n-1} \tilde{c}_p(u_{h,0}^{n-p} - u_{h,0}^{n-p-1}, u_{h,0}^n) \\
=& \frac{\tilde{c}_0}{\tau}\|u_{h,0}^n\|^2 + \frac{1}{\tau}\sum_{p=1}^{n-1}(\tilde{c}_p - \tilde{c}_{p-1})(u_{h,0}^{n-p}, u_{h,0}^n) - \frac{\tilde{c}_{n-1}}{\tau}(u_{h,0}^0, u_{h,0}^n).
\end{aligned} \tag{3.10}$$

Now we substitute (3.10) into (3.9) and use the Cauchy-Schwarz inequality, obtaining

$$\begin{aligned}
\tilde{c}_0\|u_{h,0}^n\|^2 \leq& \sum_{p=1}^{n-1}(\tilde{c}_{p-1} - \tilde{c}_p)(u_{h,0}^{n-p}, u_{h,0}^n) + \tilde{c}_{n-1}(u_{h,0}^0, u_{h,0}^n) + \tau(f^n, u_{h,0}^n) \\
\leq& \sum_{p=1}^{n-1}(\tilde{c}_{p-1} - \tilde{c}_p)\|u_{h,0}^{n-p}\|\|u_{h,0}^n\| + \tilde{c}_{n-1}\|u_{h,0}^0\|\|u_{h,0}^n\| + \tau\|f^n\|\|u_{h,0}^n\|.
\end{aligned} \tag{3.11}$$

By (3.7), the norm of the element $u_{h,0}^n$ can be estimated as

$$\|u_{h,0}^n\| \leq \alpha \sum_{p=1}^{n-1} \frac{c_{p-1}^{(1-\alpha_m)}}{p} \|u_{h,0}^{n-p}\| + \|u_{h,0}^0\| + \frac{\tau}{\tilde{c}_0} \|f^n\|, \tag{3.12}$$

and the discrete Gronwall inequality [2, Page 139] implies

$$\|u_{h,0}^n\| \leq exp\left(\alpha \sum_{p=1}^{n-1} \frac{c_{p-1}^{(1-\alpha_m)}}{p}\right)\left(\|u_{h,0}^0\| + \frac{\tau}{\tilde{c}_0}\|f^n\|\right), \tag{3.13}$$

which together with the inequality (3.1) completes the proof. □

3.2. Convergence

Here we study the convergence of the discrete weak Galerkin finite method (2.14)-(2.16) in L^2-norm. Considering the spaces S_h and $P_r(I_i)$, we set $r = k+1$ in (2.2) and define a projection operator $Q_h : H^1(I) \rightarrow W(I_h, k)$ by

$$Q_h u|_{I_i} = \left\{Q_{h,0}u, (Q_h u)_i, (Q_h u)_{i+1}\right\}, \quad i = 1, 2, \cdots, N.$$

where $Q_{h,0}$ is an L^2-projection from $L^2(I_i)$ to $P_k(I_i)$, and $(Q_h u)_i := u(x_i)$, $(Q_h u)_{i+1} := u(x_{i+1})$. It is known [21] that

$$\|Q_{h,0}u - u\|_{L^2(I_i)} \leq Ch^s \|u\|_{s,I_i}, \quad 0 \leq s \leq k+1. \tag{3.14}$$

Thus if $u \in H_0^1(I)$, then $Q_h u \in S_h^0$ and (3.14) shows that $Q_h u$ is a good approximation for functions from $H_0^1(I) \cap H^{s+1}(I)$, $s \geq 0$. Following [4], we introduce an elliptic projector $E_h : H_0^1(I) \cap H^2(I) \rightarrow S_h^0$ by

$$(d_{w,r} E_h u, d_{w,r}\chi) = (-u_{xx}, \chi_0), \quad \forall \chi \in S_h^0, \tag{3.15}$$

where $E_h u|_{I_i} = \{E_{h,0}u, (E_h u)_i, (E_h u)_{i+1}\}$, $i = 1, 2, \cdots, M$, and $u \in H_0^1(I) \cap H^2(I)$ is the exact solution of the problem (1.3).

Lemma 3.3 (cf. Gao & Mu [4]). *If $u \in H^{k+1}(I)$ is the exact solution of the problem* (1.3)*, then*

$$\|Q_{h,0}u - E_{h,0}u\| \leq Ch^{k+1}\|u\|_{k+1}. \tag{3.16}$$

By $R^{(\alpha)}(\varphi)(t_n)$ we denote the residue

$$R^{(\alpha)}(\varphi)(t_n) := I^{(\alpha)}\varphi(t_n) - H_n^{(\alpha)}(\varphi), \tag{3.17}$$

where $H_n^{(\alpha)}(\varphi)$ is defined in (2.12).

Lemma 3.4 (cf. Refs. [9,20]). *Let $\varphi = \varphi(t)$ be a real two times continuously differentiable function on the interval $(0,T)$, $T < \infty$. If φ_{tt} is integrable on $(0,T)$, then for any $t_n \in (0,T]$ the inequality*

$$\left|R^{(\alpha)}(\varphi)(t_n)\right| \le C\tau t_n^{\alpha-1}|\varphi(0)| + C\tau\int_0^{t_{n-1}}(t_n-s)^{\alpha-1}|\varphi_t(s)|\,ds + C\tau^\alpha\int_{t_{n-1}}^{t_n}|\varphi_t(s)|\,ds, \quad (3.18)$$

holds.

Theorem 3.2. *Assume that the problem* (1.3) *and equations* (2.14)−(2.16) *are solvable, and $u(x,t)$ and u_h^n are their corresponding solutions. If $u \in H^2(0,T;H^{k+1}(I))$, then*

$$\begin{aligned}\|u^n - u_{h,0}^n\| \le & C\Bigg(h^{k+1}\left(\|u(0)\|_{k+1} + \int_0^{t_n}\|u_t\|_{k+1}dt\right) \\ & + \tau\left(\|u_t(0)\| + \int_{t_{n-1}}^{t_n}\|u_{tt}\|dt + \int_0^{t_n}\|u_{tt}\|dt\right) \\ & + \tau\int_0^{t_{n-1}}(t_n-t)^{-\alpha}\left(\|u_{tt}\| + \sum_{j=1}^m d_j\|u_{tt}\|\right)dt\Bigg). \end{aligned} \quad (3.19)$$

Proof. Writing the difference $u^n - u_{h,0}^n$ in the form

$$u^n - u_{h,0}^n = \rho^n + \mu^n + e^n,$$

where $\rho^n = u^n - Q_{h,0}u^n$, $\mu^n = Q_{h,0}u^n - E_{h,0}u^n$ and $e^n = E_{h,0}u^n - u_{h,0}^n$, we recall that ρ^n and μ^n are, respectively, estimated in (3.14) and Lemma 3.3, so that

$$\|\rho^n\| \le Ch^{k+1}\|u^n\|_{k+1} \le Ch^{k+1}\left(\|u(0)\|_{k+1} + \int_0^{t_n}\|u_t\|_{k+1}dt\right), \quad (3.20)$$

$$\|\mu^n\| \le Ch^{k+1}\|u^n\|_{k+1} \le Ch^{k+1}\left(\|u(0)\|_{k+1} + \int_0^{t_n}\|u_t\|_{k+1}dt\right). \quad (3.21)$$

In order to estimate the term e^n, we write

$$\begin{aligned}&\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t e^{n-p}, v_0) + (d_{w,r}e^n, d_{w,r}v)_h \\ =&\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t E_h u^{n-p}, v_0) - \sum_{p=0}^{n-1}\tilde{c}_p(\delta_t u_h^{n-p}, v_0) + (d_{w,r}E_h u^n, d_{w,r}v)_h - (d_{w,r}u_h^n, d_{w,r}v)_h, \end{aligned} \quad (3.22)$$

and use the relations (2.14) and (3.15) to obtain

$$\begin{aligned}
&\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t e^{n-p},v_0)+(d_{w,r}e^n,d_{w,r}v)_h\\
=&\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t E_h u^{n-p},v_0)-\big(u_{xx}(t_n),v_0\big)-(f^n,v_0)\\
=&\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t E_h u^{n-p},v_0)-\big(\tilde{P}_{\alpha,\alpha_1,\dots,\alpha_m}(D_t)u(t_n),v_0\big)\\
=&\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t E_h u^{n-p},v_0)-\left(I^{(1-\alpha)}u_t(t_n)+\sum_{j=1}^{m}d_jI^{(1-\alpha_j)}u_t(t_n),v_0\right).
\end{aligned}\tag{3.23}$$

The identity $\rho^n+\mu^n=u^n-E_{h,0}^n$ implies that

$$\begin{aligned}
&\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t e^{n-p},v_0)+(d_{w,r}e^n,d_{w,r}v)_h\\
=&\sum_{p=0}^{n-1}\tilde{c}_p\big(\delta_t u^{n-p}-u_t(t_{n-p}),v_0\big)-\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t\rho^{n-p},v_0)-\sum_{p=0}^{n-1}\tilde{c}_p(\delta_t\mu^{n-p},v_0)\\
&-\left(I^{(1-\alpha)}u_t(t_n)+\sum_{j=1}^{m}d_jI^{(1-\alpha_j)}u_t(t_n)-\sum_{p=0}^{n-1}\tilde{c}_pu_t(t_{n-p}),v_0\right).
\end{aligned}\tag{3.24}$$

Recalling the relations (2.12), (3.17) and (2.17), we rewrite the third line in (3.24) as

$$\begin{aligned}
&I^{(1-\alpha)}u_t(t_n)+\sum_{j=1}^{m}d_jI^{(1-\alpha_j)}u_t(t_n)-\sum_{p=0}^{n-1}\tilde{c}_pu_t(t_{n-p})\\
=&I^{(1-\alpha)}u_t(t_n)-H_n^{(1-\alpha)}(u_t)+\sum_{j=1}^{m}d_j\left(I^{(1-\alpha_j)}u_t(t_n)-H_n^{(1-\alpha_j)}(u_t)\right)\\
=&R^{(1-\alpha)}(u_t)(t_n)+\sum_{j=1}^{m}d_jR^{(1-\alpha_j)}(u_t)(t_n).
\end{aligned}\tag{3.25}$$

Now Theorem 3.1, Lemma 3.2 and triangle inequality lead to the inequality

$$\|e^n\|\le C\left(\|e^0\|+\frac{\tau}{\tilde{c}_0}(\Pi_4^n+\Pi_1^n+\Pi_2^n+\Pi_3^n)\right).\tag{3.26}$$

On the other hand, the terms Π_l^n, $l=1,2,3,4$ can be estimated as follows

$$\Pi_1^n=\left\|\sum_{p=0}^{n-1}\tilde{c}_p\delta_t\rho^{n-p}\right\|=\left\|\sum_{p=1}^{n}\frac{\tilde{c}_{n-p}}{\tau}\int_{t_{p-1}}^{t_p}\rho_t\,dt\right\|\le\frac{\tilde{c}_0}{\tau}\int_0^{t_n}\|\rho_t\|dt, \tag{3.27}$$

$$\Pi_2^n=\left\|\sum_{p=0}^{n-1}\tilde{c}_p\delta_t\mu^{n-p}\right\|=\left\|\sum_{p=1}^{n}\frac{\tilde{c}_{n-p}}{\tau}\int_{t_{p-1}}^{t_p}\mu_t\,dt\right\|\le\frac{\tilde{c}_0}{\tau}\int_0^{t_n}\|\mu_t\|dt, \tag{3.28}$$

$$\begin{aligned}\Pi_3^n&=\left\|R^{(1-\alpha)}(u_t)(t_n)+\sum_{j=1}^{m}d_jR^{(1-\alpha_j)}(u_t)(t_n)\right\|\\&\le\left\|R^{(1-\alpha)}(u_t)(t_n)\right\|+\sum_{j=1}^{m}d_j\left\|R^{(1-\alpha_j)}(u_t)(t_n)\right\|,\end{aligned} \tag{3.29}$$

$$\begin{aligned}\Pi_4^n&=\left\|\sum_{p=0}^{n-1}\tilde{c}_p\big(\delta_tu(t_{n-p})-u_t(t_{n-p})\big)\right\|\le\tilde{c}_0\sum_{p=1}^{n}\left\|\delta_tu(t_p)-u_t(t_p)\right\|\\&=\frac{\tilde{c}_0}{\tau}\left\|\sum_{p=1}^{n}\int_{t_{p-1}}^{t_p}(t-t_{p-1})u_{tt}\,dt\right\|\le\tilde{c}_0\sum_{p=1}^{n}\int_{t_{p-1}}^{t_p}\|u_{tt}\|dt=\tilde{c}_0\int_0^{t_n}\|u_{tt}\|dt,\end{aligned} \tag{3.30}$$

It follows from (3.18) and (3.20)−(3.21) that

$$\Pi_1^n\le C\tau^{-1}\tilde{c}_0h^{k+1}\int_0^{t_n}\|u_t\|_{k+1}dt,\qquad\Pi_2^n\le C\tau^{-1}\tilde{c}_0h^{k+1}\int_0^{t_n}\|u_t\|_{k+1}dt, \tag{3.31}$$

and

$$\begin{aligned}\Pi_3^n\le&C\tau\left(t_n^{-\alpha}+\sum_{j=1}^{m}d_jt_n^{-\alpha_j}\right)\|u_t^0\|+C\left(\tau^{1-\alpha}+\sum_{j=1}^{m}d_j\tau^{1-\alpha_j}\right)\int_{t_{n-1}}^{t_n}\|u_{tt}\|dt\\&+C\tau\left(\int_0^{t_{n-1}}(t_n-t)^{-\alpha}\|u_{tt}\|dt+\sum_{j=1}^{m}d_j\int_0^{t_{n-1}}(t_n-t)^{-\alpha_j}\|u_{tt}\|dt\right).\end{aligned} \tag{3.32}$$

Besides, according to (2.16), we have

$$\|e^0\|=\|E_{h,0}u^0-u_{h,0}^0\|=0. \tag{3.33}$$

The relations (3.30), (3.31)−(3.33) and (3.25) yield

$$\begin{aligned}\|e^n\|\le&C\tau\left(\int_{t_{n-1}}^{t_n}\|u_{tt}\|dt+\int_0^{t_n}\|u_{tt}\|dt\right)+C\frac{\tau}{\tilde{c}_0}\left(t_n^{-\alpha}+\sum_{j=1}^{m}d_jt_n^{-\alpha_j}\right)\|u_t^0\|\\&+C\frac{\tau}{\tilde{c}_0}\left(\int_0^{t_{n-1}}(t_n-t)^{-\alpha}\|u_{tt}\|dt+\sum_{j=1}^{m}d_j\int_0^{t_{n-1}}(t_n-t)^{-\alpha_j}\|u_{tt}\|dt\right)\\&+Ch^{k+1}\int_0^{t_n}\|u_t\|_{k+1}dt.\end{aligned} \tag{3.34}$$

Note that if $0<\alpha_m\le\cdots\le\alpha_1<\alpha<1$, then

$$\begin{aligned}\frac{\tau}{\tilde{c}_0}\left(t_n^{-\alpha}+\sum_{j=1}^m d_j t_n^{-\alpha_j}\right)&=\frac{\tau}{\tilde{c}_0}\left(n^{-\alpha}\tau^{-\alpha}+\sum_{j=1}^m d_j n^{-\alpha_j}\tau^{-\alpha_j}\right)\\&\le\frac{\tau}{\tilde{c}_0}n^{-\alpha_m}\left(\tau^{-\alpha}+\sum_{j=1}^m d_j\tau^{-\alpha_j}\right)=\tau n^{-\alpha_m}\le\tau,\end{aligned}\tag{3.35}$$

$$\frac{\tau}{\tilde{c}_0}\le\frac{\tau}{\tau^{-\alpha_m}(1+\sum\limits_{j=1}^m d_j)}=C\tau^{1+\alpha_m}\le C\tau,\tag{3.36}$$

and the term e^n can be now estimated as follows

$$\begin{aligned}\|e^n\|\le&C\tau\left(\int_{t_{n-1}}^{t_n}\|u_{tt}\|dt+\int_0^{t_n}\|u_{tt}\|dt\right)+C\tau\|u_t(0)\|+Ch^{k+1}\int_0^{t_n}\|u_t\|_{k+1}dt\\&+C\tau\left(\int_0^{t_{n-1}}(t_n-t)^{-\alpha}\|u_{tt}\|dt+\sum_{j=1}^m d_j\int_0^{t_{n-1}}(t_n-t)^{-\alpha_j}\|u_{tt}\|dt\right).\end{aligned}\tag{3.37}$$

Finally, using the triangle inequality

$$\|u^n-u_{h,0}^n\|\le\|\rho^n\|+\|\mu^n\|+\|e^n\|,$$

and (3.20) − (3.21), (3.37), we obtain the estimate (3.19). ☐

4. Numerical Experiments

Example 4.1. As an example, we consider the problem (1.3) with the parameters: $m=1$, $d_j=1$, $L=1$, $T=1$. The right-hand side f and initial condition are

$$\begin{aligned}&f(x,t)=2(x-x^2)\left(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)}+t^2+1\right),\\&\psi(x)=x-x^2.\end{aligned}$$

The exact solution of this problem is

$$u(x,t)=(x-x^2)(t^2+1),\quad 0<x<1,\quad 0\le t\le 1.$$

Let us also use the notation

$$\mathrm{E}(h,\tau)=\|u^n-u_{h,0}^n\|,\quad \mathrm{R}^x=\log_2\left(\frac{\mathrm{E}(2h,\tau)}{\mathrm{E}(h,\tau)}\right),\quad \mathrm{R}^t=\log_2\left(\frac{\mathrm{E}(h,2\tau)}{\mathrm{E}(h,\tau)}\right).$$

Table 1 presents L^2-errors and the convergence rates R^x of the weak Galerkin finite element scheme (2.14) − (2.16) for $\alpha_1=0.2$ and various choices of α. For $k=0$ and $k=1$ the convergence rates are approximately equal to 1 and 2, consistent with Theorem 3.2. The time convergence rates R^t for $\alpha_1=0.2$, presented in Table 2 are also consistent with theoretical analysis.

Table 1: L^2-errors and convergence rate of the weak Galerkin finite element method, $N = 1024$, $\alpha_1 = 0.2$.

α	M	$k=0$		$k=1$	
		E	R^x	E	R^x
0.25	8	2.82964e-02	$*$	1.59180e-03	$*$
	16	1.42152e-02	0.99318	3.97974e-04	1.99991
	32	7.11595e-03	0.99831	9.95878e-05	1.99863
	64	3.55902e-03	0.99958	2.52707e-05	1.97850
0.5	8	2.82964e-02	$*$	1.59182e-03	$*$
	16	1.42152e-02	0.99318	3.98025e-04	1.99974
	32	7.11595e-03	0.99831	9.97884e-05	1.99591
	64	3.55903e-03	0.99957	2.60503e-05	1.93757
0.75	8	2.82964e-02	$*$	1.59184e-03	$*$
	16	1.42152e-02	0.99318	3.98125e-04	1.99940
	32	7.11596e-03	0.99830	1.00188e-04	1.99051
	64	3.55904e-03	0.99957	2.75428e-05	1.86297

Table 2: The L^2-norm errors and time convergence orders when $M = 5000$ and $\alpha_1 = 0.2$.

α	N	$k=0$		$k=1$	
		E	R^t	E	R^t
0.25	4	1.28399e-03	$*$	1.28300e-03	$*$
	8	6.09060e-04	1.07595	6.07174e-04	1.07934
	16	2.98566e-04	1.02854	2.94890e-04	1.04193
	32	1.52399e-04	0.97020	1.45252e-04	1.02161
0.5	4	2.16668e-03	$*$	2.16609e-03	$*$
	8	1.03873e-03	1.06067	1.03762e-03	1.06182
	16	5.09430e-04	1.02786	5.07284e-04	1.03241
	32	2.54941e-04	0.99872	2.50734e-04	1.01664
0.75	4	3.19865e-03	$*$	3.19825e-03	$*$
	8	1.55744e-03	1.03828	1.55671e-03	1.03879
	16	7.69151e-04	1.01784	7.67731e-04	1.01982
	32	3.83992e-04	1.00219	3.81212e-04	1.01001

5. Conclusion

In this paper, we show the stability and convergence of the weak Galerkin finite element method for multi-term time-fractional diffusion equations. Numerical results are consistent with theoretical analysis.

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