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A Weak Galerkin Finite Element Method for Multi-Term Time-Fractional Diffusion Equations

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Abstract. The stability and convergence of a weak Galerkin finite element method for multi-term time-fractional diffusion equations with one-dimensional space variable are proved. Numerical experiments are consistent with theoretical analysis.

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Key words: Multi-term time-fractional diffusion equation, weak Galerkin finite element method, stability.

1. Introduction

Let $\tilde{P}_{\alpha,\alpha_1,\cdots,\alpha_n}(D_t)$ be the differential operator,

$$\tilde{P}_{\alpha,\alpha_1,\cdots,\alpha_m}(D_t) = D_t^{\alpha} + \sum_{j=1}^{m} d_j D_t^{\alpha_j}, \tag{1.1}$$

where d_j, $j = 1, \cdots, m$ are positive numbers, $0 < \alpha_m \le \cdots \le \alpha_1 < \alpha < 1$,

$$D_t^r u(t) := \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} u'(s) ds, \quad 0 < r < 1, \tag{1.2}$$

is the Caputo fractional derivative of order r with respect to variable t — cf. Ref. [17], and Γ the Γ-function.

Fractional differential equations often arise in applications [3, 12, 18]. It is not always possible to find an analytic solution of such equations, hence numerical methods have to

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be used. In this work, we apply a weak Galerkin finite element method to initial boundary value problem for multi-term time-fractional diffusion equation with one-dimensional space variable

$$\begin{aligned}
&\tilde{P}_{\alpha,\alpha_1,\ldots,\alpha_m}(D_t)u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\\
&u(0,t) = 0, \quad u(L,t) = 0,\\
&u(x,0) = \psi(x),\\
&x \in (0,L), \quad t \in (0,T],
\end{aligned} \tag{1.3}$$

where $f(x,t)$ is a sufficiently smooth function.

Time fractional diffusion equations are studied in [8,13]. For numerical solution various methods have been proposed — e.g. Liu *et al.* [8] employed a finite difference method and developed a fractional predictor-corrector method, Zhao *et al.* [22] constructed a finite element method in space and finite difference method in time, Lopez-Marcos [5] and Lubich [7] investigated the spectral and finite element methods.

Weak Galerkin finite element method was initially introduced by Wang and Ye [19] to solve the second order elliptic problems. The main idea behind this method consists in the replacement of classical derivatives in standard variational equations by weak ones. This allows using of totally discontinuous finite elements with inferior values not related to the boundaries — cf. Ref. [16]. Nowadays, the method is widely recognised — e.g. Zhang *et al.* [21] applied it to elliptic problems with one-dimensional space variable, Chen and Zhang [1] to one-dimensional Burgers' equation, Li and Wang to parabolic equations [6], Mu *et al.* to Stokes [14] and Maxwell equations [15]. However, to the best of our knowledge, so far the weak Galerkin finite element method has not been employed in multi-term time-fractional diffusion equations. Here, we use the weak Galerkin finite element method in space and the backward Euler method in time. The corresponding integral terms are discretised by first-order convolution quadratures. We also prove the stability of the method, its convergence in L^2-norm, and derive error estimates.

The paper is organized as follows. In Section 2, we describe the weak Galerkin finite element method and write down a fully discrete weak Galerkin finite element equations. Section 3 is devoted to the stability and convergence of the method. In Section 4, we present results of numerical experiments and discuss their correlation with theoretical analysis. Our conclusion is in Section 5.

2. Weak Galerkin Finite Element Method

Let I refer to the interval $(0,L)$ and let $H^s(I)$ and $L^2(I)$ be, respectively, the usual Sobolev space and the space of square summable functions on I. The norms in $H^s(I)$ and $L^2(I)$ are denoted by $\|\cdot\|_s = \|\cdot\|_{H^s(I)}$ and $\|\cdot\|$, while $(\cdot,\cdot)$ is the inner product in $L^2(I)$. Moreover, we also consider the space $H_0^1(I) = \{v : v \in H^1(I), v(0) = 0\}$.

Let us rewrite the problem (1.3) in the weak variational form — cf. Theorem 2.2 and Theorem 2.3 in Ref. [5]. Multiplying the equation (1.3) by $v \in H_0^1(I)$ and integrating the